Notre Dame Journal of Formal Logic
Volume 25, Number 1, January 1984

Decomposable Collections of Sets

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The notion of a *weave* was first defined by Gaisi Takeuti as an approach to the problem of Borel Determinateness. In a paper co-authored by Burd and Takeuti [1] some of the game-like properties of weaves were explored. A weave is a set-theoretic object that corresponds to a two-person game in which each player presents a *choice* of moves from a set of possibilities rather than a single move. In another paper [2] Green and Takeuti used the weave idea to prove a theorem about Boolean polynomials.

The Green-Takeuti paper gives sufficient conditions enabling a Boolean polynomial to be factored into a Boolean statement in which no atom appears more than once. Consider a Boolean polynomial to be a collection of sets, each atom being an element, each term being a set of elements. In this context the Green-Takeuti theorem is a theorem about the decomposition of collections of sets.

In this paper we present a new proof of the Green-Takeuti theorem, and extend the proof to cover the case where the collection of sets is infinite (i.e., the Boolean polynomial is infinitary). We do this using the notion of a weave.

Definition 1 Let P be a set. Let $\mathbf{W}$ and $\mathbf{W}'$ be nonempty subsets of $\mathcal{P}(P) - \{\phi\}$. The pair $\langle\mathbf{W},\mathbf{W}'\rangle$ is a Weave of P iff both:

(1) for each set W in $\mathbf{W}$, and each set W' in $\mathbf{W}'$, the intersection $W \cap W'$ is a singleton set
(2) for each element p in P, there is a set W in $\mathbf{W}$ and a set W' in $\mathbf{W}'$ such that $W \cap W' = \{p\}$.

The set P is called the set of Points of the weave. We denote this by writing $P = \text{Points}(\langle\mathbf{W},\mathbf{W}'\rangle)$. Notice that Clause 2 in the definition of a weave

*The author wishes to thank Gaisi Takeuti, Harriet Ritter, and Alan Candiotti for their aid and support.

Received August 23, 1982; revised November 22, 1982

implies that the set Points($\langle \mathbf{W},\mathbf{W}' \rangle$) = $\cup\mathbf{W} = \cup\mathbf{W}'$. Thus we will abbreviate the notation by writing Points($\mathbf{W}$) instead of Points($\langle \mathbf{W},\mathbf{W}' \rangle$).

Example 1: Let $\mathbf{W} = \{\{a,d\}, \{b,c\}\}$ and let $\mathbf{W}' = \{\{a,b\}, \{c,d\}, \{a,c\}, \{b,d\}\}$. The pair $\langle \mathbf{W},\mathbf{W}' \rangle$ is a Weave. The set of Points ($\langle \mathbf{W},\mathbf{W}' \rangle$) is $\{a,b,c,d\}$.

Definition 2 A weave $\langle \mathbf{W},\mathbf{W}' \rangle$ is called Normal iff for each set $X \subset$ Points($\mathbf{W}$) either

(a) there is a set W in $\mathbf{W}$ such that $W \subset X$, or
(b) there is a set W' in $\mathbf{W}'$ such that $W' \subset$ Points($\mathbf{W}$) $- X$.

A weave is really a two-person game. Given a weave $\langle \mathbf{W},\mathbf{W}' \rangle$ with points set P, let P be the set of possible next-moves. In order to determine the next move, Player I votes by choosing a set W in $\mathbf{W}$, and simultaneously Player II votes by choosing a set W' in $\mathbf{W}'$. The intersection $W \cap W'$ is the next move. Let X by the set of moves which are considered a win for Player I. Then a normal weave is one in which either Player I or Player II has a sure-fire way of winning. Thus a normal weave is essentially a determined two-person game.

Notice that the weave of Example 1 is normal. The weave in the following example is not normal.

Example 2: Let $\mathbf{W} = \{\{a,d\}, \{b,c\}\}$ and let $\mathbf{W}' = \{\{a,b\}, \{c,d\}, \{a,c\}\}$.

As a goal we wish to prove that every normal weave can be conveniently decomposed. We must first define the operators to be used in the decomposition.

Definition 3 Let $\mathbf{W}$ be a family of sets. For each i in I (an index set) let $\mathbf{W}_i$ be a family of sets. We define:

(a) (disjunction) $\mathbf{W} = \vee_{i\epsilon I}\mathbf{W}_i$ iff $\mathbf{W} = \cup_{i\epsilon I}\mathbf{W}_i$
(b) (direct disjunction) $\mathbf{W} = \nabla_{i\epsilon I}\mathbf{W}_i$ iff $\mathbf{W} = \vee_{i\epsilon I}\mathbf{W}_i$ and for each i,j in I, if $i \neq j$, Points($\mathbf{W}_i$) $\cap$ Points ($\mathbf{W}_j$) = ϕ
(c) (conjunction) $\mathbf{W} = \wedge_{i\epsilon I}\mathbf{W}_i$ iff $\mathbf{W} = \{$sets $W | W = \cup_{i\epsilon I} W_i$, where W_i is in $\mathbf{W}_i$ for each i in $I\}$ (i.e., $\mathbf{W}$ is the collection of all sets that may be obtained by forming a union consisting of one set from each of the $\mathbf{W}_i$)
(d) (direct conjunction) $\mathbf{W} = \Delta_{i\epsilon I}\mathbf{W}_i$ iff $\mathbf{W} = \wedge_{i\epsilon I}\mathbf{W}_i$ and for each i,j in I, if $i \neq j$, Points($\mathbf{W}_i$) $\cap$ Points($\mathbf{W}_j$) = ϕ.

