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Vector Spaces and Binary Quantifiers

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1 Introduction Caicedo [1] and others [3] have observed that monadic quantifiers cannot count the number of classes of an equivalence relation. This implies the existence of a binary quantifier which is not definable by monadic quantifiers. The purpose of this paper is to show that binary quantifiers cannot count the dimension of a vector space. Thus we have an example of a ternary quantifier which is not definable by binary quantifiers.

The general form of a binary quantifier is

$$Qx_1y_1 \ldots x_ny_n\phi_1(x_1,y_1) \ldots \phi_n(x_n,y_n).$$

An example of such a quantifier is (in addition to all monadic quantifiers) the similarity quantifier:

$$Sx_1y_1x_2y_2\phi_1(x_1,y_1)\phi_2(x_2,y_2) \longleftrightarrow \phi_1(\cdot,\cdot) \text{ and } \phi_2(\cdot,\cdot) \text{ are isomorphic as binary relations.}$$

We let $\mathcal{L}(Q)$ denote the extension of first-order logic by the quantifier Q. Recall the definition of $\Delta(\mathcal{L}(Q))$ from [2]. It is proved in [4] that $\Delta(\mathcal{L}(S))$ is equivalent to second-order logic. Even monadic quantifiers can have very powerful Δ-extensions. Thus, simple syntax (such as $\mathcal{L}(Q)$) is no guarantee for simple model theory.

2 Vector spaces—the main lemma Let K be an infinite field. We shall consider vector spaces

$$\mathcal{V} = \langle V, +, \cdot, 0; K\rangle$$

over K. Here + denotes addition of vectors, $\cdot$ denotes multiplication of vectors by an element of the field, and 0 is the zero vector. Thus $\mathcal{V}$ should be considered as a two-sorted structure. Let L denote the language associated with $\mathcal{V}$

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consisting of symbols $\underline{+}$, $\underline{\cdot}$, $\underline{0}$ for the vector operations, a constant symbol $\underline{c}$ for each $c \in K$, and symbols for the field operations. The *linear type* of an n-tuple $a_1, \ldots, a_n$ of elements of V is the set of linear equations

$$c_1 x_1 + \ldots + c_n x_n = 0$$

satisfied by $a_1, \ldots, a_n$ $(c_1, \ldots, c_n \in K)$.

Main Lemma *Let $\mathcal{V}$ and $\mathcal{V}'$ be two vector spaces over K of dimensions d and d' respectively. Let $a_1, \ldots, a_n$ be an n-tuple from $\mathcal{V}$ and $a'_1, \ldots, a'_n$ an n-tuple from $\mathcal{V}'$ of the same linear type. Suppose*

$$n + 2 \leqslant d, d' \leqslant |K|.$$

Then there is a bijection f: $\mathcal{V} \to \mathcal{V}'$ such that $(x, y, a_1, \ldots, a_n)$ has the same linear type in $\mathcal{V}$ as $(fx, fy, a'_1, \ldots, a'_n)$ in $\mathcal{V}'$, whatever $x, y \in V$.

Proof: Let H be the subspace of $\mathcal{V}$ generated by $a_1, \ldots, a_n$ and H' the respective subspace of $\mathcal{V}'$. Let G be a subspace of $\mathcal{V}$ such that $\mathcal{V} = H \oplus G$ and G' a similar subspace of $\mathcal{V}'$. Note that G and G' have dimensions of at least 2, since $d, d' \geqslant n + 2$. Let W be a maximal subset of G with respect to the property

$$x \neq y \;\&\; x, y \in W \Rightarrow \{x, y\} \text{ free}.$$

Then every vector in G has the representation λw for unique $\lambda \in K$ and $w \in W$. Let W' be defined similarly in G'.

From $d \leqslant |K|$ it follows that $|V| = |K|$ (recall that K is infinite). Similarly $|H| = |G| = |K|$. Clearly $|W| \geqslant |K|$. Thus $|W| = |K|$. By symmetry, $|W'| = |W|$.

