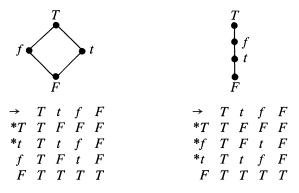
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A Note on R₊ Matrices

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Consider the following two DeMorgan monoids, in the sense of [1], for the system R of relevant implication, with lattice operations & and v defined according to the following Hasse diagrams, and \rightarrow tables defined by the following matrices (with designated elements starred).¹



Since $\sim a$ is definable in every case as $a \rightarrow f$ (and should be typographically obvious anyway), and other DeMorgan monoid operations are definable from the above (e.g., fusion (or cotenability) $a \circ b$ as $\sim (a \rightarrow \sim b)$), with t as monoid

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identity, we get a complete set of matrices for R in either case, or, saying the same thing from another viewpoint, we have defined in each case a DeMorgan monoid.

The matrix on the left I shall call the 4-diamond; that on the right, the 4-chain. (The 4-diamond is especially venerable; it is due to Church, who in [4] invented relevant implication.) What strikes the eye, however, is something else; namely, insofar as their \rightarrow tables are concerned, the 4-diamond and the 4-chain are the same. If we wrote down the tables for the other intensional operations, such as DeMorgan negation \sim , fusion \circ , fission +, and coimplication \leftrightarrow , these would also turn out to be the same in the two structures.

Clearly, there is something going on here of almost metaphysical import. For how can a chain be a diamond? (The converse question is easily answered, for, as is well known, a diamond can be a chain when one has presented it to the wrong lady.) But there are two answers to our question.

First, the 4-chain and the 4-diamond are certainly not *isomorphic* DeMorgan monoids. But, for the purposes of relevant logic, they are *to some* extent isomorphic—namely, they are isomorphic in what one might wish to identify as their intensional parts. Nor is either of these monoids even a homomorphic copy of the other, for lattice operations & and v, associated with the corresponding connectives of R, are not preserved. But the transformation h suggested by our notation is nonetheless an order homomorphism, preserving the partial order \leq in the usual sense.

Second, let us forget about DeMorgan monoids and simply think about the 4-chain and the 4-diamond as matrices, concentrating on the \rightarrow tables. From this viewpoint, we have passed from the 4-diamond to the 4-chain in a single move, simply taking f as a further "designated" value. My main purpose here will be to examine the consequences of such a move, with emphasis on the actual use of matrices and their associated algebras to give us a better picture of systems like R_{\rightarrow} .

Let us consider a practical problem: for example, finding a matrix M to refute a given nontheorem A of R_{\rightarrow} . We should like to accomplish this task as economically as possible. Let us understand economy in the sense that we should like to find a matrix which is as small as possible, and with as few undesignated values as possible, which will do the job. The two points are connected; the more values we designate, the more often we shall, in general, have designated both $a \rightarrow b$ and $b \rightarrow a$, whence, as a general principle for dealing with R_{\rightarrow} matrices of an appropriate sort, we may identify a and b, shrinking the matrix. Or we may simplify the order, as we did in passing from the 4-diamond to the 4-chain.

In fact, for R_{\rightarrow} , every nontheorem may be maximally refuted, in the sense that, where A is a nontheorem of R_{\rightarrow} , there is a finite matrix M, and an interpretation of R_{\rightarrow} in M, on which A takes a value u which is naturally understood as the greatest undesignated value in M. Moreover, M is the smallest matrix, satisfying the axioms and rules of R_{\rightarrow} , in which A is refutable. So, if we are looking for a matrix that will do in A, we might as well look for M. This property of maximal refutation continues to hold for the intensional fragment R_{\rightarrow} of R (which Dr. M. McRobbie has taught me to call R_i , as I shall henceforth), as well as for all extensions, in the same connectives, of these sys-

tems. It holds moreover for some well-known further systems, such as the intuitionist logic J and the classical logic K. But there are other well-known systems for which, in a suitable sense, the maximal refutation property holds; for example, the semirelevant extension RM of R. But the property cannot be asserted for the full system R, or for other relevant logics closely related to R. First, it is not yet known whether R has the finite model property, which we have built into our characterization of maximal refutability. Moreover, even if we were to change the characterization, or prove that R does have a finite model property, there are complications induced by the &I rule. But the intensional part of R is itself of interest, both because it is in R_i that Church's original insights into relevant implication take their clearest form, and also because it is their theory of implication which forms the most distinctive and novel aspect of the relevant logics. So we shall concentrate in this note on proving a maximal refutation lemma for R_{\rightarrow} , extending it as applicable to further systems.

I We begin with some familiar terminology. By a *matrix* for a formal language *L*, we mean a structure $M = \langle M, O, D \rangle$, where *M* is a nonempty set, *O* is a set of operations on *M* similar to the connectives of *L*, and *D* is a subset of (designated elements of) *M*. (We abuse language to the extent of often using '*M*' for both the structure and its base set.) Here, we always take \rightarrow as one of the connectives of *L*, as an implication connective appropriate to *L*; accordingly, each matrix *M* will also have a binary implication operation \rightarrow defined upon it. And implication, in turn, gives rise to a binary relation \leq , to be identified as *matrix entailment*, as follows, for all *a*, *b* in *M*.

 $D \leq a \leq b$ iff $a \rightarrow b \in D$.

Matrix entailment fathers in turn a binary relation \simeq of *matrix equivalence*, which is nothing but two-sided entailment.

 $D \simeq a \simeq b$ iff both $a \leq b$ and $b \leq a$.

A matrix is *reduced*, for our purposes, just in case no distinct elements of M are equivalent; i.e.,

(1) *M* is reduced iff, for all $a, b \in M$, $a \simeq b$ iff a = b.

We shall prefer, where possible, reduced matrices. And we shall prefer even more those matrices that respect modus ponens, which we shall call *closed*. In the \leq notation, with \Rightarrow as metalogical *if*,

(2) *M* is closed iff, for all $a, b \in M$, $a \le b \Rightarrow a \in D \Rightarrow b \in D$.