Example 3: Consider the weave of Example 1. In this case, $\mathbf{W} = (\{a\} \,\Delta\, \{d\}) \,\nabla\, (\{b\} \,\Delta\, \{c\})$. Observe also that $\mathbf{W}' = (\{a\} \,\nabla\, \{d\}) \,\Delta\, (\{b\} \,\nabla\, \{c\})$. Both $\mathbf{W}$ and $\mathbf{W}'$ can be decomposed into single-point sets, and the decompositions of $\mathbf{W}$ and $\mathbf{W}'$ are dual to one another. Note that this weave is normal.

Example 4: Consider the weave of Example 2. We can decompose $\mathbf{W}$ exactly as we did in Example 3 (since it is the same family of sets) but now $\mathbf{W}'$ has no decomposition. Note that this weave is not normal.

Let p be a point, W be a set, and $\mathbf{W}$ be a family of sets. The expression $W \,\nabla\, \mathbf{W}$ is an abbreviation for $\{W\} \,\nabla\, \mathbf{W}$, and the expression $p \,\nabla\, \mathbf{W}$ is an abbreviation for $\{\{p\}\} \,\nabla\, \mathbf{W}$. Similar abbreviations hold for the operator Δ.

Definition 4 Let $\mathbf{W}$ be a nonempty family of sets. The set C is a Choice Set for $\mathbf{W}$ iff C is a subset of Points($\mathbf{W}$) and, for each set W in $\mathbf{W}$, $C \cap W \neq \phi$. The set G is a Weft of $\mathbf{W}$ iff G is a subset of Points($\mathbf{W}$) and, for each set W in $\mathbf{W}$, $G \cap W$ is a singleton set.

The family of choice sets for $\mathbf{W}$ is denoted $c(\mathbf{W})$. The family of wefts of $\mathbf{W}$ is denoted $wf(\mathbf{W})$.

Lemma 1 *If A and B are both wefts of $\mathbf{W}$, and A is a subset of B, then $A = B$.*

Proof: Assume point b is in $B - A$. Since B is a subset of Points($\mathbf{W}$), there is a set W in $\mathbf{W}$ such that $b \in W$. So $B \cap W = \{b\}$. But then, since b is not in A, we have $A \cap W = \phi$. This contradicts the fact that A is a weft of $\mathbf{W}$.

Let $\langle \mathbf{W},\mathbf{W}' \rangle$ be a weave. It is clear from the definition that every set in $\mathbf{W}$ is a weft of $\mathbf{W}'$, and every set in $\mathbf{W}'$ is a weft of $\mathbf{W}$. This leads to the following corollary.

Corollary 1 *If W_1' and W_2' are in $\mathbf{W}'$, and W_1' is a subset of W_2', then $W_1' = W_2'$. Similarly, if W_1 and W_2 are in $\mathbf{W}$, and W_1 is a subset of W_2, then $W_1 = W_2$.*

Lemma 2 *Let $\langle \mathbf{W},\mathbf{W}' \rangle$ be a normal weave. Then $\mathbf{W}' = wf(\mathbf{W})$, and $\mathbf{W} = wf(\mathbf{W}')$.*

Proof: By Clause 1 in the definition of a weave, $\mathbf{W}' \subset wf(\mathbf{W})$. We need to show that $wf(\mathbf{W}) \subset \mathbf{W}'$.

Let G be a weft of $\mathbf{W}$. Then there is no set W in $\mathbf{W}$ such that $W \subset$ Points($\mathbf{W}$) $- G$ (because otherwise W would not meet G). Thus, by normality, there is a set W' in $\mathbf{W}'$ such that $W' \subset G$. But W' and G are both wefts of $\mathbf{W}$. So, by Lemma 1, $W' = G$.

Definition 5 Let $\mathbf{W}$ be a nonempty family of sets. We call $\mathbf{W}$ a Warp iff there exists a nonempty family $\mathbf{W}'$ such that the pair $\langle \mathbf{W},\mathbf{W}' \rangle$ is a weave. If $\mathbf{W}'$ exists so as to make $\langle \mathbf{W},\mathbf{W}' \rangle$ normal, then $\mathbf{W}$ is called a Normal Warp.

Observe that in Example 2, $\mathbf{W}$ is a normal warp, whereas $\mathbf{W}'$ is a non-normal warp.

Lemma 3 *A warp $\mathbf{W}$ is normal if and only if every choice set, C, for $\mathbf{W}$ has a subset G which is a weft of $\mathbf{W}$.*

Proof: Let X be Points($\mathbf{W}$) $- C$. Then C is a choice set for $\mathbf{W}$ if and only if there is no set W in $\mathbf{W}$ satisfying $W \subset X$. By the definition of normality, this is equivalent to the existence of a set G in $\mathbf{W}'$ satisfying $G \subset C$. By Lemma 2, this set G is a weft of $\mathbf{W}$.

Lemma 3 is the turning point in our theory. Every collection $\mathbf{W}$ can be interpreted as a *DNF* statement in an infinitary propositional calculus. A choice set for $\mathbf{W}$ is then a selection of atoms that can be used to demonstrate the falsehood of statement $\mathbf{W}$. Lemma 3 says that $\mathbf{W}$ is normal iff its falsehood can be demonstrated simply by selecting one atom from each term. The lemma will be used many times in the proofs that follow.

