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A Pair of Nonisomorphic $\equiv_{\infty\lambda}$ Models of Power λ for λ Singular with $\lambda^\omega = \lambda$

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Let $L_{\infty\lambda}$ denote the infinitary logic that is formed by allowing arbitrary conjunctions of formulas and either existential or universal quantifications over sets of fewer than λ variables. The formulas are taken to be those having fewer than λ free variables. It is easy to see that any structure of cardinality less than λ is characterized up to isomorphism by a sentence of $L_{\infty\lambda}$. Scott [5] showed that if $M \equiv_{\infty\omega} N$ and $\|M\| = \|N\| = \aleph_0$, then $M \cong N$, where $\equiv_{\infty\omega}$ denotes elementary equivalence in $L_{\infty\omega}$. Chang [1] generalized this by showing that if cf $\lambda = \omega$ then $M \equiv_{\infty\lambda} N$, $\|M\| = \|N\| = \lambda$, still implies $M \cong N$. However, Morley, gave an early unpublished example which showed that for any regular $\lambda > \omega$, there are structures M,N such that $\|M\| = \|N\| = \lambda$, $M \equiv_{\infty\lambda} N$, but $M \ncong N$. His structures were trees of height λ, one with a branch of length λ and the other without. The reader may wish to consult Dickmann [2], Nadel [3], Nadel and Stavi [4], or Stavi [13] for more work in this direction.

The above results leave open the situation for singular λ, with cf $\lambda > \omega$. The purpose of this paper is to show that if $\lambda^\omega = \lambda$, then there are structures M,N with $\|M\| = \|N\| = \lambda$ such that $M \equiv_{\infty\lambda} N$ but $M \ncong N$. Under the GCH, cf $\lambda > \omega$, $\lambda > \omega$, implies $\lambda^\omega = \lambda$, so, in this situation, the entire picture would be known.

The reader may also be interested in consulting Shelah [8], [9], [11], and [12], which deal with the question of the cardinality of $\{N/\cong : N \equiv_{\infty\omega} M, \|M\| = \|N\| = \lambda\}$, or Shelah [6], Theorem 1, [7], Chapter XIII §1, and [10], which are concerned with finding models of particular theories. In [12] our main result is proved for many more λ of cofinality $>\aleph_0$ (but for different structures).

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1 Recall the well-known back and forth property for $L_{\infty\lambda}$ (cf., e.g., [2]).

Theorem 1.1 *$M \equiv_{\infty\lambda} N$ iff there is a nonempty family F of partial isomorphisms from M to N such that*

(i) $(\forall f \in F)(\forall X \subseteq |M|)[\|X\| < \lambda \to (\exists g \in F)(f \subseteq g \;\&\; X \subseteq dom\, g)]$
(ii) $(\forall f \in F)(\forall Y \subseteq |N|)[\|Y\| < \lambda \to (\exists g \in F)(f \subseteq g \;\&\; Y \subseteq range\, g)]$.

We say that a back and forth set as in the above theorem *exemplifies* $M \equiv_{\infty\lambda} N$.

The examples to be constructed below will consist of pairs of trees of height $\omega + 1$. Specifically, we call a model $M = \langle |M|, <_M, P_i^M \rangle_{i \leqslant \omega}$ an *ω-tree*, if

(i) $(|M|, <_M)$ is a tree, i.e., $<$ partially orders $|M|$ and for each $x \in |M|$, $<_M$ linearly orders the $<$-predecessors of x
(ii) for each $i \in \omega + 1$, P_i^M is the set of nodes on the i^{th} level, and $|M| = \bigcup_{i \leqslant \omega} P_i^M$
(iii) there is a single minimal element called the root of the tree
(iv) if $a \neq b \in P_\omega^M$ then there is some $x \in |M|$ such that $x <_M a$ but not $x <_M b$, i.e., there is only one node above each branch of length ω.

First some basic definitions concerning these trees. In most cases we will suppress the use of M as superscript or subscript except when we feel some special emphasis is required.

