

## On Certain Lattices of Degrees of Interpretability

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**1 Preliminaries** All theories  $S, T, A, B$ , etc., considered in what follows are primitive recursive (Craig's theorem).  $A, B$ , etc., are reflexive extensions of Peano arithmetic  $P$ . We write  $S \vdash X$  or  $X \vdash S$ , where  $X$  is a set of sentences, to mean that  $S \vdash \phi$  for every  $\phi \in X$ . Thus  $S \vdash T$  means that  $S$  is a subtheory of  $T$ .  $S$  is an  $X$ -subtheory of  $T$ ,  $S \vdash_X T$ , if  $S \vdash \phi$  implies  $T \vdash \phi$  for every  $\phi \in X$ .  $S \leq T$  will be used to indicate that  $S$  is (relatively) interpretable in  $T$ .  $S < T$  iff  $S \leq T$  and  $S \not\leq T$  and  $S \equiv T$  iff  $S \leq T \leq S$ .  $X \upharpoonright k = \{n \in X: n \leq k\}$ . Thus  $A$  is reflexive iff  $A \vdash \text{Con}_{A \upharpoonright n}$  for every  $n$ .  $A$  is essentially reflexive if every extension of  $A$  in the language of  $A$  is reflexive.  $\phi^i$  is  $\phi$  if  $i = 0$  and  $\neg\phi$  if  $i = 1$ . Terminology and notation not explained here are standard (cf. [1]).

All proofs below of the existence of interpretations are applications, directly or indirectly, of the following basic result established by Feferman [1]:

**Lemma 1** *If  $P \vdash T$  and  $\sigma(x)$  numerates  $S$  in  $T$ , then  $S \leq T + \text{Con}_\sigma$ .*

This is proved by showing that the denumerable case of the Henkin completeness proof can be formalized in  $P$ .

For any formula  $\sigma(x)$ , let  $\sigma^*(x)$  be the formula

$$\sigma(x) \wedge \text{Con}_{\sigma(y) \wedge y \leq x}^y.$$

This definition and the following lemma are again due to Feferman [1].

**Lemma 2**

- (i) *If  $P \vdash T$  and  $\sigma(x)$  binumerates  $S$  in  $T$  and for every  $n$ ,  $T \vdash \text{Con}_{S \upharpoonright n}$ , then  $\sigma^*(x)$  binumerates  $S$  in  $T$ .*
- (ii)  *$P \vdash \text{Con}_{\sigma^*}$ .*

*Proof:* (i) is obvious. To prove (ii) we argue in  $P$  as follows: If  $Con_\sigma$ , then  $Con_{\sigma^*}$ . So suppose  $\neg Con_\sigma$ . Then there is a  $z$  such that  $\neg Con_{\sigma(x) \wedge x \leq z}$ . Let  $z_0$  be the least such  $z$ . Then  $Con_{\sigma(x) \wedge x < z_0}$  and  $\sigma^*(x) \rightarrow \sigma(x) \wedge x < z_0$ . Thus  $Con_{\sigma^*}$  in this case too.

The following key lemma is all but stated explicitly in the work of Feferman [1] and Orey [6]. It was resurrected and formulated explicitly by Hájek [3].

**Lemma 3**  $S \leq A$  iff  $A \vdash Con_{S \upharpoonright n}$  for every  $n$ .

*Proof:* Suppose first  $S \leq A$ . Then for every  $n$ , there is an  $m$  such that  $S \upharpoonright n \leq A \upharpoonright m$ . But then, by Theorem 6.4 of [1],  $P \vdash Con_{A \upharpoonright m} \rightarrow Con_{S \upharpoonright n}$ , whence  $A \vdash Con_{S \upharpoonright n}$ , since  $A$  is reflexive. This proves “only if”. To prove “if” suppose  $A \vdash Con_{S \upharpoonright n}$  for all  $n$ . Let  $\sigma(x)$  be any formula binumerating  $S$  in  $A$ . Then, by Lemma 2(i),  $\sigma^*(x)$  binumerates  $S$  in  $A$ . Hence, by Lemma 1 and Lemma 2(ii),  $S \leq A$ .

One immediate consequence of Lemma 3 is the following (cf. [6]):

**Lemma 4** (Orey’s compactness theorem)  $S \leq A$  iff  $S \upharpoonright k \leq A$  for every  $k$ .

If  $P + \phi \leq A$  and  $\phi$  is  $\Pi_1^0$ , then  $A \vdash \phi$ . Hence, by Lemma 3, we get (cf. [2], [3], [4]):

**Lemma 5**  $A \leq B$  iff  $A \vdash_{\Pi_1^0} B$ .

A sentence  $\phi$  is  $X$ -conservative over  $T$  if  $T + \phi \vdash_X T$ . Thus, by Lemma 5,  $A + \phi \leq A$  iff  $\phi$  is  $\Pi_1^0$ -conservative over  $A$ . In the following,  $\Gamma$  is either  $\Sigma_{n+1}^0$  or  $\Pi_{n+1}^0$  and  $\bar{\Gamma}$  is the dual of  $\Gamma$ . By an obvious modification of the proof of Theorem 1 [5] (due to Guaspari [2]), we get

**Lemma 6** Suppose  $P \vdash T$  and let  $X$  be any r.e. set. Then there is a  $\Gamma$  formula  $\eta(x, y)$  such that for all  $k$  and  $\phi$ ,

- (i) if  $k \in X$ , then  $T + \phi \vdash \neg \eta(\bar{k}, \bar{\phi})$
- (ii) if  $k \notin X$ , then  $\eta(\bar{k}, \bar{\phi})$  is  $\bar{\Gamma}$ -conservative over  $T + \phi$ .

A set  $X$  of sentences is said to be *monoconsistent* with  $T$  if  $T + \phi$  is consistent for every  $\phi \in X$ .

**Lemma 7** Suppose  $P \vdash T$  and  $X$  is r.e. and monoconsistent with  $T$ . Then there is a  $\Gamma$  sentence  $\psi \notin X$  which is  $\bar{\Gamma}$ -conservative over  $T$ .

*Proof:* Let  $\eta(x, y)$  be as in Lemma 6 and let  $\psi$  be such that  $P \vdash \psi \leftrightarrow \eta(\bar{\psi}, \bar{0} = \bar{0})$ . If  $\psi \in X$ , then, by Lemma 6(i),  $T \vdash \neg \eta(\bar{\psi}, \bar{0} = \bar{0})$ , whence  $T \vdash \neg \psi$ , which is impossible. Thus  $\psi \notin X$ . But then, by Lemma 6(ii),  $\psi$  is as desired.

**2 Degrees of interpretability** Throughout the rest of this paper  $T$  is a consistent primitive recursive essentially reflexive extension of  $P$ , e.g.,  $P$  or  $ZF$ , and  $A, B$ , etc., are extensions of  $T$  in the language of  $T$ . Thus  $A, B$ , etc., are essentially reflexive. Clearly  $\equiv$  (mutual interpretability) is an equivalence relation. Its equivalence classes  $\{B: B \equiv A\}$  will be called *degrees (of interpretability)* and will be written  $a, b$ , etc.  $d(A)$  is the degree of  $A$ . Let  $a \leq b$

mean that  $A \leq B$  for  $A \in a$  and  $B \in b$ . Finally let  $D_T$  be the partially ordered set of degrees thus defined.

We now define the operations  $\downarrow$  and  $\uparrow$  on theories as follows. Let

$$\begin{aligned} A^T &= T \cup \{Con_{A \upharpoonright n} : n \in \omega\}, \\ A \downarrow B &= \{\phi \vee \psi : \phi \in A \text{ \& } \psi \in B\}, \\ A \uparrow B &= A^T \cup B^T. \end{aligned}$$

Thus  $Th(A \downarrow B) = Th(A) \cap Th(B)$ , where  $Th(A) = \{\phi : A \vdash \phi\}$ . The following lemma is then an immediate consequence of Lemma 3.