The next lemma is a generalization of Lemma 2.

Lemma 4 *Let* $\mathbf{W}$ *be a warp. Then*

(1) $wfwf(\mathbf{W}) \supset \mathbf{W}$, *and*
(2) $wfwfwf(\mathbf{W}) = wf(\mathbf{W})$.

Proof: (1) If W is a set in $\mathbf{W}$, and G is in $wf(\mathbf{W})$, then $W \cap G$ is a singleton set. Thus, W is a weft of $wf(\mathbf{W})$. So W is in $wfwf(\mathbf{W})$.

(2) Apply the result of part (1) to the nonempty family $wf(\mathbf{W})$. In so doing we get that $wfwfwf(\mathbf{W}) \supset wf(\mathbf{W})$. We need to show that "subset" holds in the other direction.

Let X be a set in $wfwfwf(\mathbf{W})$. If W is in $\mathbf{W}$, then W is in $wfwf(\mathbf{W})$, by part (1). By definition of the family $wfwfwf(\mathbf{W})$, X meets each set in $wfwf(\mathbf{W})$ at one and only one point. Thus $X \cap W$ is a singleton set.

We have just shown that each set X in $wfwfwf(\mathbf{W})$ meets each set W in $\mathbf{W}$ at one and only one point. Therefore, X is in $wf(\mathbf{W})$.

Consider the collection $\mathbf{W}$ to be an infinitary statement in disjunctive normal form. Any choice set of $\mathbf{W}$ can be used to demonstrate the falsehood of $\mathbf{W}$. But if we narrow our view so as to admit only wefts of $\mathbf{W}$ as witnesses to the falsehood of $\mathbf{W}$, then Lemma 4 gives us the following axioms:

$$\mathbf{W} \rightarrow \sim\sim\mathbf{W}$$
$$\sim\mathbf{W} \equiv \sim\sim\sim\mathbf{W}.$$

These are the negation axioms for intuitionist logic.

We are now ready to show how wefts are effected by the decomposition operators. In essence, the next lemma is DeMorgan's law recast in the terminology of weaves.

Lemma 5 *Let G be any set.*

(a) *The set* G *is a weft of* $\nabla_{i\epsilon I}\mathbf{W}_i$ *iff* G *is a subset of Points*$(\nabla_{i\epsilon I}\mathbf{W}_i)$, *and, for each* i_0 *in* I, $G \cap$ *Points*$(\mathbf{W}_{i_0})$ *is a weft of* $\mathbf{W}_{i_0}$.
(b) *The set* G *is a weft of* $\Delta_{i\epsilon I}\mathbf{W}_i$ *iff* G *is a weft of* $\mathbf{W}_{i_0}$ *for some* i_0 *in* I. (*If such an* i_0 *exists, it is unique.*)

Proof: (a) $\Longrightarrow$: Let G be a weft of $\nabla_{i\epsilon I}\mathbf{W}_i$. By the definition of weft, G is a subset of Points$(\nabla_{i\epsilon I}\mathbf{W}_i)$.

Choose an element, i_0, of I. The set G meets every set in $\nabla_{i\epsilon I}\mathbf{W}_i$ at one and only one point. Each set in $\mathbf{W}_{i_0}$ is a set in $\nabla_{i\epsilon I}\mathbf{W}_i$. Therefore G meets every set in $\mathbf{W}_{i_0}$ at one and only one point. Thus $G \cap$ Points$(\mathbf{W}_{i_0})$ is a weft of $\mathbf{W}_{i_0}$.

$\Longleftarrow$: Let $G \cap$ Points$(\mathbf{W}_{i_0})$ be a weft of $\mathbf{W}_{i_0}$ (for each i_0 in I). Clearly the union $\cup_{i\epsilon I}$ $(G \cap$ Points$(\mathbf{W}_i))$ is a choice set for $\nabla_{i\epsilon I}\mathbf{W}_i$. Since the disjunction is direct, this union is in fact a weft of $\nabla_{i\epsilon I}\mathbf{W}_i$. Since G is a subset of Points$(\nabla_{i\epsilon I}\mathbf{W}_i)$, this union is precisely the set G.

(b) $\Longrightarrow$: Let G be a weft of $\Delta_{i\epsilon I}\mathbf{W}_i$. Since Points$(\Delta_{i\epsilon I}\mathbf{W}_i) = \cup_{i\epsilon I}$ Points$(\mathbf{W}_i)$, there is an element i_0 in I such that $G \cap$ Points$(\mathbf{W}_{i_0}) \neq \phi$. We must show that this element i_0 is unique.

Assume, on the contrary, that G meets two disjoint sets, W_{i_0} in $\mathbf{W}_{i_0}$, and W_{i_1} in $\mathbf{W}_{i_1}$. By definition of conjunction, there is a set W in $\Delta_{i\epsilon I}\mathbf{W}_i$ such that $W = W_{i_0} \cup W_{i_1} \cup$ Other sets. So G meets W at more than one point, contradicting the fact that G is a weft of $\Delta_{i\epsilon I}\mathbf{W}_i$.

Now choose *any* set W_{i_0} in $\mathbf{W}_{i_0}$. Find a set W in $\Delta_{i\epsilon I}\mathbf{W}_i$ such that $W = W_{i_0} \cup$ Other sets. We know that G intersects W at one and only one point. By the uniqueness of i_0, this point must be an element of W_{i_0}.