Definition 1.2 Let M be an ω-tree. Then

(i) For $a \in |M|, M_{[a]} = M \restriction \{b : b \leqslant a \text{ or } a \leqslant b\}$.
(ii) Suppose $a \in P_i$; then for $j < i$, $a(j)$ denotes the unique predecessor of a in P_j. We use P as shorthand for $\bigcup_{n \in \omega} P_n$.
(iii) $A \subseteq |M|$ is said to be a *closed* subset of M if whenever $a \in A$ and $M \models b < a$, then $b \in A$, and if whenever $a \in P_\omega^M$ and $a(i) \in A$ for each $i \in \omega$, then $a \in A$.
(iv) We write $N \subseteq_c M$ iff N is a submodel of M and $|N|$ is closed in M. We also speak of A as being closed in B where $A \subseteq B \subseteq |M|$. The meaning of this should be clear.

Given an ω-tree M and some $A \subseteq |M|$ we may define the closure of A in M, denoted $cl^M(A)$ as the smallest subset containing A and closed in M. This is obtained by closing downward and adding all nodes $a \in P_\omega^M$ such that $a(i) \in A$ for an unbounded set of i. The reader should convince himself of the following two easy facts about closure.

Lemma 1.3 *Suppose $N \subseteq_c M$ are ω-trees and $A \subseteq |N|$. Then*

(i) $cl^N(A) = cl^M(A)$
(ii) *If* $|A \cap P^M| < \lambda$ *then* $|cl^M(A) \cap P^M| < \lambda$.

Definition 1.4 Let $\mathfrak{A}$ be a family of ω-trees. An ω-tree M is said to be *$(\lambda, \mathfrak{A})$-full* iff for every $N \in \mathfrak{A}$, $a \in |M|$, $b \in |N|$ on the same level, there are closed submodels M_i of $M_{[a]}$ for $i < \lambda$, such that for $i < j < \lambda$, $|M_i| \cap |M_j| = \{c : c \leqslant a\}$, and $M_i \cong N_{[b]}$.

Definition 1.5 Let $\mathfrak{A}$ be a family of ω-trees. An ω-tree M is a $(\lambda, \mathfrak{A})$*-model* iff for every $A \subseteq |M|$ with $|A| < \lambda$, there is some $N \subseteq_c M$, with $A \subseteq |N|$ and N isomorphic to a member of $\mathfrak{A}$.

Roughly speaking $(\lambda,\mathfrak{A})$-fullness tells you that above any point there are many disjoint copies of corresponding parts of trees in $\mathfrak{A}$, while being a $(\lambda,\mathfrak{A})$-model guarantees that any subset can be expanded to a copy of something in $\mathfrak{A}$ which sits nicely in M.

The above notions are important to us here because of the following result.

Theorem 1.6 *Let $\mathfrak{A}$ be a family of ω-trees. Suppose M and N are both ω-trees which are $(\lambda,\mathfrak{A})$-full and $(\lambda,\mathfrak{A})$-models. Then $M \equiv_{\infty\lambda} N$.*

Proof: Let $G = \{g : g$ is a partial isomorphism from M to N with $\mathrm{dom}(g)$ a closed subset of M, $\mathrm{ra}(g)$ a closed subset of N, and $|\mathrm{dom}(g) \cap P^M| < \lambda\}$. We claim that G exemplifies $M \equiv_{\infty\lambda} N$.

Clearly G is nonempty since it contains the function taking the root of M to the root of N. Suppose $g \in G$ and $A \subseteq |M|$, $|A| < \lambda$. We show how to extend g to $h \in G$ with $A \subseteq \mathrm{dom}(h)$. The case for $A \subseteq |N|$ is similar.