**Lemma 8**

- (i)  $A \leq B$  iff  $A^T \vdash B$ . Thus  $A^T \equiv A$  and  $A \leq B$  iff  $A^T \vdash B^T$ .
- (ii)  $A \leq B, C$  iff  $A \leq B \downarrow C$ .
- (iii)  $A, B \leq C$  iff  $A \uparrow B \leq C$  iff  $A \uparrow B \vdash C$ .

For  $A \in a$  and  $B \in b$ , let  $a \cap b = d(A \downarrow B)$  and  $a \cup b = d(A \uparrow B)$ . By Lemma 8,  $\cap$  and  $\cup$  are well-defined,  $a \cap b$  is the glb of  $a$  and  $b$ , and  $a \cup b$  is the lub of  $a$  and  $b$ . Thus we have proved part of the following (cf. [4]):

**Theorem 1**  $D_T$  is a distributive lattice.

To prove distributivity it suffices, by Lemma 8, to verify that

$$A^T \downarrow (B \uparrow C) \vdash (A^T \downarrow B^T) \uparrow (A^T \downarrow C^T).$$

But this follows at once from the next lemma whose proof is obvious.

**Lemma 9**

- (i) For every  $k$ , there is an  $m$  such that

$$P \vdash Con_{(A \downarrow B) \upharpoonright m} \rightarrow Con_{A \upharpoonright k} \vee Con_{B \upharpoonright k}.$$

- (ii) For every  $m$ , there is a  $k$  such that

$$P \vdash Con_{A \upharpoonright k} \vee Con_{B \upharpoonright k} \rightarrow Con_{(A \downarrow B) \upharpoonright m}.$$

In [8] Švejdar introduced the lattice  $V_T$  consisting of all degrees of the form  $d(T + \phi)$ . By Theorem 11 of [5] or Theorem 3 of [4],  $V_T = D_T$ .

Clearly  $D_T$  has a minimal element  $0 = d(T)$  and a maximal element  $1$ , the common degree of all inconsistent theories. Suppose  $T$  is  $\Sigma_1^0$ -sound and  $a, b < 1$ . Then  $A \uparrow B$ , where  $A \in a$  and  $B \in b$ , is consistent and so  $a \cup b < 1$ . However, if  $T$  is not  $\Sigma_1^0$ -sound, this is not necessarily true. In fact we have the following (cf. [4]):

**Theorem 2**  $T$  is not  $\Sigma_1^0$ -sound iff there are degrees  $a_i < 1$  such that  $a_0 \cup a_1 = 1$  (and  $a_0 \cap a_1 = 0$ ).

To prove this we first prove the following simple but sometimes useful lemma (cf. [4]):

**Lemma 10** If  $X$  is r.e. and monoconsistent with  $Q$ , then there is a true  $\Pi_1^0$  sentence  $\psi$  such that  $\psi, \neg\psi \notin X$ .

*Proof:* Let  $R(k, m)$  be a primitive recursive relation such that  $X = \{k: \exists m R(k, m)\}$  and let  $\rho(x, y)$  be a PR binumeration of  $R(k, m)$ . Let  $\psi$  be such that

$$Q \vdash \psi \longleftrightarrow \forall z(\rho(\bar{\psi}, z) \rightarrow \exists u \leq z \rho(\bar{\neg\psi}, u)).$$

It is then easily verified that  $\psi$  is as desired.

**Lemma 11** *Suppose  $X$  is r.e. and monoconsistent with  $P$  and let  $\theta$  be any true  $\Pi_1^0$  sentence. There are then  $\Pi_1^0$  sentences  $\theta_i$  such that*

- (i)  $P \vdash \theta_0 \vee \theta_1$
- (ii)  $P \vdash \theta_0 \wedge \theta_1 \rightarrow \theta$
- (iii)  $\theta_i^j \notin X, i, j = 0, 1$ .

*Proof:* We may assume that if  $\phi \in X$  and  $P \vdash \phi \rightarrow \psi$ , then  $\psi \in X$ . By Lemma 10, there is a true  $\Pi_1^0$  sentence  $\psi \notin X$ . Thus, if necessary replacing  $\theta$  by  $\psi \wedge \theta$ , we may assume that  $\theta \notin X$ . Let  $\theta$  be  $\forall y \gamma(y)$ , where  $\gamma(y)$  is PR. Next let  $\delta_0(x)$  and  $\delta_1(x)$  be the formulas

$$\begin{aligned} \forall z(\neg\gamma(z) \rightarrow \exists u \leq z \text{Prf}_Q(x, u)), \\ \forall z(\text{Prf}_Q(x, z) \rightarrow \exists u < z \neg\gamma(u)). \end{aligned}$$

Then

- (1)  $P \vdash \delta_0(x) \vee \delta_1(x)$ ,
- (2)  $P \vdash \delta_0(x) \wedge \delta_1(x) \rightarrow \theta$ .

Let  $X_i = \{\phi: \delta_i(\bar{\phi}) \in X\}$ . Suppose  $Q \vdash \neg\phi$ . Then, since  $P \vdash \text{Con}_Q$ ,  $P \vdash \delta_0(\bar{\phi}) \rightarrow \theta$ , whence  $\delta_0(\bar{\phi}) \notin X$ , whence  $\phi \notin X_0$ . Moreover,  $\theta$  being true,  $P \vdash \neg\delta_1(\bar{\neg\phi})$ , whence  $\phi \notin X_1$ . Thus  $X_0 \cup X_1$  is monoconsistent with  $Q$ . But then, by Lemma 10, there is a sentence  $\psi$  such that  $\psi^i \notin X_0 \cup X_1, i = 0, 1$ . Let  $\theta_i$  be  $\delta_i(\bar{\neg\psi})$ . Then  $\theta_i \notin X$  and (i) and (ii) follow at once from (1) and (2). Finally, by (i),  $\neg\theta_i \notin X$ .

*Proof of Theorem 2:* Suppose  $T$  is not  $\Sigma_1^0$ -sound and let  $\theta$  be a true  $\Pi_1^0$  sentence such that  $T \vdash \neg\theta$ . Let  $\theta_i$  be as in Lemma 11 with  $X = Th(T)$  and let  $a_i = d(T + \theta_i)$ . Then  $a_i < 1$  and  $a_0 \cap a_1 = 0$ . By Lemma 5 and Lemma 11(ii),  $(T + \theta_0) \uparrow (T + \theta_1) \vdash \theta$ . Since  $T \vdash \neg\theta$ , it follows that  $a_0 \cup a_1 = 1$ .

By Theorem 2, if  $T$  is  $\Sigma_1^0$ -sound, then no degree, except trivially 0 and 1, has a complement, whereas if  $T$  is not  $\Sigma_1^0$ -sound, some do. The existence of pseudocomplements will be discussed later (Theorem 7).