We have thus shown that G intersects any set in $\mathbf{W}_{i_0}$ at one and only one point. Therefore G is a weft of $\mathbf{W}_{i_0}$.

$\Longleftarrow$: Let G be a weft of $\mathbf{W}_{i_0}$. Then $G \subset \text{Points}(\mathbf{W}_{i_0})$ so $G \cap \text{Points}(\mathbf{W}_{i_1}) = \phi$ for each $i_1 \neq i_0$.

Every set W in $\Delta_{i\epsilon I}\mathbf{W}_i$ is a union of some set W_{i_0}, from $\mathbf{W}_{i_0}$, and other sets. Clearly $G \cap W_{i_0}$ will be a singleton set, and G will meet none of the other sets of which W is composed. Thus $G \cap W$ is a singleton set.

Corollary 2 *The conjuncts (or disjuncts) of a warp are themselves warps.*

Proof: Let $\mathbf{W} = \mathbf{W}_1 \,\Delta\, \mathbf{W}_2$, and choose $p \in \text{Points}(\mathbf{W}_1)$. We must show that some weft of $\mathbf{W}_1$ contains p. There is a weft, G, of $\mathbf{W}$ that contains p. By Lemma 5, G is a weft of $\mathbf{W}_1$ or of $\mathbf{W}_2$. Clearly $p \in G \in wf(\mathbf{W}_1)$.

Corollary 3

(a) $wf(\nabla_{i\epsilon I}\mathbf{W}_i) = \Delta_{i\epsilon I}\, wf(\mathbf{W}_i)$
(b) $wf(\Delta_{i\epsilon I}\mathbf{W}_i) = \nabla_{i\epsilon I}\, wf(\mathbf{W}_i)$.

The next lemma states a fact which must be true in order for this structure to mimic the structure of statements in the propositional calculus. It states that no warp $\mathbf{W}$ can be both a direct disjunction and a direct conjunction of families of sets. In the statement of the lemma we assume that the various disjunctions and conjunctions are nonvacuous. For example, if $\mathbf{W} = \nabla_{i\epsilon I}\mathbf{W}_i$, we are assuming that I has more than one element, and that $\mathbf{W}_i$ is nonempty, for each i in I.

Lemma 6

(a) *Let* $\mathbf{W} = \nabla_{i\epsilon I}\mathbf{W}_i$. *Then there is no collection* $\{\mathbf{W}_j^*\}_{j\epsilon J}$ *such that* $\mathbf{W} = \Delta_{j\epsilon J}\mathbf{W}_j^*$
(b) *Let* $\mathbf{W} = \Delta_{j\epsilon J}\mathbf{W}_j^*$. *Then there is no collection* $\{\mathbf{W}_i\}_{i\epsilon I}$ *such that* $\mathbf{W} = \nabla_{i\epsilon I}\mathbf{W}_i$.

Proof: Assume that $\mathbf{W} = \nabla_{i\epsilon I}\mathbf{W}_i$ and $\mathbf{W} = \Delta_{j\epsilon J}\mathbf{W}_j^*$, and that both of these expressions are nonvacuous.

Choose an i in I, and let $\mathbf{W}_i$ be $\mathbf{W}_1$. Then $\mathbf{W}$ is of the form $\mathbf{W}_1 \,\nabla\, \mathbf{W}_2$. Similarly, choose j in J and let $\mathbf{W}_j^*$ be $\mathbf{W}_3^*$. Then $\mathbf{W}$ is of the form $\mathbf{W}_3^* \,\Delta\, \mathbf{W}_4^*$.

Let W_1 be a set in $\mathbf{W}_1$. Then W_1 is in $\mathbf{W}$, so W_1 is of the form $W_3^* \cup W_4^*$, where $W_3^* \in \mathbf{W}_3^*$ and $W_4^* \in \mathbf{W}_4^*$. Thus $W_1 \cap W_3^*$ is nonempty. Let $p \in W_1 \cap W_3^*$.

Similarly, let $q \in W_2 \cap W_4^*$ for two sets W_2 in $\mathbf{W}_2$ and W_4^* in $\mathbf{W}_4^*$.

Since $p \in \text{Points}(\mathbf{W}_1)$ and $q \in \text{Points}(\mathbf{W}_2)$, there is no set W in $\mathbf{W}$ such that $p,q \in W$. But p and q are both in $W_3^* \cup W_4^*$, which is a set in $\mathbf{W}$. This is a contradiction.

Before we state and prove the decomposition theorem, we define the degenerate case.

Definition 6 The warp $\mathbf{W}$ is called an Atom if it consists of one and only one set, and that set is a singleton set.

Example 5: Let $\mathbf{W} = \{\{a\}\}$. Then $\mathbf{W}' = wf(\mathbf{W})$ must be $\{\{a\}\}$. Clearly, neither $\mathbf{W}$ nor $\mathbf{W}'$ can be decomposed into smaller units.

The object now is to prove the following theorem.

Theorem *Let* $\mathbf{W}$ *be a normal, nonatomic warp. Then there is an index set,* I, *with cardinality greater than 1, and warp* $\mathbf{W}_i$ *for each* i *in* I, *such that either* $\mathbf{W} = \nabla_{i\in I}\mathbf{W}_i$ *or* $\mathbf{W} = \Delta_{i\in I}\mathbf{W}_i$.