Let us assume for convenience that A is disjoint from $\mathrm{dom}(g)$. Consider some $a \in A$. Since $\mathrm{dom}(g)$ is closed in M, there must be some $a' \in \mathrm{dom}(g)$ which is the highest point in $\mathrm{dom}(g)$ below a. Clearly a' is on some finite level. Let $A_{a'} = \{x \in A : x' = a'\}$. Since $|A_{a'}| < \lambda$ and M is an $(\mathfrak{A},\lambda)$-model, there is some closed submodel $X(a')$ of M with $A_{a'} \subseteq |X(a')|$ and $X(a')$ is isomorphic to some member of $\mathfrak{A}$.

Now, since N is $(\mathfrak{A},\lambda)$-full, there are N_i, $i < \lambda$, closed submodels of $N_{[g(a')]}$ such that for $i \neq j$, $N_i \cap N_j = \{b : b \leqslant g(a')\}$ and each N_i is isomorphic to $X(a')_{[a']}$. Since $\mathrm{ra}(g) \cap P^N$ has cardinality $<\lambda$, and the N_i are closed downward, there is some $i < \lambda$ such that $N_i \cap \mathrm{ra}(g) = \{b : b \leqslant g(a')\}$. Denote some such N_i by $N_{a'}$. We wish to extend g to $cl^M(A_{a'}) = cl^{X(a')}(A_{a'})$. To do this simply use the restriction to $cl^M(A_{a'})$ of some isomorphism from $X(a')_{[a']}$ onto $N_{a'}$. Since all nodes in $A_{a'}$ are related only to those nodes in $\mathrm{dom}(g)$ below a' and since their images are related only to those nodes in $\mathrm{ra}(g)$ below $g(a')$ this mapping, which we call $g_{a'}$, will be a partial isomorphism. It is also easy to see that both $\mathrm{dom}(g_{a'})$ and $\mathrm{ra}(g_{a'})$ are closed in their respective ω-trees.

Now, for a, $b \in A$ either $g_{a'} = g_{b'}$ (if $a' = b'$) or $\mathrm{dom}(g_{a'}) \cap \mathrm{dom}(g_{b'}) = \mathrm{dom}(g)$, $\mathrm{ra}(g_{a'}) \cap \mathrm{ra}(g_{b'}) = \mathrm{ra}(g)$. In fact if we let $h = \bigcup_{a \in A} g_{a'}$ then h is seen to be a partial isomorphism with $\mathrm{dom}(h)$ closed in M and $\mathrm{ra}(h)$ closed in N. This last part follows since whenever $a_1 < a_2 < \ldots$ are in $\mathrm{dom}(h)$ then for some a, $a_1, a_2, \ldots$ are all in $\mathrm{dom}(g_{a'})$ or $a_1, a_2, \ldots$ are already all in $\mathrm{dom}(g)$.

It only remains to show that $|\mathrm{dom}(h) \cap P^M| < \lambda$. However this follows from Lemma 1.3(ii) since $\mathrm{dom}(h) \cap P^M \subseteq cl^M((\mathrm{dom}(g) \cap P^M) \cup A) \cap P^M$ (actually they are equal) and $|(\mathrm{dom}(g) \cap P^M) \cup A| < \lambda$ by assumption.

2 In this section we obtain an approximation to the main result of the paper by using an extra hypothesis. The argument here is much less involved and much of it can be applied again with the more intricate construction of the next section.

Theorem 2.1 *Suppose λ is a strong limit cardinal such that $\lambda^\omega = \lambda$. Then there are ω-trees M and N of power λ such that $M \equiv_{\infty\lambda} N$ but $M \not\cong N$.*

Our first task is to describe an appropriate collection $\mathfrak{A}$ of ω-trees.

Definition 2.2 An ω-tree M is said to be *canonical* if $|M|$ is a set of sequences of length $\leqslant \omega$, $\nu \leqslant_M \eta$ iff ν is an initial subsequence of η, and the k^{th} level P_k^M consists of sequences of length k.