It is easily seen that no  $a < 1$  is meet-irreducible. For suppose  $A \in a$  and let  $X = \{\phi: Q + \phi \leq A\}$ . Then  $X$  is r.e., since  $Q$  is finite, and monoconsistent with  $Q$ . Thus, by Lemma 10, there is a  $\psi$  such that  $\psi^i \notin X$ . Let  $a_i = d(A + \psi^i)$ . Then  $a < a_i$  and  $a_0 \cap a_1 = a$ . The question arises if there are join-irreducible degrees  $a \notin \{0, 1\}$ . That the answer is negative follows from Theorem 3 (cf. [4]):

**Theorem 3** *Suppose  $b < 1$  and  $a_k \not\leq a < b \not\leq b_k$  for  $k \leq n$ . Then there are degrees  $c_i$  such that  $a < c_i < b$  and  $a_k \not\leq c_i \not\leq b_k$  for  $i = 0, 1$  and  $k \leq n$ ,  $c_0 \cap c_1 = a$ , and  $c_0 \cup c_1 = b$ .*

*Proof:* Let  $a_{n+1} = b$ ,  $b_{n+1} = a$ ,  $A \in a$ ,  $B \in b$ ,  $A_k \in a_k$ , and  $B_k \in b_k$ . By Orey's compactness theorem, there are sentences  $\psi_k$  such that  $A_k \vdash \psi_k$  and  $\psi_k \not\leq A$ .

Moreover, by Lemma 3, there is an  $m$  such that  $B_k \nVdash \text{Con}_{B \upharpoonright m}$ . Let  $\beta(x)$  be a PR binumeration of  $B$  and set

$$X = \{ \phi: \psi_k \leq A + \neg \phi \text{ for some } k \leq n+1 \} \cup \\ \{ \phi: B_k \vdash \phi \vee \text{Con}_{B \upharpoonright m} \text{ for some } k \leq n+1 \} .$$

Then  $X$  is r.e. and monoconsistent with  $P$ . Hence, by Lemma 11, there are  $\Pi_1^0$  sentences  $\theta_i$  such that

- (1)  $P \vdash \theta_0 \vee \theta_1$ ,
- (2)  $P \vdash \theta_0 \wedge \theta_1 \rightarrow \text{Con}_\beta$ ,
- (3)  $\theta_i^j \notin X, i, j = 0, 1$ .

Let  $d_i = d(A + \theta_i)$ . Then  $a \leq d_i$  and  $a_k \not\leq d_i$ , since  $\neg \theta_i \notin X$ . Also, by (1),  $d_0 \cap d_1 = a$ . By (2) and Lemmas 1 and 5,  $d_0 \cup d_1 \geq b$ . Now set  $c_i = d_i \cap b$ . Then  $a \leq c_i \leq b$ . Also  $a_k \not\leq c_i$  and so, in particular,  $c_i < b$ . Suppose  $c_i \leq b_k$ . Then  $B \downarrow (A + \theta_i) \leq B_k$ . But  $\theta_i \vee \text{Con}_{B \upharpoonright m}$  is  $\Pi_1^0$  and provable in  $B \downarrow (A + \theta_i)$ . Hence, by Lemma 5,  $B_k \vdash \theta_i \vee \text{Con}_{B \upharpoonright m}$ , contradicting (3). Thus  $c_i \not\leq b_k$ , whence  $a < c_i$ . Clearly  $c_0 \cap c_1 = a$ . Finally, by distributivity,  $c_0 \cup c_1 = b \cap (d_0 \cup d_1) = b$ .

Let

$$\text{COMPL}_{a,b} = \{ c: \text{there is a } d \text{ such that } c \cap d = a \text{ and } c \cup d = b \} .$$

As is well-known, since  $D_T$  is distributive, to each  $c \in \text{COMPL}_{a,b}$ , there is a unique  $c^* \in \text{COMPL}_{a,b}$  such that  $c \cap c^* = a$  and  $c \cup c^* = b$ . In fact

$$BA_{a,b} = (\text{COMPL}_{a,b}, \cap, \cup, *)$$

is a Boolean algebra.

**Corollary 1** *If  $a < b < 1$ , then  $BA_{a,b}$  is a denumerable atomless Boolean algebra. Thus if  $c < d < 1$ , then  $BA_{a,b}$  and  $BA_{c,d}$  are isomorphic.*

*Proof:* We need only show that  $BA_{a,b}$  is atomless. Suppose  $c \in \text{COMPL}_{a,b}$  and  $a < c$ . Then, by Theorem 3, there is a  $d \in \text{COMPL}_{a,b}$  such that  $c \not\leq d \not\leq c^*$ . Let  $e = c \cap d$ . Then  $e \in \text{COMPL}_{a,b}$  and  $a < e < c$ .

Let  $[a, b] = \{ c: a \leq c \leq b \}$ ,  $[a, b) = \{ c: a \leq c < b \}$ , and let  $(a, b]$  and  $(a, b)$  be defined in the obvious way. It is now natural to ask if  $\text{COMPL}_{a,b} = [a, b]$  provided that  $a < b < 1$ . We are going to show that the answer is negative. We define the relations  $\ll_j$  and  $\ll_m$  as follows:  $a \ll_j b$  iff  $a < b$  and for every  $c$ , if  $a \cup c \geq b$ , then  $c \geq b$ ;  $a \ll_m b$  iff  $a < b$  and for every  $c$ , if  $b \cap c \leq a$ , then  $c \leq a$ .

#### Theorem 4

- (i) *If  $0 < a \not\leq a_k, k \leq n$ , then there is a  $b$  such that  $0 < b \ll_j a$  and  $b \not\leq a_k$  for  $k \leq n$ .*
- (ii) *If  $a_k \not\leq a < 1, k \leq n$ , then there is a  $b$  such that  $a \ll_m b < 1$  and  $a_k \not\leq b$  for  $k \leq n$ .*

Part (ii) of the theorem is proved in [4].

*Proof of (i):* Let  $A \in a$  and  $A_k \in a_k$ . By Lemma 5, there is a  $\Pi_1^0$  sentence  $\theta$  such that  $A \vdash \theta$  and  $A_k \nVdash \theta$ . Let  $X = \bigcup \{ Th(A_k + \neg \theta): k \leq n \}$ . Then  $X$  is r.e. and

monoconsistent with  $T + \neg\theta$ . Hence, by Lemma 7, there is a  $\Pi_1^0$  sentence  $\psi \notin X$  such that  $\psi$  is  $\Sigma_1^0$ -conservative over  $T + \neg\theta$ . Let  $B = T + \psi \vee \theta$  and  $b = d(B)$ . Then  $0 < b \not\leq a_k$  and  $b \leq a$ . Suppose  $b \cup c \geq a$ . Let  $C \in c$ . Then, by Lemma 5, there is an  $m$  such that  $T + \psi + \text{Con}_{C \upharpoonright m} \vdash \theta$ , whence  $T + \neg\theta + \psi \vdash \neg \text{Con}_{C \upharpoonright m}$ , whence, by the choice of  $\psi$ ,  $T + \neg\theta \vdash \neg \text{Con}_{C \upharpoonright m}$ , whence  $C \vdash \theta$ . Thus  $c \geq b$  and so  $c = c \cup b \geq a$ .

From Lemmas 3 and 9 we get at once the following:

**Lemma 12**  $A \downarrow B \leq C$  iff for every  $n$ ,  $A \leq C + \neg \text{Con}_{B \upharpoonright n}$ .

Our next lemma is an immediate consequence of Lemma 5.

**Lemma 13** If  $A \leq B$  and  $\sigma$  is  $\Sigma_1^0$ , then  $A + \sigma \leq B + \sigma$ .

If  $\sigma$  is  $\Sigma_1^0$ , let  $a[\sigma]$  be the degree of  $A + \sigma$ , where  $A \in a$ . By Lemma 13,  $a[\sigma]$  is well-defined. Let  $a[\Sigma_1^0] = \bigcup \{a[\sigma] : \sigma \text{ is } \Sigma_1^0\}$ .

**Lemma 14** The following conditions are equivalent:

- (i) For every  $c$ , if  $b \cap c \leq a$ , then  $c \leq a$ .
- (ii)  $a[\Sigma_1^0] \cap [b, 1] = \phi$ .