Now we shall define the mapping f. We let f be the identity on K. Let f map W one-one onto W'. As $\vec{a}$ and $\vec{a}'$ have the same linear type, we have

$$(H, \vec{a}) \cong (H', \vec{a}')$$

and we can let f map H isomorphically onto H' such that $f(a_i) = a'_i$ $(i = 1, \ldots, n)$. Now if $v \in V$, then v has a unique representation

$$v = \lambda w + h,$$

where $\lambda \in K$, $w \in W$, and $h \in H$, and we can define

$$f(v) = \lambda f(w) + f(h).$$

This clearly makes f onto. To prove the claim concerning linear type, let

$$\mu_1 x_1 + \mu_2 x_2 + \mu_3 z_1 + \ldots + \mu_{n+2} z_n = 0$$

be an equation satisfied by $(b_1, b_2, a_1, \ldots, a_n)$ in $\mathcal{V}$. Let

$$b_i = \lambda_i w_i + h_i, \qquad (i = 1, 2).$$

Thus

$$\mu_1 \lambda_1 w_1 + \mu_2 \lambda_2 w_2 + \mu_1 h_1 + \mu_2 h_2 + \mu_3 a_1 + \ldots + \mu_{n+2} a_n = 0.$$

As $G \cap H = \{0\}$, we must have

$$\mu_1 \lambda_1 w_1 + \mu_2 \lambda_2 w_2 = 0.$$

By the very definition of W, either $\mu_1\lambda_1 = \mu_2\lambda_2 = 0$ or $w_1 = w_2$ (and $\mu_1\lambda_1 + \mu_2\lambda_2 = 0$). We also have

$$\mu_1 h_1 + \mu_2 h_2 + \mu_3 a_1 + \ldots + \mu_{n+2} a_n = 0.$$

Now in any case

$$\mu_1\lambda_1 f(w_1) + \mu_2\lambda_2 f(w_2) = 0$$

and

$$\mu_1 f(h_1) + \mu_2 f(h_2) + \mu_3 a_1' + \ldots + \mu_{n+2} a_n' = 0,$$

whence

$$\mu_1 f(b_1) + \mu_2 f(b_2) + \mu_3 a_1' + \ldots + \mu_{n+2} a_n' = 0,$$

as desired. The converse is entirely similar.

3 Equivalence of vector spaces We show that the dimension of vector spaces cannot be distinguished in certain logics.

Let Q be a *binary* quantifier, that is, a quantifier of type

(*) $\quad Qx_1y_1 \ldots x_ny_n\phi_1(x_1, y_1, \vec{z}) \ldots \phi_n(x_n, y_n, \vec{z}).$

Let $\mathcal{L}_{\infty\omega}$ denote the infinitary language over the language L defined in Section 2. If $\phi(\vec{z})$ is a formula of type (*), where each $\phi_i(x_i, y_i, \vec{z})$ is a quantifier-free formula of $\mathcal{L}_{\infty\omega}$, and T a linear type of m-tuples, let

$$\pi_K(\phi(\vec{z}), T)$$

be the true propositional symbol, if the statement (**) below holds, and the falsity symbol otherwise:

(**) There is a vector space $\mathcal{V}$ over K of dimension d, $m + 2 \leqslant d \leqslant |K|$ ($\vec{z} = (z_1, \ldots, z_m)$) which satisfies $\phi(\vec{a})$ for some m-tuple $\vec{a}$ of linear type T.

Let $\mathcal{L}_{\infty\omega}(Bin)$ be the extension of $\mathcal{L}_{\infty\omega}$ by all binary generalized quantifiers.

Elimination Lemma *Suppose $\phi(\vec{x})$ is in $\mathcal{L}_{\infty\omega}(Bin)$ and α is a cardinal exceeding the number of free variables in any subformula of $\phi(\vec{x})$. Then there is a quantifier free $\phi^*(\vec{x})$ in $\mathcal{L}_{\infty\omega}$ such that*

$$\forall\vec{x}(\phi(\vec{x}) \longleftrightarrow \phi^*(\vec{x}))$$

holds in any vector space over K of dimension d, $\alpha + 1 \leqslant d \leqslant |K|$.