For the purposes of this section we will take $\mathfrak{A}_\alpha$ to be the family of all submodels of the canonical model with universe ${}^{\omega+1}\alpha$ and let $\mathfrak{A} = \bigcup_{\alpha<\lambda} \mathfrak{A}_\alpha$. It is clear that $\mathfrak{A}$ contains an isomorphic copy of every ω-tree of power less than λ. Since λ is a strong limit, if $M \in \mathfrak{A}$, then $\|M\| < \lambda$, and $|\mathfrak{A}| = \lambda$.

Lemma 2.3 *For every ω-tree M of power $\leqslant\lambda$ there is an ω-tree M' of power $\leqslant\lambda$ such that $M \subseteq_c M'$ and M' is $(\lambda, \mathfrak{A})$-full.*

Proof: First, for each node ν of M add λ disjoint copies of each $N \in \mathfrak{A}$ directly above it to form a new ω-tree M_1. (Because $\mathfrak{A}$ is so rich, this works in place of the condition demanded in the definition of $(\lambda, \mathfrak{A})$-full.) It is easy to see that $M \subseteq_c M_1$ and $\|M_1\| = \lambda$. Iterating this procedure ω times will give the desired M'.

Lemma 2.4 *Let M be an ω-tree. Then M is a $(\lambda, \mathfrak{A})$-model.*

Proof: Suppose $A \subseteq M$, $|A| < \lambda$. The closure of A in M can have power at most $|A|^\omega \leqslant (2^{|A|})^\omega < \lambda$. Let N be the ω-tree formed from the closure of A in M. Then clearly N is isomorphic to some ω-tree in $\mathfrak{A}$.

Theorem 2.1 will follow easily from the next lemma which works for any collection $\mathfrak{A}$ and requires only that $\lambda^\omega = \lambda$.

Lemma 2.5 *Let M be an ω-tree of power λ. There is an ω-tree N of power λ such that N is not isomorphic to a closed submodel of M.*

Proof of Theorem 2.1: By Lemma 2.3 there is a $(\lambda, \mathfrak{A})$-full ω-tree M of power λ. By Lemma 2.5 there is an ω-tree N of power λ not isomorphic to a closed submodel of M. By Lemma 2.3 there is N' of power λ such that $N \subseteq_c N'$ and N' is $(\mathfrak{A}, \lambda)$-full. Then, by Lemma 2.4 and Theorem 1.6, $M \equiv_{\infty\lambda} N'$. However, clearly $M \not\cong N'$ since otherwise M would have a closed submodel isomorphic to N.

Proof of Lemma 2.5: This follows from Shelah [6], VIII, 2.5, p. 440, but we repeat it here.

The model N will be obtained from M by removing the points in M on the ω^{th} level and adding new points on the ω^{th} level above branches of M that did not have limit points. We also add a new root below the rooted M. Specifically, let N be the canonical model with universe $\{\langle\,\rangle\} \cup \{\langle a_0, \ldots, a_n\rangle : n < \omega, a_i \in P_i^M,$

and $a_j <_M a_i$ for $j < i \leqslant n\} \cup \{\langle a_0, \ldots, a_n \ldots\rangle_{n<\omega}$: $a_i \in P_i^M$, $a_j <_M a_i$ for $j < i < \omega$, and there is no $a \in P_\omega^M$ such that $a_i <_M a$ for $i < \omega\}$. Since $\lambda^\omega = \lambda$, $\|N\| = \lambda$.

Now suppose that f is an isomorphic embedding of N into a closed submodel of M. It is obvious that if $x \in P_i^M$ then $f(x) \in P_i^N$.

We define by induction on $i < \omega$, $a_i \in P_i^M$ such that for $j < i$ $a_i <_M a_j$.

For a_0 we take the root of M. Notice that $\langle a_0\rangle \in P_1^N$. For a_1 we take $f(\langle a_0\rangle)$. Then $a_1 \in P_1^M$. In general, we let $a_{i+1} = f(\langle a_0, \ldots a_i\rangle)$.

It follows inductively that $a_i \in P_i^M$ and that for $j < i$, $a_j <_M a_i$.