*Proof:* Suppose (i) holds. Let  $\sigma$  be a  $\Sigma_1^0$  sentence such that  $b \leq a[\sigma]$ . Let  $A \in a$  and  $B \in b$ . Then  $B \downarrow (A + \neg\sigma) \leq A$ . But then, by (i),  $A + \neg\sigma \leq A$ . But  $\neg\sigma$  is  $\Pi_1^0$ . Hence  $A \vdash \neg\sigma$  and so  $a[\sigma] = 1$ .

Next, suppose (ii) holds. Let  $c$  be such that  $b \cap c \leq a$ . Let  $A \in a$ , etc. Then, by Lemma 12,  $B \leq A + \neg \text{Con}_{C \upharpoonright m}$ . But  $\neg \text{Con}_{C \upharpoonright m}$  is  $\Sigma_1^0$ . Hence, by (ii),  $A \vdash \text{Con}_{C \upharpoonright m}$ . But this holds for every  $m$  and so  $c \leq a$ .

*Proof of Theorem 4(ii):* Let  $A \in a$  and  $A_k \in a_k$ . By Orey's compactness theorem, there are sentences  $\psi_k$  such that  $A_k \vdash \neg\psi_k \not\leq A$ . Let  $X = \{\phi : \psi_k \leq A + \neg\phi \text{ for some } k \leq n\}$ . Then  $X$  is r.e. and monoconsistent with  $A$ . But then, by Lemma 7, there is a  $\Sigma_1^0$  sentence  $\sigma \notin X$  such that  $\sigma$  is  $\Pi_1^0$ -conservative over  $A$ . Let  $B = A + \neg\sigma$  and  $b = d(B)$ . Then  $a_k \not\leq b < 1$ . Suppose now  $\phi$  is  $\Sigma_1^0$  and  $b \leq a[\phi]$ . Then  $A + \phi \vdash \neg\sigma$ , whence  $A + \sigma \vdash \neg\phi$ , whence  $A \vdash \neg\phi$ , i.e.,  $a[\phi] = 1$ . Now apply Lemma 14.

We write  $a \ll^* b$  to mean that there is a  $\Pi_1^0$  sentence  $\theta$  such that  $B \vdash \theta$  and  $A + \neg\theta \leq A$ , where  $A \in a$  and  $B \in b$ . The above proof of Theorem 4(ii) yields a degree  $b$  such that  $a \ll^* b$ . It is also easily shown that  $a \ll^* b$  implies  $a \ll_m b$ . This leads to the question of whether or not the converse is true. This is answered negatively in the following:

**Corollary 2** For every  $a < 1$ , there is a  $b$  such that  $a \ll_m b$  but not  $a \ll^* b$ .

*Proof:* Let  $A \in a$ ,  $X = \{\phi : a \ll_m d(A + \phi)\}$ , and  $Y = \{\phi : a \ll^* d(A + \phi)\}$ . By Lemma 3,  $\{\phi : A + \phi \leq B\}$  is  $\Pi_2^0$ . Hence, by Lemma 14,  $X$  is  $\Pi_3^0$ . But, by Corollary 3 of [5],  $Y$  is a complete  $\Sigma_3^0$  set and so is not  $\Pi_3^0$ . Finally, as mentioned above,  $Y \subseteq X$ . It follows that  $X \not\subseteq Y$ . Let  $\psi \in X - Y$  and let  $b = d(A + \psi)$ . Then  $b$  is as desired.

It is of some interest to note that if  $A$  is consistent and  $\alpha(x)$  is a *PR* binumeration of  $A$ , then  $d(A) \ll^* d(P + \text{Con}_\alpha)$ . This follows from Lemma 1 and the fact, proved by Feferman [1], that  $A + \neg \text{Con}_\alpha \leq A$ .

Theorem 4 can be relativized as follows.

**Corollary 3**

- (i) If  $a < b$ , then there is a  $c$  such that  $a < c < b$  and for every  $d \geq a$ , if  $c \cup d \geq b$ , then  $d \geq b$   
(ii) If  $a < b$ , then there is a  $c$  such that  $a < c < b$  and for every  $d \leq b$ , if  $c \cap d \leq a$ , then  $d \leq a$ .

*Proof:* (i) By Theorem 4(i), there is an  $e \not\leq a$  such that  $e <_j b$ . Let  $c = e \cup a$ . If  $d \geq a$  and  $d \cup c \geq b$ , then  $e \cup a \cup d \geq b$ , whence  $e \cup d \geq b$ , whence  $d \geq b$ . The proof of (ii) is similar.

Corollary 3 can be applied to obtain information on  $COMPL_{a,b}$  as follows.

**Corollary 4** Suppose  $a \leq c < d \leq b$ . Then there are  $c_0, d_0$  such that  $c < c_0 \leq d_0 < d$  and  $COMPL_{a,b} \cap ((c, c_0] \cup [d_0, d)) = \emptyset$ .

*Proof:* First note that

$$(1) \quad COMPL_{a,b} \cap [c, d] \subseteq COMPL_{c,d}.$$

By Corollary 3(i), there is a  $c_1$  such that  $c < c_1 < d$  and if  $c \leq e$  and  $c_1 \cup e \geq d$ , then  $e \geq d$ . By Corollary 3(ii), there is a  $d_1$  such that  $c < d_1 < d$  and if  $e \leq d$  and  $e \cap d_1 \leq c$ , then  $e \leq c$ . Let  $c_0 = c_1 \cap d_1$  and  $d_0 = c_1 \cup d_1$ . Then, by (1),  $c_0$  and  $d_0$  are as desired.

Let  $G$  be a set of degrees. Then  $c$  is isolated from  $G$  in  $(a, b)$  if  $c \in (a, b)$  and to any  $a_0, b_0$  such that  $a \leq a_0 < c < b_0 \leq b$ , there are  $a_1, b_1$  such that  $a_0 \leq a_1 < c < b_1 \leq b_0$  and  $G \cap [a_1, b_1] = \emptyset$ .

**Corollary 5** If  $c \in [a, b] - COMPL_{a,b}$ , then  $c$  is isolated from  $COMPL_{a,b}$  in  $(a, b)$ .

*Proof:* Let  $c$  be as assumed and suppose  $a \leq a_0 < c < b_0 \leq b$ . By Corollary 4, there is an  $a_1$  such that  $a_0 \leq a_1 < c$  and  $COMPL_{a,b} \cap [a_1, c] = \emptyset$ . By Theorem 3, there are  $c_i, i = 0, 1$ , such that  $c < c_i < b_0$  and  $c_0 \cap c_1 = c$ . It suffices to show that  $COMPL_{a,b} \cap [a_1, c_i] = \emptyset$  for  $i = 0$  or  $i = 1$ . Suppose not and let  $d_i, i = 0, 1$ , be counterexamples. Then  $d_0 \cap d_1 \in COMPL_{a,b} \cap [a_1, c]$ , contrary to the choice of  $a_1$ .

Theorem 4(i) suggests the problem if to any  $a < 1$ , there is a  $b$  such that  $a <_j b < 1$ . The dual of this is obviously false. We show that the answer is negative.

**Theorem 5** There is a degree  $a < 1$  such that if  $a \leq b < 1$ , then there is a  $c < b$  such that  $a \cup c = b$ .

