Notre Dame Journal of Formal Logic
Volume 24, Number 4, October 1983

Manifolds Allowing RET Arithmetic

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In cardinal arithmetic, two cardinal numbers are added by using the union of disjoint sets representing the two numbers, the disjointness ensuring that addition is well-defined. In the arithmetic of recursive equivalence types (RETs) on the natural numbers, addition is handled similarly. But for RETs, the representative sets need to be chosen disjoint in a recursive sense; such choice is easily accomplished. In [1], the theory of RETs was generalized to manifolds, and in that setting matters are more complex. To permit addition of RETs, some fairly strong conditions were imposed on the manifolds involved. This paper will show the necessity for imposing such conditions.

To keep the paper self-contained, this paragraph briefly reviews the relevant terminology from [1]. A mapping α from $\mathbf{N}$, the set of natural numbers, onto a set B is called an *enumeration* of B. For some index set P, let $A = \bigcup_{p \in P} A_p$ and $\mathcal{A} = \{\alpha_p\}_{p \in P}$, where A_p is enumerated by α_p for each p. Further, assume that for each $p,q \in P$, $\alpha_q^{-1}(A_p)$ is r.e. and that for each $p,q \in P$ there exists a partial recursive $f_{p,q}:\alpha_q^{-1}(A_p) \to \alpha_p^{-1}(A_q)$ with $\alpha_q = \alpha_p \circ f_{p,q}$. Under these conditions, we say that $\langle A, \mathcal{A}\rangle$ is a *recursively enumerable manifold* (REM). $\mathcal{A}$ is the *atlas* of the manifold. If each $\alpha_q^{-1}(A_p)$ is recursive, we call $\langle A, \mathcal{A}\rangle$ a *recursive manifold* (RM). Any REM (respectively, RM) with every α_p an *indexing*—that is, an injective map—is said to be an IREM (respectively, IRM). If each A_p intersects at most finitely many other A_q's, $\langle A, \mathcal{A}\rangle$ is *finitary*. $S \subseteq A$ is *$\mathcal{A}$-r.e.* iff, for all p, $\alpha_p^{-1}(S)$ is r.e. Let $\langle B, \mathcal{B}\rangle = \langle B, \{\beta_q\}_{q \in Q}\rangle$ be another manifold. A function f from A to B is *$\mathcal{A}$-$\mathcal{B}$-p.r.* iff: (1) its domain S is $\mathcal{A}$-r.e. and (2) for every $p \in P$, $q \in Q$, there exists p.r. $f_{p,q}:\alpha_p^{-1}(S \cap f^{-1}(B_q)) \to \mathbf{N}$ with $f \circ \alpha_p = \beta_q \circ f_{p,q}$. We also define f to be *$\mathcal{A}$-$\mathcal{B}$-compact* iff each $f(A_p)$ is contained in a finite union of B_q's. If f is $\mathcal{A}$-$\mathcal{B}$-p.r. and 1-1 with f $\mathcal{A}$-$\mathcal{B}$-compact and f^{-1} $\mathcal{B}$-$\mathcal{A}$-compact, f is an *embedding*. (As a very simple example, let $I(n) = n$ for all $n \in N$. Then $\langle N, \{I\}\rangle$ is a finitary IRM, the $\{I\}$-r.e. sets are

Received September 3, 1982

just the r.e. sets, and the $\{I\}$-$\{I\}$-embeddings are just the 1-1 p.r. functions on $\mathbf{N}$.) For $S,T \subseteq A$, $S \underset{f}{\cong} T$ means that f is an $\mathcal{A}$-$\mathcal{A}$-embedding with $f(S) = T$. $[S]$, the RET of S, is defined as $\{T \mid \exists f \text{ with } S \underset{f}{\cong} T\}$.

To have well-defined addition of RETs on a manifold $\langle A, \mathcal{A} \rangle$, any two RETs must have separable (i.e., contained in disjoint $\mathcal{A}$-r.e. supersets) representatives. In particular, to add $[A] + [A]$, there must exist disjoint B and C, where $A \underset{f}{\cong} B$ and $A \underset{g}{\cong} C$ for suitable $\mathcal{A}$-$\mathcal{A}$-embeddings f,g. Thus there exist embeddings $f,g:A \to A$ with disjoint ranges. Conversely, if such a pair of "separating embeddings" exists, any two RETs may be added, as $f(D)$ and $g(E)$ would be separable representatives of $[D]$ and $[E]$.

Thus the question of performing addition boils down to that of the existence of separating embeddings. In [1], the manifold was assumed to be a finitary IRM; this proved sufficient, but appeared somewhat strict. However, we now show that neither the recursivity nor the injectivity may be dropped, i.e., there will be a finitary IREM and a finitary RM, neither of which possesses separating embeddings.