Thus the closed matrices are just those in which D is "closed up" under the matrix entailment \leq .

In general, let us take a subset $G \subseteq M$ to be a matrix filter iff G is closed up, and a matrix ideal iff G is closed down; i.e.,

(3) G is a matrix ideal iff, for all $a, b \in M$, $a \le b \Rightarrow b \in G \Rightarrow a \in G$,

with the dual condition, generalizing (2), for filters, of which D is required

to be one in closed matrices. Letting, now and henceforth, U be the set M-D of *undesignated* elements of M, trivially equivalent to (2) is the requirement that U be an ideal. Each element $x \in M$ determines a corresponding *principal ideal* and *principal filter*, which we shall symbolize respectively by $[\leq x]$ and $[x\leq]$, so that $y \in [\leq x]$ iff $y \leq x$, and $y \in [x\leq]$ iff $x \leq y$. Where $G \subseteq M$, we shall also say that a member x of G is *maximal* in G iff it precedes in G only its matrix equivalents; i.e., x is maximal in G iff, $x \in G$ and, for all $y \in G$, $x \leq y \Rightarrow x \approx y$. And a maximal element x in G is G-greatest iff, moreover, for all y in G, $y \leq x$. Evidently given G is an ideal, x is G-greatest iff $G = [\leq x]$. In a reduced matrix, there is clearly at most one G-greatest element for any $G \subseteq M$. On the obvious duals, we may speak also of minimal and least elements of G.

What we wish to prove is, I trust, now shaping up: namely, that in R_{\rightarrow} and similar systems, there is for each nontheorem A a suitable finite, closed, reduced matrix M, such that M is as small as possible and A is refutable at a U-greatest element of M. This will count as a "best refutation" of A; it is the one that we should look for if we were, say, programming a computer to refute nontheorems of R_{\rightarrow} (with, perhaps, another computer busy proving theorems). But let us return, briefly, to general considerations. We may take a theory T on a sentential language L simply as a subset of L; and we take T to be *closed* just in case it is closed under modus ponens, again disinteresting ourselves in any theories that are not so closed. Then T itself may be thought of as a closed matrix-namely, the so-called Lindenbaum matrix $\langle L, O, T \rangle$, where L is taken as the set of all formulas, O as the set of connectives and constants of the language, and T as the set of theorems. (Note that, in this special case, we reverse our usual ambiguity, referring to a Lindenbaum matrix by the name 'T' of its set of theorems-i.e., of its designated elements, not all its elements.) Note that only in exceptional circumstances will a Lindenbaum matrix be reduced; but there are instances; the substantial achievement of Martin's [6] was to show precisely that the Lindenbaum matrix of the system P-W (T_{\star} -W in [1]) is reduced, disposing affirmatively of a long nagging open problem ([1], p. 95).

We now revert to the familiar. Let $M = \langle M, O, D \rangle$ and $M' = \langle M', O', D' \rangle$ be matrices for some sentential language L. A homomorphism from M to M' is a homomorphism in the algebraic sense, preserving the operations in O. A matrix homomorphism h is a homomorphism that preserves also D, in the sense that $a \in D \Rightarrow ha \in D'$. Note that every matrix homomorphism must preserve matrix entailment also, since $a \leq b$ iff $a \Rightarrow b \in D$, whence $h(a \Rightarrow b) =$ $ha \Rightarrow hb \in D'$, so that $ha \leq hb$ in M' if $a \leq b$ in M.

Let T now be a theory, identified with its Lindenbaum matrix $\langle L, O, T \rangle$, and let M be a matrix for L. Then M is a T-matrix just in case every homomorphism from T to M is a matrix homomorphism, or, as it is more ordinarily put, all theorems of T are "true" (or "designated") in M on all interpretations. Since our concern is with closed theories and closed matrices, we may assume that the rule of modus ponens is built in on both sides.

I conclude this section with a slight digression. Relevant logics have been equipped, through the work of Dunn and others, with an "algebraic analysis", to which we referred at the outset. Since the relation between the "algebras" of relevant logics and their "matrices" is familiar to workers in the field, as part of the (conscious or unconscious) "folk wisdom" of the subject, it may be useful to spell out here exactly what that relation is. Briefly, the "algebras" in which relevant logicians have taken a special interest may themselves be identified as special kinds of matrices for relevant theories, tied rather closely to a specific choice of primitive vocabulary. What is special about these matrices is that they are taken as both closed and reduced (with, when & is present in the vocabulary, the usual further requirement that the adjunction rule &I also be respected, in the sense that D shall be closed under &). What is specific about them is that the sentential constant t and the fusion operation \circ play important roles "algebraically", so that, when we count the relevant algebras as matrices for logics, we must be sure to count t and \circ as among the (primitive or defined) logical particles of the relevant logic in question.

Let us give some examples. The Church monoids of [1] (p. 376) are just the closed reduced R'_{+} matrices (where R'_{+} is just the result of adding t and \circ , with their governing principles) to R_{+} . In the same sense, Dunn monoids are closed, reduced R^{+} matrices, and DeMorgan monoids are closed, reduced R matrices (with \circ and t in the vocabulary and with D respecting the &I rule in both cases). Providing relevant logics in general with Dunn-style algebras was a project long stymied, on the point that the crucial fusion connective \circ is indefinable in E and other weak sisters of R. But Routley and I in [7] found a way to overcome this problem, whence there are now E^{+} algebras that are the closed, reduced E^{+} matrices, T algebras that are the closed, reduced T matrices (where T is the "ticket entailment" system of [1]), and so forth in general, on close analogy to the corresponding algebraic situation for R and its fragments.