*Proof:* If  $T$  is not  $\Sigma_1^0$ -sound, this is obvious, by Theorem 2. So suppose  $T$  is  $\Sigma_1^0$ -sound. Let  $\tau(x)$  be a PR binumeration of  $T$  and let  $a = d(T + Con_\tau)$ . Then  $a < 1$ . Suppose now  $a \leq b < 1$ . Let  $\beta(x)$  be a PR binumeration of a theory of degree  $b$ . Next let  $\phi$  be such that

$$P \vdash \phi \leftrightarrow \forall z (Prf_\tau(\bar{\phi}, z) \rightarrow \exists u \leq z Prf_\beta(\overline{\neg 0 = 0}, u)) .$$

Finally let  $\hat{\phi}$  be the sentence

$$\forall z (Prf_\beta(\overline{\neg 0 = 0}, z) \rightarrow \exists u < z Prf_\tau(\bar{\phi}, u)) .$$

Then, by standard arguments,

- (1)  $T \not\vdash \phi$ ,
- (2)  $P \vdash \phi \vee \hat{\phi}$ ,
- (3)  $P \vdash \phi \wedge \hat{\phi} \rightarrow \text{Con}_\beta$ .

Clearly  $P \vdash \neg\phi \rightarrow \text{Pr}_\tau(\neg\phi)$ . Since  $\neg\phi$  is  $\Sigma_1^0$ , it follows, by Corollary 5.5 of [1], that  $P \vdash \neg\phi \rightarrow \text{Pr}_\tau(\neg\phi)$ . Thus

- (4)  $P \vdash \text{Con}_\tau \rightarrow \phi$ .

Let  $d = d(T + \hat{\phi})$ . Then, since  $\hat{\phi}$  and  $\text{Con}_\tau$  are  $\Pi_1^0$ , it follows from (3), (4), and Lemma 1 that  $a \cup d \geq b$ . Suppose  $a \leq d$ . Then  $T + \hat{\phi} \vdash \text{Con}_\tau$ . Hence, by (2) and (4),  $T \vdash \phi$ , contradicting (1). Thus  $a \not\leq d$ . Now let  $c = d \cap b$ . Then  $c < b$ . Finally  $a \cup c = (a \cup d) \cap (a \cup b) = b$ .

Let  $\bigcup G$  ( $\bigcap G$ ), where  $G$  is a set of degrees, be the supremum (infimum) of  $G$ , if it exists. Somewhat surprisingly the following infinitary distributive laws hold.

### Theorem 6

- (i) If  $\bigcup G$  exists, then  $\bigcup G \cap b = \bigcup \{a \cap b : a \in G\}$ .
- (ii) If  $\bigcap G$  exists, then  $\bigcap G \cup b = \bigcap \{a \cup b : a \in G\}$ .

To prove (ii) we need the following:

**Lemma 15**  $A \uparrow B \geq C$  iff for every  $(\Sigma_1^0)$  sentence  $\theta$  and every  $m$ , if  $A^T + \neg\text{Con}_{C \upharpoonright m} \vdash_{\Sigma_1^0} T + \theta$ , then  $B \vdash \neg\theta$ .

*Proof:* Suppose first  $A \uparrow B \geq C$ . Let  $\theta$  and  $m$  be such that  $A^T + \neg\text{Con}_{C \upharpoonright m} \not\vdash_{\Sigma_1^0} T + \theta$ . There is a  $k$  such that  $A^T + \text{Con}_{B \upharpoonright k} \vdash \text{Con}_{C \upharpoonright m}$ . It follows that  $T + \theta \vdash \neg\text{Con}_{B \upharpoonright k}$ , whence  $B \vdash \neg\theta$ . This proves “only if”. To prove “if” suppose  $A \uparrow B \not\geq C$ . Then there is an  $m$  such that for every  $k$ ,  $A^T + \text{Con}_{B \upharpoonright k} \not\vdash \text{Con}_{C \upharpoonright m}$ . But then, by Theorem 5 of [5], there is a  $\Sigma_1^0$  sentence  $\theta$  such that  $A^T + \neg\text{Con}_{C \upharpoonright m} \vdash_{\Sigma_1^0} T + \theta$  and  $T + \theta \not\vdash \neg\text{Con}_{B \upharpoonright k}$  for every  $k$ . Since  $\neg\theta$  is  $\Pi_1^0$ , it follows that  $B \not\vdash \neg\theta$  and so the proof is complete.

*Proof of Theorem 6:* (i) Let  $c = \bigcup G$ . It suffices to show that for every  $d$ , if  $a \cap b \leq d$  for every  $a \in G$ , then  $c \cap b \leq d$ . Let  $B \in b$ , etc. If  $A \downarrow B \leq D$  for every  $A$  with  $d(A) \in G$ , then, by Lemma 12,  $A \leq D + \neg\text{Con}_{B \upharpoonright n}$  for every such  $A$  and every  $n$ . But then for every  $n$ ,  $C \leq D + \neg\text{Con}_{B \upharpoonright n}$ . Hence, again by Lemma 12,  $C \downarrow B \leq D$ .

(ii) Let  $c = \bigcap G$ . It suffices to show that if  $d \leq a \cup b$  for  $a \in G$ , then  $d \leq c \cup b$ . Again let  $B \in b$ , etc. Suppose  $D \leq A \uparrow B$  for every  $A$  such that  $d(A) \in G$ . Then, by Lemma 15, for all such  $A$ , every  $m$ , and every  $\Sigma_1^0$  sentence  $\theta$ , if  $B^T + \neg\text{Con}_{D \upharpoonright m} \vdash_{\Sigma_1^0} T + \theta$ , then  $A \vdash \neg\theta$ . But then for every  $m$  and every  $\Sigma_1^0$  sentence  $\theta$ , if  $B^T + \neg\text{Con}_{D \upharpoonright m} \vdash_{\Sigma_1^0} T + \theta$ , then  $C \vdash \neg\theta$ . Hence, again by Lemma 15,  $D \leq C \uparrow B$ .

The following corollaries are immediate.

### Corollary 6

- (i) If  $d = \bigcup \{b : b \cap a \leq c\}$ , then  $d \cap a \leq c$ .
- (ii) If  $d = \bigcap \{b : b \cup a \geq c\}$ , then  $d \cup a \geq c$ .



**Corollary 7**

- (i) If  $a \not\leq c$ , then there is a  $d < 1$  such that if  $a \cap b \leq c$ , then  $b \leq d$ .  
(ii) If  $c \not\leq a$ , then there is a  $d > 0$  such that if  $a \cup b \geq c$ , then  $b \geq d$ .

By (i) of our next result, to every  $c < 1$ , there is an  $a > c$  which has no pseudocomplement relative to  $c$ .

**Theorem 7**

- (i) If  $c < 1$ , then there is an  $a > c$  such that  $\{b: b \cap a \leq c\}$  has no supremum.  
(ii) If  $0 < c < 1$  and there is a  $\Pi_1^0$  sentence  $\theta$  such that  $c = d(T + \theta)$ , then there is a degree  $a < c$  such that  $\{b: b \cup a \geq c\}$  has no infimum.

The proof of Theorem 7 is deferred to the end of the paper. All examples, known so far, of degrees  $a_i, c_i$  such that  $\bigcup \{b: b \cap a_0 \leq c_0\}$  and  $\bigcap \{b: b \cup a_1 \geq c_1\}$  exist can be obtained in a straightforward manner from Lemma 14 and the proof of Theorem 4(i).

Let  $A[X] = \{d(A + \phi): \phi \in X\}$  and  $a[X] = \bigcup \{A[X]: A \in a\}$ . (For  $X = \Sigma_1^0$  this definition of  $a[X]$  is equivalent to the one given earlier.) By the proof of Theorem 11 of [5],  $A[\Sigma_2^0] = A[\Pi_2^0] = \{d(B): B \vdash A\}$  and so  $a[\Sigma_2^0] = a[\Pi_2^0] = [a, 1]$ . Moreover, by Lemma 13,  $a[\Sigma_1^0] = A[\Sigma_1^0]$  for  $A \in a$  and, by Theorem 11 of [5],  $A[\Sigma_1^0] = \{d(A \cup X): X \text{ r.e. and } \subseteq \Sigma_1^0\}$ .