Let $T \subseteq \mathbf{N}$ be a maximal set. That is, T is r.e., and its complement $\overline{T}$ is an infinite set such that for any r.e. W, either $W \cap \overline{T}$ or $\overline{W} \cap \overline{T}$ is finite. We set $A = (T \times \{1\}) \cup (\overline{T} \times \{0,2\})$ and $\mathcal{A} = \{\alpha_1, \alpha_2\}$, where

$$\alpha_1(n) = \begin{cases} (n,0) \text{ if } n \notin T \\ (n,1) \text{ if } n \in T \end{cases}$$
$$\alpha_2(n) = \begin{cases} (n,1) \text{ if } n \in T \\ (n,2) \text{ if } n \notin T. \end{cases}$$

$\langle A, \mathcal{A} \rangle$ is clearly a finitary IREM. We will reach a contradiction by assuming it allows separating embeddings.

Let $f,g:A \to A$ be embeddings with disjoint ranges. Since $f(A)$, $g(A)$ are disjoint $\mathcal{A}$-r.e. sets, $\alpha_2^{-1}(f(A))$ and $\alpha_2^{-1}(g(A))$ are disjoint r.e. sets. T was chosen to be maximal, hence the intersection of any r.e. set with $\overline{T}$ must be either finite or cofinite in $\overline{T}$. Not both of $\alpha_2^{-1}(f(A)) \cap \overline{T}$, $\alpha_2^{-1}(g(A)) \cap \overline{T}$ can be cofinite in $\overline{T}$ since they would then have nonempty intersection. Thus one, say $\alpha_2^{-1}(f(A)) \cap \overline{T}$, is finite. But this implies $f(A) \subseteq A_1 \cup \{a_0, a_1, \ldots, a_k\}$, where the $a_i \in A_2 - A_1$. Again, since $f(A)$ is $\mathcal{A}$-r.e., $\alpha_1^{-1}(f(A))$ is r.e.; hence there exists a 1-1 onto p.r. $h:\alpha_1^{-1}(f(A)) \to \mathbf{N} - \{0, 1, \ldots, k\}$. Defining h^* by

$$\begin{cases} h^*(x) = h(\alpha_1^{-1}(x)) & x \in f(A) \cap A_1 \\ h^*(a_i) = i & i = 0, 1, \ldots, k, \end{cases}$$

$h^* \circ f$ is an embedding from A onto $\langle \mathbf{N}, \{I\} \rangle$. Then α, the indexing of A defined to be $(h^* \circ f)^{-1}$, is an $\{I\}$-$\mathcal{A}$-embedding. Let β be the enumeration of A given by

$$\begin{cases} \beta(2n) = \alpha_1(n) \\ \beta(2n+1) = \alpha_2(n). \end{cases}$$

As an atlas, $\{\beta\}$ is matched with $\mathcal{A}$; that is, the atlases induce the same recursive notions.

I claim that, as enumerations, α and β are equivalent, in other words, that there exist recursive functions s and t such that $\alpha = \beta \circ s$ and $\beta = \alpha \circ t$. We may easily define t as $\alpha^{-1} \circ \beta$. The condition $\alpha = \beta \circ s$ is, of course, equivalent to the condition that $(\alpha^{-1} \circ \beta) \circ s$ be the identity. Since $\alpha^{-1} \circ \beta$ is a recursive map from $\mathbf{N}$ onto $\mathbf{N}$, setting $s(n)$ = the least m such that $(\alpha^{-1} \circ \beta)(m) = n$ will work.

Thus the enumeration β is equivalent to an indexing. This implies that $\sim$ is a recursive predicate, where $m \sim n \Leftrightarrow \beta(m) = \beta(n)$. Thus $\{(m,n)|\beta(m) = \beta(n)\}$ is recursive. Intersecting with $\{2m|m \in \mathbf{N}\} \times \{2n + 1|n \in \mathbf{N}\}$, we have the recursivity of

$$\{(2m, 2n + 1)|\beta(2m) = \beta(2n + 1)\} = \{(2n, 2n + 1)|n \in T\}.$$

But this implies that T would be recursive, in contradiction to T's maximality.

Thus we have found a finitary IREM which does not possess separating embeddings. Since $\{\beta\}$ is matched with $\mathcal{A}$, the finitary RM $\langle A,\{\beta\}\rangle$ likewise lacks separating embeddings.

REFERENCE

[1] Harkleroad, L., "Recursive equivalence types on recursive manifolds," *Notre Dame Journal of Formal Logic*, vol. 20 (1979), pp. 1-31.

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