What is it that produces these close ties between relevant algebras, which one thinks of as having smooth properties and an interesting mathematical character, and the mere matrices for relevant theories, which one tends to think of as simply devices, of a more or less *ad hoc* character? The chief clues lie in Belnap's [3] and Dunn's [5], and they build on important connections between the matrix entailment relation \leq (definable on any matrix with a suitable \rightarrow , or, for that matter, an unsuitable \rightarrow) and the set *D* of matrix "truths". For, where *T* is a logic, the "matrix truths" according to a *T*-matrix *M* are of course parasitic upon the logical truths according to *T*. So, where *T* has a decent \rightarrow , matrix \leq begins to take on some familiar properties in any *T*-matrix. For example, if $p \rightarrow p$ is a theorem of *T*, *T*-matrix \leq must be reflexive. If $q \rightarrow r \rightarrow . p \rightarrow q \rightarrow . p \rightarrow r$ is a theorem of *T*, \leq must also be transitive, in any *closed T*-matrix. And so forth.

We have been viewing \leq as deriving its *T*-matrix properties from the \rightarrow of *T*, and, thereby, through the elements of *M* that are required to be designated. For, matrix wise, *D* determines \leq . But the converse point of view is also possible, and it is in fact characteristic of the "algebraic" approach, which starts out with a nice relation \leq on *M* (e.g., a partial order) and which, in effect, cooks up *D* from \leq . And it is here, in fact, that the (Ackermann) sentential constant *t* has been most helpful. For the properties of *t* (in *R*, say) suffice precisely to make it *D*-least in any closed *R* matrix, whence *D* becomes the principal filter [$t \leq$]. If, accordingly, we begin with *t* and \leq , we may simply

define D as $[t \le]$, which, in effect, is what Dunn does in [5]. And the crucial link between these two ways of thinking is secured by the principle $t \le a \rightarrow b$ iff $a \le b$, which, when one reflects on it, is exactly right, on the intuition that it is *true* that a implies b iff, in fact, a implies b. (This intuition is central, and, though worked out in the form now being considered in [5], is more appropriately to be credited to [3].)

There is one more vital point, which Dunn has stressed in conversation. It lies in the fact that relevant algebras are to be taken as *reduced* matrices; i.e., if $a \simeq b$ in a matrix that counts also as an algebra, a and b shall be the same element. (Put otherwise, the matrix entailment \leq shall be antisymmetric, which, on the minimal conditions already noted that make it reflexive and transitive, assure that \leq will be a partial order in relevant algebras.) But reduction, unlike the other properties that decent logics require of their closed matrices, does not happen automatically. To be sure, we can cause it to happen, on the following familiar plan. If \leq is reflexive and transitive, \simeq is an equivalence relation, partitioning the members of M into distinct equivalence classes. By identifying the members of the classes, we may pass to a "quotient matrix" M/\simeq , which will be reduced.

Well, we may be able to play this trick. First, let us look at its logical significance, on the principle that, however dimly, the properties that our favored matrices come to enjoy will be parasitic on their parent logics. The logical principle underlying the reduction step just traced is "replacement of equivalents". And the idea, perhaps, is that if $A \rightarrow B$ and $B \rightarrow A$ are both *true*, relative to some interpretation, then, from the point of view of that interpretation, A and B are also *indistinguishable*; they express, it might be said, the *same* proposition, on the interpretation. And this thought does call for reduction, for if we think of matrices, or algebras, as offering propositional models of our logics (as [3] might be held to suggest), it is an offense against Ockham to have, in the model, distinct entities with the same propositional content.

Let us, however, return to the logical point, since the place of the replacement principle in relevant logics is itself subtle. Perhaps the most straightforward form of this principle is just

(4) $(A \leftrightarrow B) \rightarrow C(A) \leftrightarrow C(B),$

where C(B) results from C(A) by one or more replacements of occurrences of A with B. But (4) holds in R and other relevant logics only in *intensional contexts*—that is, where C(A) is built up from A and propositional variables using only intensional connectives, and not using, in particular, the truthfunctional connectives & and v. But weaker forms of (4) hold quite generally; for example,

(5) $(A \leftrightarrow B) \& t \rightarrow C(A) \leftrightarrow C(B),$

is a theorem scheme of R, licensing in particular the usual replacement *rules* for this system and its extensions.

It might be argued that, in view of (5), relevant logics have the "usual" replacement properties. But, strictly speaking, this is not correct, a matter affecting the principal result of this note. For what, mathematically, allows

us to "collapse" an abstract algebra "modulo" some equivalence \simeq ? Central is the requirement that the equivalence in question shall be a *congruence* with respect to all relevant operations: i.e., where \circ is any *n*-ary operation of the algebra, the requirement is

(6)
$$a_1 \simeq b_1 \& \ldots \& a_n \simeq b_n \Rightarrow \circ (a_1, \ldots, a_n) \simeq \circ (b_1, \ldots, b_n),$$

for all elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of the algebra in question.

Next let us note that, if (4) is a theorem scheme of a logic T, we may reasonably expect that the "congruence principle" (6) will hold for all closed T-matrices (after a little further manipulation relating conditionals and biconditionals, on plausible assumptions). But we cannot so easily come to this conclusion if replacement only holds in the form (5). For consider the particular case of a closed R-matrix in which $b \simeq c$, and let us ask ourselves, as an application of (6), whether $a \circ b \simeq a \circ c$. The answer is, "Yes". For, by definition, $b \to c$ and $c \to b$ are both designated. In $R, p \to q \to q \to p \to p \to p \leftrightarrow q$ is a theorem, whence $b \leftrightarrow c$ is also designated, by the $\rightarrow E$ principle (2) for closed matrices. But, since o is an *intensional* connective, the appropriate instance $p \leftrightarrow q \rightarrow r \circ p \leftrightarrow r \circ q$ of (4) is an *R*-theorem, whence, appealing again to (2), $a \circ b \leftrightarrow a \circ c$ must be designated also, after which, appealing still to nothing but R-theorems and (2), it readily follows that $a \circ b \simeq a \circ c$. Other cases are similar, for all intensional connectives of R (i.e., the connectives of R_i). So, \simeq is a congruence on any closed R_i -matrix. Accordingly, given a closed R_i -matrix, we may straightforwardly reduce it, identifying congruent elements and getting a quotient matrix $M/\simeq = \langle M/\simeq, O/\simeq, D/\simeq \rangle$ defining the operations in O/\simeq by "representatives", and noting in particular that no confusion resulted in the definition of D/\sim , since, again applying (2), if $a \simeq b$ in a closed matrix then $a \in D$ iff $b \in D$.