Outline and motivation for the proof: Every warp $\mathbf{W}$ is naturally a disjunction—it is a disjunction of its sets:

$$\mathbf{W} = W_1 \vee W_2 \vee W_3 \vee W_4 \vee W_5 \vee W_6$$

where $W_1, \ldots, W_6$ are the sets in $\mathbf{W}$. The problem is that this disjunction may not be direct. For instance, the point p may appear in both W_1 and W_2. In this case we begin by "factoring out" p to get

$$p \,\Delta\, (W_1^* \vee W_2^*) \vee W_3 \vee \ldots \vee W_6.$$

In fact, we can find a weft $\{p,q,r\}$ and factor it out, obtaining

$$p \,\Delta\, (W_1^* \vee W_2^*) \vee q \,\Delta\, (W_3^* \vee W_4^*) \vee r \,\Delta\, (W_5^* \vee W_6^*).$$

Now we could proceed recursively if the outermost disjunctions were all direct. Unfortunately this is not always the case. The families $W_1^* \vee W_2^*$ and $W_3^* \vee W_4^*$ may have points in common. We must show that, in such a case, the points that these two families have in common can be factored out, and they can be factored out of both families in the same way.

$$[p \,\Delta\, (W_1^{**} \vee W_2^{**}) \,\nabla\, q \,\Delta\, (W_3^{**} \vee W_4^{**})] \,\Delta\, \mathbf{W}^* \vee r \,\Delta\, (W_5^* \vee W_6^*)$$

Still the outermost disjunction may not be direct. We need an extra lemma to show that any set of points which $W_1^{**} \vee W_2^{**}$ and $W_5^* \vee W_6^*$ have in common can be factored out (Lemma 11).

Finally we must show that this factoring process stops after finitely many steps.

Now we proceed with the proof of the theorem.

Lemma 7 *Let* S *and* T *be nonempty subsets of* $Points(\mathbf{W})$ *satisfying* $S \cap T = \phi$. *Assume further that each weft,* G, *of* $\mathbf{W}$ *is either a subset of* S *or a subset of* T. *Then* $wfwf(\mathbf{W}) = \mathbf{W}_S \,\Delta\, \mathbf{W}_T$, *where* $S = Points(\mathbf{W}_S)$ *and* $T = Points(\mathbf{W}_T)$.

Proof: Let $\mathbf{W}'_S$ be the family of all wefts of $\mathbf{W}$ that are subsets of S. Similarly define $\mathbf{W}'_T$. Then $wf(\mathbf{W}) = \mathbf{W}'_S \,\nabla\, \mathbf{W}'_T$. By Corollary 3, $wfwf(\mathbf{W}) = wf(\mathbf{W}'_S) \,\Delta\, wf(\mathbf{W}'_T)$. Let $\mathbf{W}_S$ be $wf(\mathbf{W}'_S)$ and $\mathbf{W}_T$ be $wf(\mathbf{W}'_T)$.

Corollary 4 *If, in addition to the hypotheses of Lemma 7, we have that* $\mathbf{W}$ *is normal, then* $\mathbf{W} = \mathbf{W}_S \,\Delta\, \mathbf{W}_T$.

Proof: If $\mathbf{W}$ is normal, then by Lemma 2 $\mathbf{W} = wfwf(\mathbf{W})$.

For Lemmas 8 through 10 we assume that $\mathbf{W}$ is normal and is of the form $\vee_{i\in I}(a_i \,\Delta\, \mathbf{W}_i)$, where $\{a_i | i \in I\}$ is a weft of $\mathbf{W}$ and, for each $i \neq j$, $a_i \neq a_j$.

Lemma 8

(1) *For any* $J \subset I$, *the family* $\vee_{i\epsilon J}(a_i \triangle \mathbf{W}_i)$ *is normal.*
(2) *For any* $i \in I$, *the family* $a_i \triangle \mathbf{W}_i$ *is normal.*
(3) *For any* $i \in I$, *the family* $\mathbf{W}_i$ *is normal.*

Proof: (1) Let C be a choice set for $\vee_{i\epsilon J}(a_i \triangle \mathbf{W}_i)$. Let X be $\{a_i | i \notin J\}$. Consider the set $C' = C \cup X$. This set C' is a choice set for $\mathbf{W}$. Since $\mathbf{W}$ is normal, there is a set $G' \subset C'$ such that G' is a weft of $\mathbf{W}$. The set $G' - X$ is a subset of C. It is also a choice set for $\vee_{i\epsilon J}(a_i \triangle \mathbf{W}_i)$, since X and Points$(\vee_{i\epsilon J} a_i \triangle \mathbf{W}_i)$ have no elements in common.

Let W be a set in $\vee_{i\epsilon J}(a_i \triangle \mathbf{W}_i)$. If we view W as a set in $\mathbf{W}$, we see that $G' \cap W$ has at most one element. Thus $(G' - X) \cap W$ has at most one element. Therefore $G' - X$ is a weft of $\vee_{i\epsilon J}(a_i \triangle \mathbf{W}_i)$.

(2) This is a corollary of (1).

(3) Let C be a choice set for $\mathbf{W}_i$. Then C is a choice set for $a_i \triangle \mathbf{W}_i$. So by (2), there is a set $G \subset C$ such that G is a weft of $a_i \triangle \mathbf{W}_i$. But then a_i is not in G. Thus, by Lemma 5, G is a weft of $\mathbf{W}_i$.