We now consider the branch $\{a_i : i \in \omega\}$. We obtain the desired contradiction by asking if there is some $a \in M$ above all the a_i.

If there is no such $a \in M$, then by definition $\langle a_i\rangle_{i<\omega} \in N$. But then $f(\langle a_i\rangle_{i<\omega}) \in P_\omega^M$ and $f(\langle a_i\rangle_{i<\omega}) >_M f(\langle a_0, \ldots, a_i\rangle) = a_{i+1}$ for each $i \in \omega$, a contradiction.

If on the other hand there is some $a \in P_\omega^M$ such that $a_i < a$ for all $i < \omega$, then, by definition, $\langle a_i\rangle_{i<\omega} \notin N$. But then $a \notin \mathrm{ra}(f)$ and so $\mathrm{ra}(f)$ is not a closed subset of M, again a contradiction.

Remark 2.6 From two models M_1 and M_2 as constructed above we can generate 2^λ nonisomorphic but $\equiv_{\infty\lambda}$ ω-trees of power λ. The basic idea is as follows: suppose $A \subseteq^{\omega>} \lambda$ closed under initial segments but with no infinite increasing sequences. If $f: A \to \{0, 1\}$ we define, $N_A^f = A \cup \{\eta^\frown\nu : \eta \in A,\ f(\eta) = i,\ \nu \in |M_i|\}$. It is easy to see that for any f, $G : A \to \{0, 1\}$, $N_A^f \equiv_{\infty\lambda} N_G^g$.

By choosing A carefully we could also arrange that for $f \neq g$, $N_A^f \not\cong N_A^g$. Rather than go into the necessary details here, we give a simpler construction in the next section where we construct λ^+ nonisomorphic models to begin with.

3 We now remove the assumption that λ is a strong limit cardinal and assume only that $\lambda^\omega = \lambda$. If λ happens not to be a strong limit cardinal then the construction of the previous section will produce an ω-tree of cardinality greater than λ. A new, more subtle construction is needed which will not force up cardinalities. We will simultaneously construct the desired models along with the collection $\mathfrak{A}$, rather than specifying $\mathfrak{A}$ in advance as we did previously. In particular we will define by induction on $\alpha < \lambda^+$ canonical ω-trees M_α as follows:

$$|M_0| = \{\langle \underbrace{0, \ldots, 0}_{i \text{ times}}\rangle : i \in \omega\} \ .$$

M_0 is just a single branch. Next, having defined M_β for each $\beta < \alpha$ we define M_α by building it up from closed submodels. First we define for $\beta < \alpha$.

$$\begin{aligned} |M_{\alpha,\beta}^0| = \{\langle\,\rangle\} &\cup \{\langle\langle 0, \alpha, \beta, a_0\rangle, \ldots, \langle 0, \alpha, \beta, a_n\rangle\rangle :\ a_l \in P_l^{M_\beta} \text{ and} \\ &a_l < a_{l+1}\} \cup \{\langle\langle 0, \alpha, \beta, a_0\rangle, \ldots, \langle 0, \alpha, \beta, a_n\rangle \ldots\rangle : \\ &a_l \in P_l^{M_\beta},\ a_l < a_{l+1} \text{ and such that there is no } a \in P_\omega^{M_\beta} \\ &\text{such that } a_l < a \text{ for all } l < \omega.\} \end{aligned}$$

The purpose of this step is to ensure nonisomorphism. We then let

$$M_\alpha^0 = \bigcup_{\beta<\alpha} M_{\alpha,\beta}^0 \ .$$

In the previous step and in what follows, the use of the 0, α, β, etc., is to create distinct copies and make the models essentially disjoint.