Let us now ask the same question where the matrix operation in question is truth-functional &, assuming $b \simeq c$ in a closed *R*-matrix, and wondering whether $a \& b \simeq a \& c$. The above argument breaks down on the point that the appropriate instance of (4), namely $p \leftrightarrow q \rightarrow . r \& p \leftrightarrow . r \& q$, cannot be an *R*-theorem, since & is truth-functional. And what this means, really, is that understanding the *closure* of an *R*-matrix with reference to modus ponens alone is insufficient to assure the congruence properties necessary for reduction.

There is a remedy for this breakdown of reduction, actually pursued in [5], and which readers of [1] might take to be the natural one. For R and other relevant logics are formulated not with one rule but with two, adding &I to the customary $\rightarrow E$ rule. And we have already seen that the "algebras" of relevant logics respect &I as much as they respect $\rightarrow E$; the appropriate condition to add on matrices is, of course, just

(7) $a \in D$ and $b \in D \Rightarrow a \& b \in D$.

Let us call a matrix which satisfies both (2) and (7) strongly closed, since such a matrix would ordinarily be considered a strong matrix for the relevant logics. Attending to strongly closed matrices overcomes our congruence problems, since it enables us to use (5) (or an approximation thereof, even in the case where t is not an explicit primitive) where (4) was used before, whence, e.g., the matrix \simeq is readily shown to be a congruence, with respect to all operations, whenever *M* is a strongly closed *R* matrix (or *R*+ matrix, *E* matrix, etc.). Under such circumstances we can pass to quotients; we may also express our previous observation more exactly as "The DeMorgan monoids are exactly the reduced, strongly closed *R*-matrices, where *R* is formulated with *t* explicit", an observation to be extended mutatis mutandis to other algebras and systems.

Note, incidentally, the great advantages that are conferred by sticking to reducible matrices, and then reducing them. In the first place, an unreduced reducible matrix (i.e., one on which \simeq is a nontrivial congruence) is in important ways superfluous; for any purpose for which we are likely to *want* a matrix—e.g., to reject nontheorems efficiently—is better served by a reduced matrix than by an unreduced one of which it is the quotient. For, as one sees immediately, any interpretation in the unreduced matrix is directly mirrored in its quotient, refuting and validating the same formulas. So if we had, for example, a computer program to churn out useful strongly closed *R*-matrices, we should want it to avoid the unreduced such matrices completely, lest the product be mainly a lot of waste paper. Moreover, a really important property of the matrices that have turned out to be "relevant algebras" is that they are so readily *visualizable*, via partial orders and Hasse diagrams (such as strew the pages of [1]), enabling one to take in at a glance the important structural features of a given model of a relevant logic.

However, all these advantages notwithstanding, there is something disturbing about the effects in R of the & I rule. It is, after all, a mere truth-functional rule, not supported by any more convincing thesis of the system R itself than $A \& B \rightarrow A \& B$. (See [2] for discussion of this point.) More to the immediate point, mixing the rules &I, $\rightarrow E$ in the deduction of theorems means that we lose some control of the deductive process; this is true even though, on a slight reformulation of R, all applications of the &I rule may precede any applications of $\rightarrow E$ (a point which I owe in part to Belnap). This complicates, for example, the decision problem. And even more to the immediate point, it apparently complicates also the problem of finding an efficient refutation of a given nontheorem, on which topic we are presently dwelling. So there may yet be a future, in metalogical investigations into relevant logics, for matrices that are closed but not strongly closed, despite the partial loss of congruence properties. For R_i and its fragments and subsystems, the question does not arise, since here we have only $\rightarrow E$ to worry about anyway, while, as noted, reduction via matrix equivalence is always possible.

2 Having disposed of the preliminaries and accompanying observations, we can now get on with the main business of this note: efficient refutation of nontheorems of R_{\rightarrow} and its kin. The method is that suggested by the passage from the 4-diamond to the 4-chain. The key is contained in the following lemma, which we state and prove first for R_{\rightarrow} , noting thereafter the R_{\rightarrow} properties on which it depends, that we may draw the appropriate generalizations.

Finite matrix shrinking lemma for R_{\rightarrow} Let $M = \langle M, O, D \rangle$ be a closed, finite R_{\rightarrow} matrix. Set U = M - D, and let $a \in U$. Then there is a closed, finite reduced

 R_{\rightarrow} matrix $M' = \langle M', O', D' \rangle$, with the following properties: (i) M' is a matrixhomomorphic image of M, in the sense that there exists a matrix homomorphism h from M onto M'; (ii) Setting U' = M' - D', ha $\in U'$; more than that, U' is the principal ideal [\leq ha] in M', so that, since M' is reduced, ha is the unique greatest undesignated element of M'; (iii) M' can be shrunk no further without designating a; i.e., where h' is a matrix homomorphism (but not an isomorphism) onto a finite closed R_{\rightarrow} matrix $M'' = \langle M'', O'', D'' \rangle$, then $h'(ha) \in D''$.

Proof: We shall construct M' from M. Our construction splits into two stages. First, we construct a finite sequence of matrices $M = M_0, \ldots, M_n$, keeping the same base set M and simply adding more designated elements, to form the increasing sequence $D = D_0 \subseteq \ldots \subseteq D_n$, taking care not to put in any D_i the element a which we wish to keep undesignated. When this stage is completed, we then simply reduce M_n modulo its matrix equivalence \approx , forming M'. We then must show that M' has the properties that we have claimed for it.

In the first place, it is trivial, for any *T*-matrix whatsoever, that it remains a *T*-matrix on designating more elements. For if all the theorems of *T* are confined on interpretation to some subset *D* of *M*, of course they are also confined to any superset *D'* of *D*. So our construction can only be interesting if it shows the proper respect for modus ponens by taking us from one *closed* matrix to another. And it is interesting that R_{\rightarrow} seems to have just about the right supply of theorems for the purpose, as we shall see.