Lemma 9 *Let Points*$(\mathbf{W}_1) \cap$ *Points*$(\mathbf{W}_2) = C \neq \phi$. *If* G *is a weft of* $\mathbf{W}_1$ *and* $G \cap C \neq \phi$, *then* $G \cap C$ *is a weft of* $\mathbf{W}_2$.

Proof: Consider the set $G \cup \{a_2\}$. This is a choice set for $a_1 \triangle \mathbf{W}_1 \vee a_2 \triangle \mathbf{W}_2$, which is normal. So there is a weft, G^*, of $a_1 \triangle \mathbf{W}_1 \vee a_2 \triangle \mathbf{W}_2$ satisfying $G^* \subset G \cup \{a_2\}$. Since a_2 is not an element of Points$(a_1 \triangle \mathbf{W}_1)$, we must have that $G^* - \{a_2\}$ is a weft of $a_1 \triangle \mathbf{W}_1$. Thus $G^* - \{a_2\} = G$ (by Lemma 1).

We claim that in fact $G^* = G$. If not, then $G^* = G \cup \{a_2\}$. But then we have a weft, G^*, for $a_1 \triangle \mathbf{W}_1 \vee a_2 \triangle \mathbf{W}_2$ that contains both a_2 and the elements of $G \cap C$. Let $c \in G \cap C$. Then c is in some set W_2 in $\mathbf{W}_2$. So both a_2 and c are in some set W_2^* in $a_2 \triangle \mathbf{W}_2$. Thus $G^* \cap W_2^*$ is not a singleton set. Therefore $G^* = G$.

This means that G is a weft of $a_1 \triangle \mathbf{W}_1 \vee a_2 \triangle \mathbf{W}_2$. So G meets every set in $a_2 \triangle \mathbf{W}_2$ at one and only one point. Since $a_2 \notin G$, G meets every set in $\mathbf{W}_2$ at one and only one point. So $G \cap$ Points$(\mathbf{W}_2)$ is a weft of $\mathbf{W}_2$. But $G \cap$ Points$(\mathbf{W}_2) = G \cap C$. Therefore $G \cap C$ is a weft of $\mathbf{W}_2$.

Lemma 10 *Let Points*$(\mathbf{W}_1) \cap$ *Points*$(\mathbf{W}_2) = C \neq \phi$. *Let* G *be a weft of* $\mathbf{W}_1$. *Then either* $G \subset C$ *or* $G \subset$ *Points*$(\mathbf{W}_1) - C$.

Proof: Assume that $G \cap C \neq \phi$. By the previous lemma, $G \cap C$ is a weft of $\mathbf{W}_2$. By applying the lemma to the sets C and $G \cap C$ (as a weft of $\mathbf{W}_2$) we get that $(G \cap C) \cap C$ is a weft of $\mathbf{W}_1$. Thus both $G \cap C$ and G are wefts of $\mathbf{W}_1$. By Lemma 1, $G \cap C = G$. So $G \subset C$.

Corollary 5 *Let Points*$(\mathbf{W}_1) \cap$ *Points*$(\mathbf{W}_2) = C \neq \phi$. *Then* $\mathbf{W}_1 = \mathbf{W}_1^* \triangle \mathbf{W}_C$ *and* $\mathbf{W}_2 = \mathbf{W}_2^* \triangle \mathbf{W}_C$, *where* $C =$ *Points*$(\mathbf{W}_C)$.

Proof: This follows from Corollary 4 and Lemma 10.

Lemma 11 *Let* $\mathbf{W}$ *be a normal warp of the form* $[(\mathbf{W}_1 \vee \mathbf{W}_2) \triangle \mathbf{W}_3] \vee \mathbf{W}_4$, *where* $\mathbf{W}_3$ *is not a conjunct of* $\mathbf{W}_4$. *Let Points*$(\mathbf{W}_1) \cap$ *Points*$(\mathbf{W}_4) = C \neq \phi$. *Then*

$C \subset Points(\mathbf{W}_2)$. Also $\mathbf{W}_1 \vee \mathbf{W}_2 = \mathbf{W}^* \vartriangle \mathbf{W}_C$, and $\mathbf{W}_4 = \mathbf{W}_4^* \vartriangle \mathbf{W}_C$, where $C = Points(\mathbf{W}_C)$.

Proof: We will show that $C \subset \text{Points}(\mathbf{W}_2)$. The rest of the proof is similar to the proofs of Lemmas 9 and 10, and Corollary 5.

The family $\mathbf{W}$ can be written in the form

$$(\mathbf{W}_1 \vartriangle \mathbf{W}_3) \vee (\mathbf{W}_2 \vartriangle \mathbf{W}_3) \vee \mathbf{W}_4.$$

Let G_4 be a weft of $\mathbf{W}_4$ satisfying $G_4 \cap C \neq \phi$. Then, as we have shown in the proof of Lemma 9, $G_4 \cap C$ is a weft of $\mathbf{W}_1$.

Let G_3 be a weft of $\mathbf{W}_3$. The set $G_3 \cup G_4$ is a choice set for $\mathbf{W}$. Since $\mathbf{W}_1 \vartriangle \mathbf{W}_3$ is part of our expression for $\mathbf{W}$, we must eliminate (from $G_3 \cup G_4$) either all points from G_3 or all points from G_4 in order to obtain a weft of $\mathbf{W}$.