Next, we define

$$\begin{aligned}|M_\alpha^{l+1}| = M_\alpha^l \cup \{\eta^\frown\langle\langle l+1, \alpha, \beta, \eta, \nu \restriction k\rangle \ldots \langle l+1, \alpha, \beta, \eta, \nu \restriction m\rangle\rangle :\\ \eta \in M_\alpha^l,\ \beta < \alpha,\ l(\eta) = k-1,\ l(\nu) \geq k \text{ and } \nu \in M_\beta;\\ m = l(\nu)+1 < \omega \text{ or } m = l(\nu) = \omega\}\ .\end{aligned}$$

The purpose of this is to ensure $(\lambda, \mathfrak{A})$-fullness. Finally, we let

$$M_\alpha = \bigcup_{l<\omega} M_\alpha^l$$

and

$$\mathfrak{A} = \{M_\alpha : \alpha < \lambda\}\ .$$

Note that M_α^l is closed in M_α for each $l < \omega$ and $M_{\alpha,\beta}^0$ is closed in M_α^0.

Lemma 3.1 For each $\alpha < \lambda^+$, $\|M_\alpha\| \leq (|\alpha| + \aleph_0)^{\aleph_0}$. Hence, for α such that $\lambda \leq \alpha < \lambda^+$, $\|M_\alpha\| = \lambda$, and for $\alpha < \lambda$, $\|M_\alpha\| \leq \lambda$.

Proof: The first assertion is easily checked by induction on the stages of the construction. The exponent results from the first step of the construction. The second assertion follows since $\lambda^\omega = \lambda$ and $\|M_\alpha\|$ is obviously at least $|\alpha|$.

Remark 3.2 For M an ω-tree let M' denote the ω-tree formed by restricting to P^M, i.e., removing any nodes on the ω^{th} level. It is then easy to check that M'_α is a Σ_1 definable function of α, or even that M'_α is a primitive recursive set function of α. The point is that the relation $T = M'_\alpha$ is absolute for standard models of a very weak part of ZF. Of course, this is not true of M_α since the nodes on the ω^{th} level depend upon the set of all branches.

Theorem 3.3 *For* $\lambda \leq \beta < \alpha < \lambda^+$, $\|M_\alpha\| = \|M_\beta\| = \lambda$; $M_\alpha \equiv_{\infty\lambda} M_\beta$ *and* $M_\alpha \not\cong M_\beta$.

The above is an immediate consequence of Lemma 3.1, Theorem 1.6, and the three lemmas to follow.

Lemma 3.4 *For* $\beta < \alpha < \lambda^+$, $M_\alpha \not\cong M_\beta$.

Proof: We can show that M_α cannot even be embedded as a closed submodel of M_β just as in the proof of Lemma 2.5, where we now use the model $M_{\alpha,\beta}^0$ which is built into M_α in the first inductive step.

Lemma 3.5 *For* $\alpha \geq \lambda$, M_α *is* $(\lambda, \mathfrak{A})$-*full.*

Proof: Let $N \in \mathfrak{A}$. Then $N = M_\beta$ for some $\beta < \lambda \leq \alpha$. Let $\eta \in P_i^{M_\alpha}$ and $\nu \in P_i^N$. By the construction of M_α, we have $\eta \in |M_\alpha^l|$ for some $l \in \omega$. Now, our construction of M_α^{l+1} already embeds $N_{[\nu]}$ in M_α^{l+1} and hence in M_α as a closed submodel. To see that there are actually λ such closed submodels with pairwise intersection $\{\eta \restriction j : j \leq i\}$ we note that we actually put in a distinct copy of $N_{[\nu]} \backslash \{\nu \restriction j : j \leq i\}$ for each γ such that $\beta \leq \gamma < \alpha$.

The next lemma has the least obvious proof, since our construction did not seem to explicitly arrange that the M_α's would be $(\lambda, \mathfrak{A})$-models. However, the construction has taken care of this indirectly.

Lemma 3.6 *For $\alpha < \lambda^+$, M_α is a $(\lambda, \mathfrak{A})$-model.*

Note that if $\forall \mu < \lambda(\mu^{\aleph_0} < \lambda)$. Then the proof is easy. The proof of Lemma 3.6 will be divided into several parts.