Suppose then that we have arrived at stage *i* of the suggested construction, at which we have a closed R_{\rightarrow} matrix $M_i = \langle M, O, D_i \rangle$, where *M*, *O*, are as in our original matrix M_0 , our original $D \subseteq D_i$, while the particular element *a* that we are keeping undesignated fails to belong to D_i . The matrix equivalence \simeq is defined on M_i , partitioning *M* into equivalence classes. We keep track of these classes by using 'B' for the class which contains a particular element *b* of *M*, etc.; in particular, *A* is {*c*: *c* \simeq *a* in M_i }, for our permanent "bad guy" *a*.

It may be that a is already a greatest undesignated element of M_i ; in that case, go on to stage 2. Otherwise, I assert, there is some maximally undesignated element b of M, such that $b \notin A$. For, since M is finite, and since $U_i = M - D_i$ is nonempty (containing at least a), and since \leq is transitive (R, property) and reflexive (ditto), there exist, evidently, maximal elements of U_i . Indeed, for each c in U_i , there is some U_i -maximal b such that $c \leq b$. And if all the U_i -maximal b belong to A, a is already U_i -greatest.

So, on the assumption that we still have work to do, choose any b that is U_i -maximal but not equivalent to a, and form D_{i+1} by adding b and all its M_i -equivalents to D_i ; i.e., $D_{i+1} = D_i \cup B$. $M_{i+1} = \langle M, O, D_{i+1} \rangle$. On our trivial observation, M_{i+1} remains an R_{\rightarrow} matrix, and we need only show that it is closed; i.e., that D_{i+1} remains a filter in M_{i+1} . So let us assume that c and $c \rightarrow d$ both belong to D_{i+1} . We must show $d \in D_{i+1}$. There are four cases.

Case 1. $c \in D_i$ and $c \to d \in D_i$. In this case, since M_i is closed on assumption, $d \in D_i \subseteq D_{i+1}$.

Case 2. $c \approx b$ and $c \rightarrow d \in D_i$, where \approx is M_i equivalence. This implies $b \leq c$ and $c \leq d$ in M_i , whence, since \leq is transitive in the closed R_* matrix M_i , $b \leq d$

in M_i . But we chose b as U_i -maximal, whence either $d \in D_i$ or $d \simeq b$. In either case, $d \in D_{i+1}$.

Case 3. $c \in D_i$ and $b \simeq c \rightarrow d$ in M_i . We observe that, in virtue of the R_{\rightarrow} theorem $p \rightarrow p \rightarrow q \rightarrow q$, $c \rightarrow d \rightarrow d \in D_i$ (since D_i is a filter), whence, in M_i , we have both $b \leq c \rightarrow d$ and $c \rightarrow d \leq d$, whence, again by transitivity of $M_i \leq$ and U_i -maximality of $b, d \in D_{i+1}$ for the same reason as in the previous case.

Case 4. $c \approx b$ and $b \approx c \rightarrow d$. This is, apparently, the interesting case, but the most interesting thing about it is that it cannot arise, on account of the R_{\rightarrow} contraction principle. For, again by definition of \approx and transitivity of \leq , we have $c \leq c \rightarrow d$ in M_i ; i.e., $c \rightarrow . c \rightarrow d \in D_i$. We observe that, in virtue of the R_{\rightarrow} theorem $(p \rightarrow . p \rightarrow q) \rightarrow . p \rightarrow q$, and the fact that D_i is a filter, $c \rightarrow d \in D_i$; i.e., $c \leq d$ in M_i . But since $c \rightarrow d \in D_i$ and $c \rightarrow d \leq b$, the latter by the hypothesis of the case, $b \in D_i$, since D_i is still a filter. But we chose b as an undesignated element of M_i , whence, indeed, the case does not arise.

This exhausts the cases, and assures that we can continue enlarging our original D until we reach an $M_n = \langle M, O, D_n \rangle$ in which a is a greatest undesignated element. (Evidently we reach M_n in a finite number of steps, since M was finite to begin with, whence we shall eventually run out of further undesignated elements to add.) We now observe, for reasons noted in Section 1, that the \simeq of M_n is a congruence (with respect to the only operation of R_{\rightarrow} , namely \rightarrow), whence we may pass to the quotient matrix $M_n/\simeq = M' = \langle M', O', D' \rangle$, also for reasons noted. We must now verify the various statements of our lemma concerning M'.

First, let h be the natural homomorphism from M_n onto M'. Since M_n is closed, evidently M' is also closed; equally evidently, it is finite and reduced. Since, moreover, h is a matrix homomorphism from M_n to M', it is also a matrix homomorphism from our original matrix M to M' (since, if h preserves D_n , it preserves its subset D). This establishes (i) of the conclusion of our lemma.

To show (ii), we recall that a was U_n -greatest, whence ha will be U'-greatest (and hence unique, since M' is reduced).

Finally, we consider (iii). We need to show that any proper homomorphic image M'' of M', where M'' is a closed R_{\rightarrow} matrix and h' is a matrix homomorphism carrying M' onto M'', contains h'(ha) as a designated element. Let $D^* = h'^{-1}(D'')$; i.e., for all $b \in M'$, $b \in D^*$ iff $h'b \in D''$. Since matrix homomorphisms preserve designated elements, $D' \subseteq D^*$. First, suppose $D' = D^*$. Then, I assert, M'' is an isomorphic copy of M'. To do this, it suffices to show that if h'b = h'c in M'', b = c in M' (for then h' will be a bijection from M' to M'', preserving operations in O and, by supposition, preserving both D' and its complement). For suppose h'b = h'c. Then, in view of the R theorem $p \to p$, $h'b \to p'$ $h'c = h'(b \to c) \in D''$, whence $b \to c \in D^*$, which is D' by supposition. So $b \le c$ in M', and, by parity of reasoning, $c \leq b$; i.e., $b \simeq c$ in M' and, since M' is reduced, b = c. So, in this case, M'' is not a proper homomorphic image of M', but an isomorphic copy. For the other case, suppose that D' is a proper subset of D^* . First, we show that D^* is a filter in M'. For suppose $b \in D^*$ and $b \leq c$. Then $h'b \leq h'c$ in M'' (as we observed in Section 1), whence, since M'' is closed, $h'c \in D''$ and so $c \in D^*$. So D^* is a filter, which, as a proper superset of D', contains some undesignated element b of M'. But ha is the greatest undesignated element of M', whence $b \le ha$, whence, by filterhood of D^* , $ha \in D^*$, whence, by definition, $h'(ha) \in D''$, completing the proof of the finite matrix shrinking lemma for R_{\rightarrow} .