Proof: The proof proceeds by induction on the length of $\phi(\vec{x})$. To prove the quantifier step, consider a formula $\phi(\vec{x})$ of type (*) above. Let $\mathcal{T}$ be the set of all linear types of m-tuples. If $T \in \mathcal{T}$, let $P_T(\vec{z})$ be the conjunction of all equations

(+) $\quad c_1z_1 + \ldots + c_mz_m = 0$

which belong to T as well as of all

$$c_1z_1 + \ldots + c_mz_m \neq 0$$

such that (+) is not in T. Finally, let

$$\phi^*(\vec{z}) = \bigvee_{T \in \mathcal{T}} (P_T(\vec{z}) \wedge \pi_K(\phi(\vec{z}), T)).$$

To prove the claimed equivalence of $\phi(\vec{z})$ and $\phi^*(\vec{z})$, let $\mathcal{V}'$ be a vector space over K of dimension $>\alpha$. For a start, suppose $\mathcal{V}'$ satisfies $\phi(\vec{a}')$ where $\vec{a}'$ is an m-tuple from $\mathcal{V}'$. As it turns out in a while, we may assume the $\vec{a}'$ are all from V (and not from K). Let $T \in \mathcal{T}$ be the linear type of $\vec{a}'$. Thus $\mathcal{V}'$ satisfies $P_T(\vec{a}')$. By definition, $\pi_K(\phi(\vec{z}), T)$ is true (take $\mathcal{V} = \mathcal{V}'$ in (**)). Therefore $\phi^*(\vec{a}')$ holds in $\mathcal{V}'$. For the converse, suppose $\mathcal{V}'$ satisfies $\phi^*(\vec{a}')$. There are a $T \in \mathcal{T}$, and an m-tuple $\vec{a}$ as in (**). Now $\mathcal{V}$ satisfies $\phi(\vec{a})$ and $\vec{a}$ and $\vec{a}'$ have the same linear type T. Let $f\colon \mathcal{V} \to \mathcal{V}'$ be as in the Main Lemma. If there happened to be elements of K in $\vec{a}'$, f would be fixed on them, so they would cause no trouble.

By the conclusion of the Main Lemma, the sequences $(x, y, \vec{a})$ and $(fx, fy, \vec{a}')$ have the same linear type whatever $x, y \in V$. This implies

$$\mathcal{V} \models \phi_i(x, y, \vec{a}) \longleftrightarrow \mathcal{V}' \models \phi_i(fx, fy, \vec{a}')$$

for all $i = 1, \ldots, m$ and $x, y \in V$. By the closure of Q under isomorphisms, we get

$$\mathcal{V} \models \phi(\vec{a}) \longleftrightarrow \mathcal{V}' \models \phi(\vec{a}').$$

We have already observed that $\phi(\vec{a})$ holds in $\mathcal{V}$. Therefore $\mathcal{V}' \models \phi(\vec{a}')$ as desired.

Corollary 1 *Let ϕ be a sentence in $\mathcal{L}_{\infty\omega}(Bin)$ and let α be a cardinal greater than the number of free variables in any subformula of ϕ. Then either ϕ is true in all vector spaces over K of dimension d, $\alpha + 1 \leqslant d \leqslant |K|$, or true in none.*

This result shows that $\mathcal{L}_{\infty\omega}(Bin)$ cannot distinguish two infinite-dimensional vector spaces over $\mathbf{R}$, and $\mathcal{L}_{\omega\omega}(Bin)$ cannot distinguish finite-dimensional vector spaces over, say, Q from the infinite dimensional one.