If  $A$  is  $\Sigma_1^0$ -sound and  $b < 1$ , then there is a  $c \in A[\Pi_1^0]$  such that  $b \leq c < 1$  (cf. also Corollary 14 below). By contrast we have the following:

**Corollary 8** *To every  $b > a$ , there is a  $c \in a[\Sigma_1^0] - \{1\}$  such that  $a[\Sigma_1^0] \cap [b \cup c, 1) = \emptyset$ .*

*Proof:* Let  $A \in a$  and  $B \in b$ . There is a  $\Pi_1^0$  sentence  $\theta$  such that  $B \vdash \theta$  and  $A \not\vdash \theta$ . Let  $C = A + \neg\theta$  and  $c = d(C)$ . Now let  $d$  be any degree such that  $(b \cup c) \cap d \leq a$ . Then  $b \cap d \leq c$ . Hence, by Lemma 12,  $d \leq c$  and so  $d = (b \cup c) \cap d \leq a$ . Now apply Lemma 14.

**Corollary 9** *If  $a < b$ , then there is a  $c$  such that  $a \leq c < b$  and for every  $d \leq a$  and every  $e \in d[\Sigma_1^0]$ , if  $c \leq e$ , then  $b \leq e$ .*

*Proof:* Let  $c$  be as in Corollary 3(ii). Suppose  $d \leq a$  and  $c \leq d[\sigma]$ , where  $\sigma$  is  $\Sigma_1^0$ . Let  $f = b \cap d(D + \neg\sigma)$ , where  $D \in d$ . Then  $f \leq b$  and  $f \cap c \leq d[\sigma] \cap d(D + \neg\sigma) \leq d \leq a$ . Hence  $f \leq a \leq c$  and so  $f = f \cap c \leq d$ . Thus, by Lemma 12,  $D + \neg\sigma \leq D + \neg\text{Con}_{B \upharpoonright n}$ , whence,  $\neg\sigma$  being  $\Pi_1^0$ ,  $D + \sigma \vdash \text{Con}_{B \upharpoonright n}$ . But this holds for every  $n$  and so  $b \leq d[\sigma]$ .

**Corollary 10** *If  $c \in (a, b) - a[\Sigma_1^0]$ , then  $c$  is isolated from  $a[\Sigma_1^0]$  in  $(a, b)$ .*

The proof of this is the same as that of Corollary 5 except that Corollary 4 is replaced by Corollary 9.

Corollary 9 suggests the question if to any  $a < 1$ , there is a  $b > a$  such that  $a[\Sigma_1^0] \cap (a, b) = \emptyset$ . The following result answers this in the negative.

**Theorem 8** *If  $A \in a < b \not\leq b_k$  for  $k \leq n$ , then there is a degree  $c \in A[\Pi_1^0] \cap A[\Sigma_1^0] \cap (a, b)$  such that  $c \not\leq b_k$  for  $k \leq n$ .*

*Proof:* We may assume that  $a \leq b_k$ . Let  $B \in b$ ,  $B_k \in b_k$ , and  $B_{n+1} = A$ . Next let

$X = \bigcup \{Th(B_k) : k \leq n + 1\}$ . By Lemma 6, there is a *PR* formula  $\eta(x, z)$  such that

- (1) if  $\phi \in X$ , then  $A^T + \phi \vdash \neg \exists z \eta(\bar{\phi}, z)$ ,
- (2) if  $\phi \notin X$ , then  $\exists z \eta(\bar{\phi}, z)$  is  $\Pi_1^0$ -conservative over  $A^T + \phi$ .

There is a *PR* formula  $\delta(u)$  such that  $B \vdash \forall u \delta(u)$  and  $B_k \not\vdash \forall u \delta(u)$ . Let  $\theta$  be such that

$$P \vdash \theta \longleftrightarrow \forall u (\neg \delta(u) \rightarrow \exists z < u \eta(\bar{\theta}, z)) .$$

Finally let  $\chi$  be the sentence

$$\exists z (\eta(\bar{\theta}, z) \wedge \forall u \leq z \delta(u)) .$$

Then

- (3)  $P \vdash \chi \longleftrightarrow \exists z \eta(\bar{\theta}, z) \wedge \theta$ ,
- (4)  $P + \theta + \neg \exists z \eta(\bar{\theta}, z) \vdash \forall u \delta(u)$ .

We now show that

- (5)  $\theta \notin X$ .

Suppose  $\theta \in X$  and let  $k$  be such that  $B_k \vdash \theta$ . Then, by (1),  $A^T + \theta \vdash \neg \exists z \eta(\bar{\theta}, z)$ . But then, by (4) and since  $A^T \vdash B_k$ ,  $B_k \vdash \forall u \delta(u)$ , contrary to hypothesis.

Since  $\theta$  is  $\Pi_1^0$ , (5) implies that  $A < A + \theta \not\leq B_k$ . By (3),  $A + \theta \vdash A + \chi$ . Since  $\chi$  is  $\Sigma_1^0$ , we have  $A + \chi \leq A^T + \chi$ . By (5) and (2),  $\exists z \eta(\bar{\theta}, z)$  is  $\Pi_1^0$ -conservative over  $A^T + \theta$ . Hence, by (3),  $A^T + \chi \leq A^T + \theta$ . Since  $A^T + \theta \vdash A + \theta$  and  $A^T + \theta \vdash B$ , it now follows that  $A + \chi \equiv A + \theta \leq B$ . Let  $c = d(A + \chi) = d(A + \theta)$ . Then  $c$  is as desired.

By the proof of Theorem 4(ii), to every  $a < 1$ , there is a  $b < 1$  such that  $a[\Sigma_1^0] \cap [b, 1) = \emptyset$ . Nevertheless we have the following

**Corollary 11** *If  $a < b$ , then there are  $a_n \in a[\Sigma_1^0]$  such that for every  $n$ ,  $a_n < a_{n+1}$ , and  $\bigcup \{a_n : n \in \omega\} = b$ .*

*Proof:* Let  $c_n$ ,  $n \in \omega$ , be all degrees  $\not\leq b$ . Let  $a_0 = a$ . Now suppose  $a_n$  has been defined and  $a_n < b$ . Since  $b \not\leq c_n$ , there is a degree  $d \not\leq c_n$  such that  $a_n < d < b$ . By Theorem 8, there is an  $a_{n+1} \in a_n[\Sigma_1^0]$  such that  $a_n < a_{n+1} \leq d$  and  $a_{n+1} \not\leq c_n$ . It follows that  $a_{n+1} \in a[\Sigma_1^0]$  and  $a_{n+1} < b$ . Finally it is clear that  $\bigcup \{a_n : n \in \omega\} = b$ .

Consider  $a[\Sigma_1^0]$  as a substructure of  $D_T$ . It is a distributive lattice with meet  $\cap$  and join  $\cup$ . Clearly  $a_0 \cap a_1 = a_0 \cap a_1$ . To find  $a_0 \cup a_1$ , note that if  $\sigma$  and  $\sigma_i$  are  $\Sigma_1^0$  and  $A + \sigma_i \leq A + \sigma$ ,  $i = 0, 1$ , then  $A + \sigma_0 \wedge \sigma_1 \leq A + \sigma$ . Thus if  $a_i = d(A + \sigma_i)$ , then  $a_0 \cup a_1 = d(A + \sigma_0 \wedge \sigma_1)$ . But then it is easily verified that  $b \in a[\Sigma_1^0]$  has a complement in  $a[\Sigma_1^0]$  iff  $b \in a[\Pi_1^0]$ . Thus, from Theorem 8, we get the following corollary showing that the isomorphism type of  $D_T^a = a[\Sigma_1^0] \cap a[\Pi_1^0]$  is independent of  $a$  and  $T$  provided that  $a < 1$ .