The idea of our shrinking lemma is that, given any refutation of a nontheorem A of R_{\star} at an undesignated element a of a closed R_{\star} matrix M, we may always shrink M so that a becomes the greatest undesignated element of the shrunken matrix, in effect. Note that every application of the shrinking lemma either increases the number of designated elements, or identifies previously distinct elements; the passage from the 4-diamond to the 4-chain, taking the former as the M of the lemma and the latter as M', with F the element to be kept undesignated, is an application of the first kind. Such applications always change the matrix partial order \leq , in the "flattening" way observed in the passage from the 4-diamond to the 4-chain. For observe that, in a closed R_{\star} matrix, $b \rightarrow b \leq b$ holds iff b is designated, whence increasing the number of designated elements also makes the relation \leq hold more often.

The proof of the shrinking lemma raises a number of related questions. Those to which I know the answers turn out negative, which may furnish a clue to the remainder. Our sample case, after all, terminated with a chain. Could it be, accordingly, that, given *any* closed R_{\perp} matrix, there is some way to keep adding new designated elements, reducing as we go, until the result is a chain? No, it could not be. For consider the matrix M_0 of [1], taken simply as an R_{\perp} matrix. M_0 has a least element -3, which will remain undesignated in all closed homomorphic images of M_0 but the trivial one. Moreover, we have for the distinct elements +1, +2 of M_0 , $+1 \rightarrow +2 = +2 \rightarrow +1 = -3$. So, shrink as we will to closed nontrivial R_{\perp} matrices, +1 and +2 will remain incomparable under matrix \leqslant , whence M_0 cannot be shrunk to a chain.

At this point, some empirical evidence intrudes. John Slaney has been writing programs that find R matrices by the bucketful. And he reports that, even among rather large matrices, the number of chains is somewhat staggering; e.g., about a third of the 10×10 matrices are chains, by far the largest contribution to the supply of DeMorgan monoids from underlying DeMorgan lattices. We might interpret this phenomenon to mean that, even if the shrinking process does not invariably produce chains, it very often does. And some of the most interesting and useful matrices are chains-e.g., the Sugihara matrices of [1], which we shall discuss further below. And, while specialization to chains is not generally attractive for R, since & and v must also be catered for, one might wonder whether, in R_{\rightarrow} itself, every nontheorem is refutable in some chain. Many nontheorems are; all 1-variable nontheorems of R_{\rightarrow} , for example, are refutable in the 4-chain. To get a counterexample to the hypothesis that every nontheorem of R_{\rightarrow} is chain-refutable, consider the Church disjunction $A \oplus B =_{df} A \rightarrow B \rightarrow B \rightarrow A \rightarrow A$, introduced in [4]. Note that $(p \rightarrow q) \oplus (q \rightarrow p)$ is valid in every closed R, matrix for which matrix \leq is a total order. For one of $p \rightarrow q$, $q \rightarrow p$ must be designated in such a matrix, whence, since Church disjunctions are relevantly implied by their disjuncts, the displayed disjunction must be chain-valid. But it is refutable in M_0 on assigning +1 to p and +2 to q.

Our lemma shows that every nontheorem of R_{\rightarrow} is, in a certain sense, maximally refutable. A related question is, "Is every nontheorem minimally

refutable?", in the sense that every nontheorem can be refuted at the *least* value in some closed R_{\rightarrow} matrix. An equivalent question is, "Can every nontheorem be rejected in a closed R_{\rightarrow} matrix with a lone undesignated element?" For what it is worth, Slaney's evidence shows that such matrices are uncommonly common, while, again, all 1-variable nontheorems have this property. While, maybe, one expects that the answer is negative, the question is interesting. Consider the case of Peirce's law, $p \rightarrow q \rightarrow p \rightarrow p$. This isn't valid, either, in the intuitionist logic J, which is a supersystem of R_{\rightarrow} , so that we can try looking at a J refutation. Here's one, in the familiar matrix J3.

 \rightarrow is defined familiarly on J3 by $a \rightarrow b = T$ if $a \le b$, $T \rightarrow b = b$ always, and, otherwise, $a \rightarrow b = F$. T is the only designated value. As a J, matrix, J3 is a fortiori a closed, reduced R_{\rightarrow} matrix.

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Peirce's law is refutable in J3, but on only one assignment. For $N \rightarrow F \rightarrow N \rightarrow N = N$, yielding a maximal (and maximally efficient) refutation in terms of our shrinking lemma. Question: can we manipulate J3 further so as to refute Peirce's law at F? Answer: No, because any further manipulation, of the above sort, must designate N also, given the argument for (iii) of our lemma, whence there will be no refutation whatsoever. Moreover, in no reduced closed J matrix M whatsoever is Peirce's law refutable, if there is only one undesignated element; for the only such M is truth-tables, and Peirce's law is a truth-table tautology.

However, this does not settle the matter for R_{\star} . In fact, there are lots of R_{\star} matrices which are not J matrices. The most justly famous of them all, the 3-point Sobociński (or Sugihara) matrix S3, has a Hasse diagram just like J3, but so defines \rightarrow that $N \rightarrow N = N$, but, otherwise, $a \rightarrow b = T$ if $a \leq b$ and $a \rightarrow b = F$ if a > b, designating both T and N. This time, $N \rightarrow F \rightarrow N \rightarrow N = F$, whence Peirce's law is R-refutable at a lone F after all. In fact, since S3 is characteristic for the intensional part RM_i of RM, as Parks and I (independently) showed, those nontheorems of R_{\star} that fail also in RM_{\star} are certainly refutable at F, including, naturally, all classical nontautologies. So it would be interesting to know whether this can always be done, sharpening for R_{\star} the main result here.