Proposition *Suppose $\mathcal{V}$ and $\mathcal{V}'$ are two vector spaces over an uncountable field K of different infinite dimensions. Suppose $\mathcal{K}$ and $\mathcal{K}'$ are $PC(\mathcal{L}(Q_1))$-classes such that $\mathcal{V} \in \mathcal{K}$ and $\mathcal{V}' \in \mathcal{K}'$. Then $\mathcal{K} \cap \mathcal{K}' \neq \phi$.*

Proof: By compactness there are vector spaces $\mathcal{W} \in \mathcal{K}$ and $\mathcal{W}' \in \mathcal{K}'$ over a field K' such that $\mathcal{W}$ and $\mathcal{W}'$ have uncountable dimension. This depends on the fact that in any vector space over an uncountable field of dimension $\geqslant n$ there are uncountably many vectors, no n of which are linearly dependent ($n \geqslant 2$). (Consider vectors with coordinates $(x, x^2, x^3, \ldots, x^n)$ where x belongs to the field. No n of these vectors are linearly dependent because

$$\begin{vmatrix} x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ x_1^n & x_2^n & \ldots & x_n^n \end{vmatrix} = \prod_{1 \leqslant i < j \leqslant n} x_i(x_i - x_j) \neq 0$$

if $x_1, \ldots, x_n$ are nonzero and different.) We may assume $|\mathscr{W}| = |\mathscr{W}'| = |K'| = \aleph_1$. Thus $dim(\mathscr{W}) = dim(\mathscr{W}') = \aleph_1$ whence $\mathscr{W} \cong \mathscr{W}'$. This implies $\mathscr{K} \cap \mathscr{K}' \neq \phi$.

This proposition shows that we cannot hope to separate the vector spaces, which were proved to be inseparable by binary quantifiers, by *PC*-classes of $\mathscr{L}(Q_1)$. Other examples have to be used if one wants to show the undefinability of $\Delta(\mathscr{L}(Q_1))$ by binary quantifiers. The same applies to such extensions of $\mathscr{L}(Q_1)$ as $\mathscr{L}^{Pos}$ and $\mathscr{L}(aa)$. Thus we have:

Corollary 2 *We can replace $\mathscr{L}_{\infty\omega}(Bin)$ in Corollary 1 by $\Delta(\mathscr{L}(Q_1))$, $\Delta(\mathscr{L}^{Pos})$ and $\Delta(\mathscr{L}(aa))$.*

4 Logics which can separate vector spaces The most straightforward example of a logic capable of distinguishing infinite dimensional vector spaces from finite dimensional ones is $\mathscr{L}_{\omega_1\omega}$: consider the sentence

$$\bigwedge_{n<\omega} \exists x_1 \ldots x_n \forall f_1 \ldots f_n \in K(f_1x_1 + \ldots + f_nx_n = 0 \longleftrightarrow f_1 = \ldots = f_n = 0).$$

This sentence is in fact in the fragment $\mathscr{L}_{HYP}$ where *HYP* is the smallest admissible language containing ω. Thus we have[1]:

Proposition $\Delta(\mathscr{L}(Q_0)) \nleqslant \mathscr{L}_{\omega\omega}(Bin)$.

By considering the sentences

$$Q_1xB(x) \wedge \bigwedge_{n<\omega} \forall x_1 \ldots x_n \in B\, \forall f_1 \ldots f_n \in K$$
$$\left(\bigwedge_{1\leqslant i<j\leqslant n} x_i \neq x_j \rightarrow (f_1x_1 + \ldots + f_nx_n = 0 \longleftrightarrow f_1 = \ldots = f_n = 0)\right)$$

$$\neg Q_1xB(x) \wedge \forall x \bigvee_{n<\omega} \exists x_1 \ldots x_n \in B\, \exists f_1 \ldots f_n \in K(x = f_1x_1 + \ldots + f_nx_n),$$

and bearing in mind that $\mathscr{L}_{HYP} \leqslant \Delta(\mathscr{L}(Q_0, Q_1))$, one gets:

Proposition $\Delta(\mathscr{L}(Q_0, Q_1)) \nleqslant \mathscr{L}_{\infty\omega}(Bin)$.

Corollary 3 $\mathscr{L}_{\omega_1\omega}(Q_1) \nleqslant \mathscr{L}_{\infty\omega}(Bin)$.