We claim that we cannot always eliminate G_4. If we could, then every weft of $\mathbf{W}_3$ would be a weft of $\mathbf{W}_4$. Then we could apply Corollary 4 letting $S = \text{Points}(\mathbf{W}_3)$ and $T = \text{Points}(\mathbf{W}_4) - \text{Points}(\mathbf{W}_3)$. This would make $\mathbf{W}_3$ a conjunct of $\mathbf{W}_4$, which is a contradiction.

Thus there is a set G_3^* which is a weft of $\mathbf{W}_3$ and is not a weft of $\mathbf{W}$. We claim that *any* weft G_4^* of $\mathbf{W}_4$ that meets C is a weft of $\mathbf{W}$. This is because $G_3^* \cup G_4^*$ is a choice set for $\mathbf{W}$. In order to get a weft we have to eliminate either G_3^* or G_4^*. But G_3^* is not a weft of $\mathbf{W}$.

Our set G_4^*, being a weft of $\mathbf{W}$, must therefore be a weft of $\mathbf{W}_2 \vartriangle \mathbf{W}_3$. But $C \cap \text{Points}(\mathbf{W}_3) = \phi$. Therefore G_4^* is a weft of $\mathbf{W}_2$. Note that we have shown this for *any* such set G_4^*. Thus $C \subset \text{Points}(\mathbf{W}_2)$.

Lemma 12 *Let $\mathbf{W}$ be a warp of the form*

$$\ldots((((\mathbf{W}_0 \vartriangle \mathbf{W}_0^*) \triangledown \mathbf{W}_1) \vartriangle \mathbf{W}_1^*) \triangledown \mathbf{W}_2) \vartriangle \ldots .$$

Then $\mathbf{W}$ is not normal.

Proof: There is an ordinal Γ such that $\mathbf{W}$ is

$$\vee_{i \in I}[\mathbf{W}_i \vartriangle (\vartriangle_{i \leqslant j < \Gamma} \mathbf{W}_j^*)].$$

For $i = 1,2, \ldots$ let C_i be a weft of $\mathbf{W}_i^*$, and let C be $\cup_{0 \leqslant i < \Gamma} C_i$. The set C is a choice set for $\mathbf{W}$.

Assume the existence of a set $G \subset C$ such that G is a weft of $\mathbf{W}$. We claim that there is a unique i_0 satisfying $G \cap C_{i_0} \neq \phi$. (Otherwise we would have $c_{i_0} \in C_{i_0} \cap G$ and $c_{i_1} \in C_{i_1} \cap G$. Choose $W_{i_0}^*$ and $W_{i_1}^*$ such that $c_{i_0} \in W_{i_0}^* \in \mathbf{W}_{i_0}^*$ and $c_{i_1} \in W_{i_1}^* \in \mathbf{W}_{i_1}^*$. There is a set W^* in $\vartriangle_{0 \leqslant i < \Gamma} \mathbf{W}_i^*$ containing $W_{i_0}^*$ and $W_{i_1}^*$. Then c_{i_0} and c_{i_1} are both in G *and* in W^*, contradicting the fact that G is a weft of $\mathbf{W}$.)

Thus, by Lemma 1, $G = C_{i_0}$. But C_{i_0} is a subset of $\text{Points}(\mathbf{W}_{i_0}^*)$, which does not meet $\text{Points}(\mathbf{W}_i \vartriangle \vartriangle_{i \leqslant j < \Gamma} \mathbf{W}_j^*)$ for any $i > i_0$. This contradicts the fact that G is a weft of $\mathbf{W}$.

Therefore, $\mathbf{W}$ is not normal.

Proof of the Decomposition Theorem: The following procedure will decompose any nonatomic normal warp into the nontrivial direct conjunction (or disjunction) of two normal warps:

Step I. We begin with $\mathbf{W}$ in the form $\bigvee_{i\in I} W_i$.

Step II. Choose a weft $\{a_0, a_1, \ldots\}$ of $\mathbf{W}$, and re-order the W_i sets so that

$$\begin{array}{l} a_0 \in W_{0_0}, W_{0_1}, W_{0_2}, \ldots; \\ a_1 \in W_{1_0}, W_{1_1}, W_{1_2}, \ldots; \\ \quad\cdot \\ \quad\cdot \\ \quad\cdot \\ \quad \text{etc.} \end{array}$$

Thus $\mathbf{W}$ is in the form $\bigvee_{i\in I}\bigvee_{j\in J_i} W_{i_j}$ (where I is an index set, and J_i is an index set for each i in I). If we let $\mathbf{W}_i = \bigvee_{j\in J_i}(W_{i_j} - \{a_i\})$ then $\mathbf{W}$ is in the form

$$(*) \qquad \bigvee_{i\in I}(a_i \,\Delta\, \mathbf{W}_i).$$

Step III. If the Points sets of the $\mathbf{W}_i$ families are pairwise disjoint, we are done. If not, assume without loss of generality that $\text{Points}(\mathbf{W}_0) \cap \text{Points}(\mathbf{W}_1) = C \neq \phi$. Let α (some ordinal) be the number of families $\mathbf{W}_i$ satisfying $C \subset \text{Points}(\mathbf{W}_i)$. Re-order the disjuncts in (*) so that $C \subset \text{Points}(\mathbf{W}_i)$ for all $i < \alpha$. Then, by Corollary 5, we have $\mathbf{W}_i = \mathbf{W}_i^* \,\Delta\, \mathbf{W}_C$ for each $i < \alpha$ (where $C = \text{Points}(\mathbf{W}_C)$). Thus $\mathbf{W}$ is

$$\bigvee_{i<\alpha}[a_i \,\Delta\, \mathbf{W}_i^* \,\Delta\, \mathbf{W}_C] \vee \bigvee_{i\geq\alpha}[a_i \,\Delta\, \mathbf{W}_i].$$