Mention of J_{\rightarrow} matrices prompts the following observation:

Fact: A closed reduced R_{\rightarrow} matrix is a J_{\rightarrow} matrix iff it has exactly one designated value. The same holds, mutatis mutandis, for R+ matrices.

Reason: Suppose that M is a closed reduced R_{\rightarrow} matrix with but one designated value d. It will suffice to validate the paradox $A \rightarrow B \rightarrow B$, since, added to R_{\rightarrow} , this produces J_{\rightarrow} . For this it suffices that, for each a in M, $a \leq d$. In fact, $a \leq (a \rightarrow a \rightarrow a) \rightarrow a \rightarrow a = d$, by the identity, permutation, and contraction principles of R and the fact that d is the only undesignated element. Conversely, the noisome paradox $B \rightarrow A \rightarrow B$ assures, as is well-known, that closed, reduced J_{\rightarrow} matrices will have but one designated value, which, a fortiori, makes them R_{\rightarrow} matrices with this property.

Further Fact: A closed reduced R_i matrix with exactly one designated value is a Boolean algebra, validating exactly the classical tautologies. The same holds, mutatis mutandis, for R matrices.

Further Reason: For the same reason as above, the noisome paradox $A \rightarrow .$ $B \rightarrow A$ is valid in any R_i matrix with exactly one designated value, after which all classical tautologies in the R_i vocabulary must be valid (in view of the standard properties of the DeMorgan negation of R). But \rightarrow, \sim is a sufficient basis for classical logic, whose closed, reduced matrices are exactly the Boolean algebras. Enough said.

Dunn has suggested in conversation that the above fact and further fact may be known (mentioning Pahi in particular as an author who has thought about these things). I wouldn't be surprised. But it is at least interesting that, for a closed reduced matrix to be *properly* relevant, it is both sufficient and necessary that more than one proposition should be considered true. For the root of both the elegance and the absurdity of the conventional logical wisdom may well be located in the assumption that all Truths are One. And it is interesting to find technical reflection of this assumption in the fact that positive Rmatrices suffer intuitionist breakdown, and full R matrices suffer classical breakdown, when it is made. (In the mutatis mutandis clauses of our fact and further fact, incidentally, one should take, I suppose, the R+ and Rmatrices in question to be *strongly* closed, in which case, if there is only one designated value and the matrix is reduced, & and \circ will coincide, so that the extensional case reduces to the intensional one.)

3 In this section, we generalize our principal lemma of the last section and draw appropriate conclusions. For the proof of this lemma is quite general, and depends only in a few places on particular properties of R_{\rightarrow} , whence it may also be asserted of any supersystems of R_{\rightarrow} that retain these properties, with or without additional connectives, or constants. The *theorems* of R_{\rightarrow} to which we appealed were the following:

B axiom	$q \rightarrow r \rightarrow . p \rightarrow q \rightarrow . p \rightarrow r$
CI axiom	$p \rightarrow p \rightarrow q \rightarrow q$
W axiom	$(p \rightarrow . p \rightarrow q) \rightarrow . p \rightarrow q$
I axiom	$p \rightarrow p$.

In addition, we appealed implicitly to the fact that the set of theorems of R_{\rightarrow} is closed under substitution for sentential variables (since this is built in, more or less, to the matrix approach) and under modus ponens (since this is what gives us a preference for closed matrices). These were the only assumptions used in the part of our proof in which, given an undesignated element a of a closed matrix $\langle M, O, D \rangle$, with M finite, we were able to go on adding new designated elements until a was a greatest undesignated element, in the closed matrix $\langle M, O, D_n \rangle$. Since, in fact, the axioms to which we appealed are exactly sufficient for R_{\rightarrow} , that part of the proof will go through for any supersystem of R_{\rightarrow} whatsoever, with whatever connectives one pleases (in addition to \rightarrow). So

Corollary 1 Let S be any system among whose theorems are the B, CI, W, and I axioms above. Let $M = \langle M, O, D \rangle$ be a closed, finite S matrix. Set M - D = U, and let $a \in U$. Then there is a closed S matrix $M_n = \langle M, O, D_n \rangle$ such that $D \subseteq D_n$ and $U_n = M - D_n = [\leq a]$.

In the interesting cases, S will be, as noted, an extension of R_{+} , closed under modus ponens and substitution. Among such extensions are R_i , R, R_{+} (and the Boolean extensions CR^* , CR thereof), RM_{+} , RM, J_{+} , J, D, K, and the various extensions of these systems. But the utility of Corollary 1 is considerably less than that of the lemma, since, although we get in general a closed S matrix M_n , in which a is maximally undesignated, we may not be permitted to reduce M_n to get, usually, something simple. So let us examine the particular features of R_{+} that permit also the reduction step of the proof of our lemma. From the discussion of Section 1, we already know, pretty well, what they are. The key point is that the matrix equivalence \simeq , when we arrive at it, must be a congruence on M_n . A sufficient (and, in the usual cases, necessary) condition for this is that replacement of equivalents in the form (4) of Section 1 shall be guaranteed by a theorem scheme of the system S in question. Let us restate this condition to avoid its apparent dependence on a \leftrightarrow connective, calling it the congruence condition for the system S.

(8) p → q →. q → p →. A(p) → A(q) is a theorem of S, for each formula A(p) in the vocabulary of S in which p occurs, and for each result A(q) of substituting q for exactly one occurrence of p in A(p).

Congruence Fact: Let S be any system satisfying, without restriction, the congruence condition (8), and let M be any closed S-matrix. Let moreover the Transitivity Axiom B and the Reflexivity Axiom I above be theorems of S. Then the matrix equivalence \approx is a congruence on M, whence M may be reduced modulo this congruence to get an equivalent closed S-matrix M/\approx .