We shall now introduce a ternary quantifier Q which is not definable in $\mathscr{L}_{\infty\omega}(Bin)$. For a ternary predicate $D(x, y, z)$, constants c_0, c_1, and a unary predicate $B(x)$ consider the formulas:

$$\begin{aligned}\phi_0(x, y, u, v) \longleftrightarrow\ & x \neq u \wedge x \neq v \wedge y \neq u \wedge y \neq v \wedge \\ & ((x = y \wedge u = v) \vee (x \neq y \wedge u \neq v \wedge \neg\exists z(D(x, y, z) \wedge D(u, v, z)) \\ & \wedge \exists z((D(x, v, z) \wedge D(u, y, z)) \vee (D(x, u, z) \wedge D(y, v, z)))))\end{aligned}$$

$$\phi_1(x, y, u, v) \longleftrightarrow (\phi_0(x, y, u, v) \wedge (\exists z(D(x, u, z) \wedge D(y, v, z)) \rightarrow x = y)$$

$$\phi_+(x, y, z) \longleftrightarrow \exists uv(\phi_1(c_0, x, u, v) \wedge \phi_1(u, v, y, z))$$

$$F(x) \longleftrightarrow D(x, c_0, c_1)$$

$$\begin{aligned}\phi_.(x, y, z) \longleftrightarrow\ & x = z = c_0 \vee (x = c_1 \wedge z = y) \vee (F(x) \wedge x \neq c_0 \wedge x \neq c_1 \\ & \wedge \exists uv(\phi_0(c_1, u, x, v) \wedge \phi_0(u, y, v, z) \wedge D(u, v, c_0) \wedge D(y, z, c_0))\end{aligned}$$

$\phi_+^1(\lambda, x) \longleftrightarrow \lambda = c_0 \vee x = c_0$

$\phi_+^n(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_n) \longleftrightarrow \exists uvw(\phi_.(\lambda_1, x_1, u)$
$\wedge\ \phi_.(\lambda_2, x_2, v) \wedge \phi_+(u, v, w) \wedge \phi_+^{n-1}(c_1, \lambda_3, \ldots, \lambda_n, w, x_3, \ldots, x_n))$

$Free^n(x_1, \ldots, x_n) \longleftrightarrow \forall\lambda_1 \ldots \lambda_n((F(\lambda_1) \wedge \ldots \wedge F(\lambda_n)$
$\wedge\ \phi_+^n(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_n) \rightarrow \lambda_1 = \ldots = \lambda_n = c_0)$

$$Fr(B) \longleftrightarrow \bigwedge_{n<\omega} \forall x_1 \ldots x_n \in B\left(\bigwedge_{1 \leqslant i<j \leqslant n} x_i \neq x_j \rightarrow Free^n(x_1, \ldots x_n)\right).$$

Definition $QxyzD(x, y, z) \longleftrightarrow$ there is an uncountable set B such that $Fr(B)$ holds for some choice of $c_0 \neq c_1$.

Suppose now that V is a vector space over a field K. Define

$$D_V(x, y, z) \longleftrightarrow \exists\lambda \in K(x = \lambda y + (1 - \lambda)z)$$
("x, y and z are on the same line").

Then for this interpretation of D and any choice of $c_0 \neq c_1$, $Fr(B)$ holds if and only if B is a free set of vectors. This shows that one can separate dimensions of vector spaces using Q.

Proposition *The class of countable dimensional vector spaces is definable in* $\mathcal{L}(Q)$.

Corollary 4 $\mathcal{L}(Q) \nleqslant \mathcal{L}_{\infty\omega}(Bin)$, *that is, there is a ternary quantifier which is not definable using binary quantifiers.*

Problems: Is there an $(n + 1)$-ary quantifier not definable using n-ary quantifiers for $n > 2$? Is $\Delta(\mathcal{L}(Q_1))$ definable using binary quantifiers?

NOTE

1. Recall that $\Delta(\mathcal{L}(Q_0)) = \mathcal{L}_{HYP}$.

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