**Corollary 12** *If  $a < 1$ , then  $D_T^a$  is a denumerable atomless Boolean algebra and for every  $b > a$ ,  $b = \bigcup \{c < b : c \in D_T^a\}$ .*

From (ii) of the following result, an improvement of Theorem 14 of [4]

and Corollary 4 of [5], it follows that if  $a < 1$ , then  $D_T^a$  is a proper subset of  $a[\Sigma_1^0]$ .

**Theorem 9**

- (i) If  $d(A) \leq b < c$ , then there is a degree which is isolated from  $A[\Pi_1^0]$  in  $(b, c)$ .  
(ii) If  $a < b$ , then there is a degree  $c \in a[\Sigma_1^0]$  which is isolated from  $a[\Pi_1^0]$  in  $(a, b)$ .

To prove this we need the following rather straightforward strengthening of Lemma 10 (cf. [4]).

**Lemma 16** *If  $X$  is r.e. and monoconsistent with  $Q$ , then there is a  $\Pi_1^0$  formula  $\eta(x)$  such that for every  $n$  and every  $f \in {}^{n+1}2$ ,  $\bigvee \{\eta(\bar{k})^{f(k)} : k \leq n\} \notin X$ .*

*Proof of Theorem 9:* We prove (ii). The proof of (i) is similar but simpler. Let  $A \in a$  and  $B \in b$ . By Theorem 8, there is a  $\Sigma_1^0$  sentence  $\chi$  such that  $A < A + \chi < B$ . By Orey's compactness theorem, there is a  $p$  such that  $Q \vdash A \uparrow p$  and  $A \uparrow p + \chi \not\leq A$ . By Lemma 16 applied to the set  $\{\phi : A \uparrow p + \chi \vee \phi \leq A + \chi \vee \neg\phi\}$ , there is a  $\Sigma_1^0$  formula  $\delta(x)$  such that:

- (1)  $A \uparrow p + \chi \vee \delta(\bar{m}) \not\leq A \cup \{\chi \vee \delta(\bar{n}) : n < m\}$ .

Let  $C = A \cup \{\chi \vee \delta(\bar{n}) : n \in \omega\}$  and  $c = d(C)$ . Then  $c \in (a, b)$  and, by Theorem 11 of [5],  $c \in a[\Sigma_1^0]$ .

To show that  $c$  is isolated from  $a[\Pi_1^0]$  in  $(a, b)$ , suppose  $a \leq c_0 < c < c_1(\leq b)$ . Let  $C_i \in c_i$  and let  $q$  be such that  $C \uparrow q \not\leq C_0$  and  $C_1 \uparrow q \not\leq C$ . Let  $\psi_m$  be the sentence  $\bigwedge \{\chi \vee \delta(\bar{n}) : n < m\}$  and set

$$\begin{aligned} X &= \{\phi : C \uparrow q \leq (C \downarrow (T + \neg\phi)) \uparrow C_0\}, \\ Y &= \{\phi : (C_1 \uparrow q) \downarrow (Q + \phi) \leq C\}, \\ Z &= \{\phi : \exists m((A \uparrow p + \chi \vee \delta(\bar{m})) \downarrow (Q + \phi) \leq A + \psi_m + \neg\phi)\}. \end{aligned}$$

By (1),  $X \cup Y \cup Z$  is r.e. and monoconsistent with  $Q$ . Thus, by Lemma 10, there is a  $\Sigma_1^0$  sentence  $\psi$  such that  $\psi, \neg\psi \notin X \cup Y \cup Z$ . Let  $d_0 = (c \cap d(T + \neg\psi)) \cup c_0$  and  $d_1 = c_1 \cap d(C + \psi)$ . Clearly  $c_0 \leq d_0 \leq c \leq d_1 \leq c_1$ . Moreover  $c \not\leq d_0$ , since  $\psi \notin X$ , and  $d_1 \not\leq c$ , since  $\psi \notin Y$ . Thus  $d_0 < c < d_1$ .

It remains to show that  $a[\Pi_1^0] \cap [d_0, d_1] = \emptyset$ . It suffices to prove that:

- (2) if  $A' \in a$ ,  $\theta$  is  $\Pi_1^0$ , and  $A' + \theta \leq C + \psi$ , then  $C \downarrow (T + \neg\psi) \not\leq A' + \theta$ .

Suppose (2) is false. Then there is an  $m$  such that  $A + \psi_m + \psi \vdash \theta$ . Since  $A' \equiv A$  and  $\psi_m$  and  $\psi$  are  $\Sigma_1^0$ , it follows that  $A' + \psi_m + \psi \equiv A + \psi_m + \psi$  and so  $A' + \psi_m + \psi \vdash \theta$ . Hence  $A' + \theta \leq A + \psi_m + \psi$ , whence  $C \downarrow (T + \neg\psi) \leq A + \psi_m + \psi$ . But this is impossible, since  $\neg\psi \notin Z$ . This proves (2) and so concludes the proof.

Combining Corollary 9 and Theorem 9 we get

**Corollary 13** *If  $a < b$ , then there are  $c, d$  such that  $a \leq c < d \leq b$  and  $(a[\Pi_1^0] \cup a[\Sigma_1^0]) \cap [c, d] = \emptyset$ .*

*Proof:* By Theorem 9(ii), there are  $a', b'$  such that  $a \leq a' < b' \leq b$  and

$a[\Pi_1^0] \cap [a', b'] = \emptyset$ . By Corollary 9, there is a  $c$  such that  $a' \leq c < b'$  and  $a[\Sigma_1^0] \cap [c, b'] = \emptyset$ . Now let  $d$  be such that  $c < d < b'$ . Then  $c$  and  $d$  are as desired.

Next we show that Theorem 9(i) cannot be improved by replacing  $A[\Pi_1^0]$  by  $a[\Pi_1^0]$ .

**Corollary 14** *If  $a \leq b < 1$ , then there is a  $c$  such that  $b \leq c < 1$  and  $[c, 1] \subseteq a[\Pi_1^0]$ .*

*Proof:* Let  $A \in a$  and  $B \in b$ . Then  $A^T \nvdash B$ . By Lemma 7, there is a  $\Sigma_1^0$  sentence  $\sigma$  such that  $\sigma$  is  $\Pi_1^0$ -conservative over  $A^T$  and  $B \not\vdash \sigma$ . Let  $c = d(B + \neg\sigma)$ . Then  $b \leq c < 1$ . Suppose now  $d \geq c$  and let  $D \in d$ . Let  $A' = A^T \cup \{\neg\sigma \rightarrow \phi : \phi \in D\}$ . Then  $A' \in a$ . Moreover  $A' + \neg\sigma \vdash A^T + \neg\sigma \cup D \vdash D$ . Hence  $d = d(A' + \neg\sigma)$  and so  $d \in a[\Pi_1^0]$ .

Let us say that the infimum  $\bigcap G$  is *trivial* if there is a finite set  $H \subseteq G$  such that  $\bigcap G = \bigcap H$ .

**Theorem 10** *Suppose  $A$  is consistent.*

(i) *There is a primitive recursive set  $X$  of  $\Sigma_1^0$  sentences such that  $d(A)$  is the nontrivial infimum of  $A[X]$ .*

(ii) *There is a primitive recursive set  $Y$  of  $\Sigma_1^0$  sentences such that  $A[Y]$  has no infimum.*

*Proof:* (i) By Corollary 2, there is a  $B$  such that  $d(A) \ll_m d(B)$  but not  $d(A) \ll^* d(B)$ . Let  $X = \{\neg \text{Con}_{B \upharpoonright n} : n \in \omega\}$ . Suppose  $C \leq A + \neg \text{Con}_{B \upharpoonright n}$  for every  $n$ . Then  $C \downarrow B \leq A$  and so  $C \leq A$ . Thus  $d(A) = \bigcap A[X]$ . This infimum is nontrivial, since otherwise there would exist an  $m$  such that  $A + \neg \text{Con}_{B \upharpoonright m} \leq A$ , contradicting the fact that not  $d(A) \ll^* d(B)$ .