Proof of Lemma 3.6 (beginning): Let $\alpha < \lambda^+$, $A \subseteq |M_\alpha|$, $|A| < \lambda$. We begin by fixing some sufficiently small but nice universe which will help us find the required closed submodel $N \cong M_\beta$, $\beta < \alpha$ with $A \subseteq |N|$. Specifically, let U be an elementary submodel of $(H((2^\lambda)^+), \epsilon)$ with $A \cup \{A\} \cup \{\alpha\} \subseteq U$ and $\|U\| < \lambda$. Let f be the Mostowski collapsing isomorphism restricted to U and let U^* denote the transitive set (collapse) isomorphic to U. Using Remark 3.2 it is not hard to see that for any $\beta \in |U|, f(M_\beta^l) = M_{f(\beta)}^l$ and all the related properties are preserved. In the following sublemma we concentrate on the ω^{th} level.

Sublemma 3.7 Suppose that for all $i < \omega$, $\eta \restriction i \in M_\alpha \cap U$. Let $\eta^* = \langle f(\eta \restriction i) : i < \omega \rangle$. Then,

$$\eta \in P_\omega^{M_\alpha} \text{ iff } \eta^* \in P_\omega^{M_{f(\alpha)}} .$$

Proof: Notice first that η^* is always a "branch" in $M_{f(\alpha)}$. The only question is whether η^* is actually in $M_{f(\alpha)}$. Of course, η need not be in U. The proof proceeds by induction on α (i.e., we prove it for all ordinals $<\lambda^+$ which belong to U).

First, for each $i \in \omega$ let l_i be the smallest l such that $\eta \restriction i \in |M_\alpha^l|$. We already know that $\eta \restriction i \in M_\alpha^l$ iff $\eta^* \restriction i \in M_{f(\alpha)}^l$ so l_i applies as well to η^*. Notice that l_i is nondecreasing in i. We consider three cases:

Case 1. $\{l_i : i \in \omega\}$ is unbounded in ω. Then by the construction $\eta \notin |M_\alpha|$ and $\eta^* \notin |M_{f(\alpha)}|$.

Case 2. $\{l_i : i \in \omega\}$ has least upper bound $l > 0$. Then, for sufficiently large i, $\eta \restriction i \in |M_\alpha^{l+1}| - |M_\alpha^l|$. Thus, $\eta \in |M_\alpha|$ iff for some unique $\eta_0 \in M_\alpha^l$, $\beta < \alpha$, $\nu \in P_\omega^{M_\beta}$, and suitable i,

$$\eta = \eta_0 {}^\frown \langle \langle l+1, \alpha, \beta, \eta_0, \nu \restriction i \rangle, \langle l+1, \alpha, \beta, \eta_0, \nu \restriction i+1 \rangle \ldots \rangle .$$

Now, by our induction hypothesis the above is true iff the analogous situation holds for η^*, $f(\eta_0)$, $f(\beta)$, $\langle f(U \restriction m), m \in \omega \rangle$ and i, which is again equivalent by the definition of $M_{f(\alpha)}$ to $\eta^* \in M_{f(\alpha)}$.

Case 3. $l_i = 0$ for all $i \in \omega$. Then there is a unique $\beta < \alpha$ such that for all $i \in \omega$, $\eta \restriction i \in M_{\alpha,\beta}^0$ and η has the form $\langle \langle 0, \alpha, \beta, a_0 \rangle, \langle 0, \alpha, \beta, a_1 \rangle, \ldots \rangle$, where $a_l \in P_l^{M_\beta}$ and $a_l < a_{l+1}$. By our construction, $\eta \in M_\alpha$ iff in M_β there is no x above all the a_l. Clearly $f(a_l) \in P_l^{M_{f(\beta)}}$, $f(a_l) < f(a_{l+1})$, and $\eta^* = \langle \langle 0, f(\alpha), f(\beta), f(a_0) \rangle, \langle 0, f(\alpha), f(\beta), f(a_1) \rangle \ldots \rangle$ and $\eta^* \in M_{f(\alpha)}$ iff in $M_{f(\beta)}$ there is no y above all the $f(a_l)$. Now, applying the induction hypothesis to the branch formed by the a_l in M_β finishes the argument.