We can rewrite this as

$$(**) \qquad \bigvee_{i<\alpha}[a_i \,\Delta\, \mathbf{W}_i^*] \,\Delta\, \mathbf{W}_C \vee \bigvee_{i\geq\alpha}[a_i \,\Delta\, \mathbf{W}_i].$$

Step IV. Case 1. For each $i \geqslant \alpha$, $\text{Points}(\mathbf{W}_i) \cap \text{Points}(\mathbf{W}_C)$ is empty. Then the disjunction in (**) to the right of $\mathbf{W}_C$ is direct by Lemma 11. Go back to the beginning of Step III, this time working with all $\mathbf{W}_i$ for $i \geqslant \alpha$.

Case 2. Without loss of generality, assume that $\text{Points}(\mathbf{W}_\alpha) \cap C = K \neq \phi$. Re-order and renumber the disjuncts in (**) so that $K \subset \mathbf{W}_i$ for each i satisfying $\alpha \leqslant i < \rho$. Then by Corollary 5 we have $\mathbf{W}_C = \mathbf{W}_C^* \,\Delta\, \mathbf{W}_K$, and, for each i between α and ρ, $\mathbf{W}_i = \mathbf{W}_i^* \,\Delta\, \mathbf{W}_K$, where $K = \text{Points}(\mathbf{W}_K)$. This gives us an expression of the form

$$[\ldots \Delta\, \mathbf{W}_C^* \vee \bigvee_{\alpha\leqslant i<\rho}(a_i \,\Delta\, \mathbf{W}_i^*)] \,\Delta\, \mathbf{W}_K \vee a_\rho \,\Delta\, \mathbf{W}_\rho \vee \ldots.$$

Now we repeat Step IV for the collection of $\mathbf{W}_i$ with $i \geqslant \rho$.

By Lemma 12, the repetition of Step IV, Case 2, will not cause an infinite loop. Such a loop would lead to an expression of the form

$$\ldots ((((\mathbf{W}_0 \,\Delta\, \mathbf{W}_1) \,\nabla\, \mathbf{W}_2) \,\Delta\, \mathbf{W}_3) \,\nabla\, \mathbf{W}_4) \,\Delta \ldots$$

This expression would not be a valid decomposition of $\mathbf{W}$, since it has no outermost connective.

This completes the proof of the theorem.

The converse of the Decomposition Theorem is clearly false. We now construct a suitable counterexample.

Form the binary tree of finite sequences of 0's and 1's in the usual way:

- The empty sequence is the root node, and

- For any finite sequence x, the children of x are $x /\!/ 0$ and $x /\!/ 1$, where $/\!/$ is the concatenation operator.

Every full path on the tree represents an ω-length sequence of 0's and 1's. Let X be a set of ω-length sequences. Define an infinite game on the tree by starting at the root node, and having Players I and II alternately move from the current node to one of its children. Player I wins iff the resulting ω-length sequence is in X.

Every strategy for Player I gives us a set of ω-length sequences. Let W be a set in $\mathbf{W}$ iff W is the set of ω-length sequences associated with some strategy for Player I. Similarly $\mathbf{W}'$ is the collection of sets associated with strategies for Player II. In a play of the game, Player I chooses a I-strategy, Player II chooses a II-strategy, and the result is a single sequence of length ω. Thus the pair $\langle \mathbf{W}, \mathbf{W}' \rangle$ is a weave.

Again let x be a finite sequence of 0's and 1's. We define $\mathbf{W}_x$ recursively:

- If x is the empty sequence, let $\mathbf{W}_x = \mathbf{W}$
- If $x = y /\!/ 0$ or $y /\!/ 1$, and y is of even length, then $\mathbf{W}_x$ is the collection of sets W in $\mathbf{W}_y$ such that each sequence in W begins with x
- If $x = y /\!/ 0$ or $y /\!/ 1$, and y is of odd length, then $\mathbf{W}_x$ is the collection of all *maximal subsets* W of sets in $\mathbf{W}_y$ such that each sequence in W begins with x.

Let y be a finite sequence of even length. Then $\mathbf{W}_y = \mathbf{W}_{y/\!/0} \nabla \mathbf{W}_{y/\!/1}$. If y is of odd length, then $\mathbf{W}_y = \mathbf{W}_{y/\!/0} \,\Delta\, \mathbf{W}_{y/\!/1}$. Thus $\mathbf{W}$ is a decomposable warp.

But $\mathbf{W}$ is not normal unless we accept the Axiom of Determinateness, which states that in any infinite game with any set X (of ω-length sequences taken to be winning for Player I) either Player I or Player II has a sure-fire winning strategy. The Axiom of Determinateness is known to contradict the Axiom of Choice [3]. Thus we can safely say that the weave we have constructed is decomposable, but not normal.

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