To prove Theorem 10(ii) we need:

**Lemma 17** *If  $Z$  is an r.e. set of  $\Pi_1^0$  sentences, then  $A[Z]$  does not have a nontrivial infimum.*

*Proof:* We may assume that

(1) there is no  $k$  such that  $A + \psi \vdash \bigvee Z \upharpoonright k$  for every  $\psi \in Z$ ,

since otherwise  $A[Z]$  would have a trivial infimum. To obtain the desired conclusion it is sufficient (and necessary) to show that  $X = \{\phi \in \Pi_1^0 : A + \psi \vdash \phi \text{ for every } \psi \in Z\}$  is not r.e. We may assume that  $Z$  is primitive recursive. Let  $\xi(x)$  be a PR binumeration of  $Z$ . Next let  $\rho(x, y)$  be a PR binumeration of a relation  $R(k, m)$  such that  $Y = \{k : \forall m R(k, m)\}$  is not r.e. Finally let  $\eta(x)$  be the formula

$$\forall z (\neg \rho(x, z) \rightarrow \exists u \leq z (\xi(u) \wedge \Pi_1^0\text{-true}(u))) ,$$

where  $\Pi_1^0\text{-true}(x)$  is a partial truth definition for  $\Pi_1^0$  sentences. If  $k \in Y$ , then clearly  $\eta(\bar{k}) \in X$ . Suppose  $k \notin Y$ . Let  $m$  be such that not  $R(k, m)$ . Then  $A + \eta(\bar{k}) \vdash \bigvee Z \upharpoonright m$ , whence, by (1),  $\eta(\bar{k}) \notin X$ . Thus  $Y = \{k : \eta(\bar{k}) \in X\}$  and so  $X$  is not r.e.

*Proof of Theorem 10(ii):* By the proof of Theorem 8, there are primitive recursive functions  $g(n)$  and  $h(n)$  such that if  $\phi$  is  $\Pi_1^0$  and  $A < A + \phi$ , then  $g(\phi)$  is  $\Sigma_1^0$ ,  $h(\phi)$  is  $\Pi_1^0$ , and  $A < A + g(\phi) \equiv A + h(\phi) \leq A + \phi$ . We now define  $\sigma_n$  and  $\psi_n$  as follows. Let  $\psi_0$  be any  $\Pi_1^0$  sentence such that  $A < A + \psi_0$ . Next suppose  $\psi_n$  has been defined and  $A < A + \psi_n$ . Let  $\theta$  be a  $\Pi_1^0$  Rosser sentence for  $A + \neg\psi_n$ . Then  $A < A + \psi_n \vee \theta < A + \psi_n$ . Let  $\sigma_n = g(\psi_n \vee \theta)$  and  $\psi_{n+1} = h(\psi_n \vee \theta)$ . Then  $A + \psi_n > A + \sigma_n \geq A + \psi_{n+1} > A$ . Now let  $Y = \{\sigma_n : n \in \omega\}$ . Then, by Lemma 17,  $Y$  is as desired.

In connection with this proof it may be remarked that if  $A$  is consistent, then there is no partial recursive function  $g(n)$  such that if  $\sigma$  is  $\Sigma_1^0$  and  $A < A + \sigma$ , then  $g(\sigma)$  is a  $\Pi_1^0$  sentence such that  $A < A + g(\sigma) \leq A + \sigma$ . For assuming the contrary we would have for every  $\Sigma_1^0$  sentence  $\sigma$ :  $A + \sigma \leq A$  iff if  $g(\sigma)$  is a  $\Pi_1^0$  sentence and  $A + \sigma \vdash g(\sigma)$ , then  $A \vdash g(\sigma)$ . But this is impossible, since, by Theorem 12 of [5],  $\Sigma_1^0 \cap \{\phi : A + \phi \leq A\}$  is not  $\Sigma_2^0$ .

Finally we are ready to give the

*Proof of Theorem 7:* (i) Let  $C \in c$ . By Theorem 10(ii), there is a primitive recursive set  $Y = \{\sigma_n : n \in \omega\}$  of  $\Sigma_1^0$  sentences such that  $C[Y]$  has no infimum. Let  $A = C \cup \{\neg\sigma_n : n \in \omega\}$  and  $a = d(A)$ . Then  $B \downarrow A \leq C$  iff  $B \leq C + \sigma_n$  for every  $n$ . But then, by Corollary 6(i), a supremum of  $\{b : b \cap a \leq c\}$  would be an infimum of  $C[Y]$ .

(ii) Let  $B = T + \neg\theta$ . Then  $B$  is consistent but not  $\Sigma_1^0$ -sound. We now effectively define  $\Pi_1^0$  sentences  $\psi_n$  such that

- (1)  $B_k = B \cup \{\psi_m : m < k\}$  is consistent,
- (2)  $B_{k+1} \not\vdash_{\Sigma_1^0} B_k$ .

Suppose  $\psi_m$  has been defined for  $m < k$ . Let  $\beta_k(x)$  be a PR binumeration of  $B_k$ . Let  $\delta(x, y)$  be the formula  $\beta_k(x) \wedge \forall z \leq x \neg \text{Prf}_{\beta_k}(y, z)$ . Finally let  $\psi_k$  be such that  $P \vdash \psi_k \leftrightarrow \text{Con}_{\delta(x, \neg\psi_k)}$ . Then, by a standard argument, (1) holds with  $k$  replaced by  $k + 1$  and  $\delta(x, \neg\psi_k)$  is a PR binumeration of  $B_k$ . Since  $B_k$  is not  $\Sigma_1^0$ -sound, it follows, by a result of Smoryński [7] (Application 5, p. 197) that  $\psi_k$  is not  $\Sigma_1^0$ -conservative over  $B_k$  and so (2) holds.

Now let  $X = \{\psi_m : m \in \omega\}$  and  $a = d(T \cup X) \cap c$ . To show that  $a$  is as desired we first observe that if  $\{b : b \cup a \geq c\}$  has an infimum, then so does  $\{d(T + \psi) : \psi \text{ is } \Pi_1^0 \text{ and } T \cup X + \psi \vdash \theta\}$ . But the latter set has no infimum: If  $T \cup X + \psi \vdash \theta$ , then for some  $k$ ,  $B \cup X \upharpoonright k \vdash \neg\psi$ . Now, by (2), there is a  $\Sigma_1^0$  sentence  $\sigma$  such that  $B \cup X \vdash \sigma$  and  $B \cup X \upharpoonright k \not\vdash \sigma$ . It follows that  $T \cup X + \neg\sigma \vdash \theta$  and  $T + \neg\sigma \not\vdash \psi$ . Now apply Lemma 17. Thus  $a$  is as claimed and the proof is complete.

Theorem 7(ii) is a partial dual of Theorem 7(i). The problem of whether or not the full dual is true remains unsolved.

One major open problem, which will certainly have occurred to the reader, is this: If  $S$  is  $\Sigma_1^0$ -sound but  $T$  is not, then, by Theorem 2,  $D_S$  and  $D_T$  are not isomorphic. But supposing that  $S$  and  $T$  are both  $\Sigma_1^0$ -sound (true), does it follow that  $D_S$  and  $D_T$  are isomorphic?

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