Proof of Lemma 3.6 (conclusion): Trivially $A \subseteq |M_\alpha| \cap U$ which is in turn a subset of the closure of $M_\alpha^l \cap U$ in M_α (in the sense of the real world). Now, by the sublemma, this last ω-tree is isomorphic to the closure of $M_{f(\alpha)}^l \cap U^*$ in $M_{f(\alpha)}$, which is just the closure of $M_{f(\alpha)}^l$ in $M_{f(\alpha)}$, which is $M_{f(\alpha)}$. Thus, we have

A as a subset of a closed subset of M_α, viz. $cl^{M_\alpha}(M'_\alpha \cap U)$, which is isomorphic to a member of $\mathfrak{A}$, viz., $M_{f(\alpha)}$. This concludes the proof.

Remark 3.8 Now, having produced λ^+ pairwise $\equiv_{\infty\lambda}$ but nonisomorphic ω-trees we indicate how to get 2^λ. Actually we only need to use λ from among the λ^+, say N_i, $i < \lambda$. For each nonempty $S \subseteq \lambda$ we define an ω-tree T_S which consists of a root and above that root λ copies of each N_i for $i \in S$. It is easy to see that for nonempty $S_1 \neq S_2 \subseteq \lambda$, $T_{S_1} \equiv_{\infty\lambda} T_{S_2}$ but clearly $T_{S_1} \not\cong T_{S_2}$.

REFERENCES

[1] Chang, C. C., "Some remarks on the model theory of infinitary languages," pp. 36–63 in *The Syntax and Semantics of Infinitary Languages, Lecture Notes in Mathematics, 72*, Springer-Verlag, Berlin, 1968.

[2] Dickmann, M., "Larger infinitary languages," Chapter IX in *Model Theoretic Logic*, Springer-Verlag, Berlin, to appear.

[3] Nadel, M., "$L_{\omega_1\omega}$ and admissible fragments," Chapter VIII in *Model Theoretic Logic*, Springer-Verlag, Berlin, to appear.

[4] Nadel, M. and J. Stavi, "$L_{\infty\lambda}$-equivalence, isomorphism, and potential isomorphism," *Transactions of the American Mathematical Society*, vol. 236 (1978), pp. 51–74.

[5] Scott, D., "Logic with denumerably long formulas and finite strings of quantifiers," pp. 329–341 in *Proceedings of the Berkeley Symposium on Theory of Models, 1963*, North Holland, Amsterdam, 1965.

[6] Shelah, S., "Some remarks in model theory," Abstract 75T-E77, *Notices of the American Mathematical Society*, vol. 23 (1976), A-289.

[7] Shelah, S., *Classification Theory*, 2nd Ed., North Holland, Amsterdam, 1984.

[8] Shelah, S., "On the number of nonisomorphic models of cardinality λ $L_{\infty\lambda}$ equivalent to a fixed model," *Notre Dame Journal of Formal Logic*, vol. 22, no. 1 (1981), pp. 5–10.

[9] Shelah, S., "On the number of nonisomorphic models in $L_{\infty\lambda}$ when λ is weakly compact," *Notre Dame Journal of Formal Logic*, vol. 23, no. 1 (1982), pp. 21–26.

[10] Shelah, S., "Existence of many $L_{\infty\lambda}$-equivalent nonisomorphic models of T of power λ," to appear.

[11] Shelah, S., "On the possible number no(M)," *Notre Dame Journal of Formal Logic*, to appear.

[12] Shelah, S., "On the possible number no(M) II," to appear.

[13] Stavi, J., "On $L_{\infty\kappa}$-equivalence of Boolean algebras, rings and groups," Abstract 75T-E77, *Notices of the American Mathematical Society*, vol. 22 (1975), A-714.

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