Reason: That \simeq is a congruence means that (6) of Section 1 holds, for each operation \circ of M, and that \simeq itself is an equivalence relation. For the latter, we have already noted that Axioms B and I suffice, in closed matrices. And we merely illustrate the former, choosing \vee as a binary operation and showing that, in the presence of the congruence condition, $a \simeq c$ and $b \simeq d$ suffice for $a \lor b \simeq c \lor d$, in the closed S-matrix M. In fact, $p \rightarrow q \rightarrow . q \rightarrow p \rightarrow . (p \lor r) \rightarrow (q \lor r)$ is an instance of (8), choosing A(p) as $p \lor r$, whence $a \rightarrow c \rightarrow . c \rightarrow a \rightarrow . (a \lor b) \rightarrow (c \lor b) \in D$, since M is an S-matrix. Since the two antecedents belong to D, on assumption, and since M is closed, $(a \lor b) \leq (c \lor b)$. Similarly, $(c \lor b) \leq (a \lor b)$. Since $p \rightarrow q \rightarrow . q \rightarrow p \rightarrow . (r \lor p) \rightarrow (r \lor q)$ is likewise an instance of (8), we can also show $c \lor b \simeq c \lor d$, whence, by transitivity of \simeq , $a \lor b \simeq c \lor d$, as claimed. So the situation is familiar enough, and any further verification is left to the reader.

Corollary 2 Let S be any extension of R_{\rightarrow} , with perhaps additional connectives, for which the congruence condition (8) holds. Then the finite matrix shrinking lemma holds for S.

Proof of Corollary 2 is hardly needed, since, given Corollary 1 and the Congruence Fact, proof is exactly as of the lemma, just putting 'S' wherever

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' R_{\star} ' appears in the statement or proof of that lemma. Thus nontheorems are maximally refutable, in a finite reduced, closed matrix, in many famous and infamous systems; among them are R_{\star} , R_i , RM_i , and our old friends D, J, and K. For the old friends, the "finitude" part is established by well-known results; for R_{\star} and R_i , it is established by little known results, namely:

Finitude Fact for R_{\rightarrow} and R_i : R_{\rightarrow} and R_i have the *finite matrix* property; i.e., every nontheorem of these systems is refutable in a closed, finite matrix for the system.

Proof of the finitude fact is rather intricate, being established in the draft manuscript [8] by adaptation of the Kripke proof-theoretic argument for the decidability of R_{\star} . So I state it here merely as a fact. But note that, given the finitude fact, we can find a maximally efficient finite refutation of a given nontheorem, simply by applying our lemma. And the result will always be that a nontheorem is refuted at a greatest undesignated element in a closed, reduced matrix, since, if it isn't, we can simply cut the matrix down still further. And, of course, some refutation will also be maximally efficient, since there will be a closed, reduced matrix of smallest finite size in which the maximal refutation takes place. *Really* efficient decision procedures, of course, may be another story; even for J, the extant decision procedures (of which I know, anyway) quickly become quite cumbersome; and the situation for R_i , say, is at writing even worse. But any improvements can themselves be improved by the method sketched here.

4 In this section, I turn to the limits of these methods. An immediate one lies in the method by which we proved our shrinking lemma. To get a best refutation, we added new designated elements in no particular order, save that we always chose maximally undesignated ones to add. We can, of course, try out all orders. But, as a practical matter, it would be good to have some criterion on which we can make a "best choice", if our methods are to be at all efficient.

A second limitation is that the above reasoning only applies to reasonably strong systems, and that it depends on *both* the *CI* principle (which fails in *E*) and on the *W* principle (which fails in many systems, including, not surprisingly, R-W). These principles were needed to get us through the "cases" 3 and 4 respectively in the proof of the key lemma. Accordingly, it would be particularly interesting to know what form, if any, our lemma takes in weaker relevant logics (or irrelevant ones, for that matter).

A more important limitation is the dependence on finitude, in the matrices from which we start. In fact, we can state our lemma so that it avoids such dependence. Here is the idea. Instead of designating new matrix elements, one at a time, we may pick any matrix filter F, and designate $F \cup D$ at one fell swoop, provided that the filter in question is *directed down*; i.e., provided that any two elements of F have a lower bound in F, on the matrix entailment ordering relation \leq . On the same R_{\star} principles as before, this process takes us from closed matrices to closed matrices. So here is a plan to turn an *arbitrary* R_{\star} matrix M into one in which a given undesignated element a is maximally

undesignated, reminiscent of how completeness proofs go. D being fixed and set equal to D_0 , well-order the elements of M. Let a be set aside to be kept undesignated. Suppose that a matrix M_i has been defined, on the same base set M, with matrix entailment \leq_i . Pick any element b such that $b \leq_i a$, and let $D_{i+1} = D_i \cup [b \leq]$. This time, the reader may verify that the process takes us from closed matrices to closed matrices, letting D_{i+1} serve as the new set of designated elements, where we had an R_{\rightarrow} matrix to start with. Since we are not assuming finitude any more, every so often this process will lead us to a limit ordinal, and we will have to sum up, taking D_i at the limit as the union of all its predecessors. For familiar reasons, such D_i will still be closed, and will still lack a. At some point, we must run out of further undesignated elements to add, so that, at this last gasp, all undesignated elements will be $\leq a$, providing a maximal, but not necessarily finite, refutation of any nontheorem of R_{\star} refutable at a. And this situation also can be generalized in the spirit of Corollaries 1 and 2, with the *reducibility* of our last M_i governed by the universal theoremhood of the formulas (8).

We can, in particular, go through the process just sketched with respect to the Lindenbaum matrix of whatever extension S of R_{+} we are considering. This time, however, the process of choosing the b whose principal filters are to be added becomes a real headache. For we should like to make, at each stage, a finitizing choice, if we can; that is, we should like to arrive at a D_i such that, if we reduced M_i modulo a suitable congruence (which can be matrix equivalence when S satisfies the congruence condition), the result would be a closed, finite matrix for S. It is not immediately apparent, however, how we can make such a choice, except through some other argument, if there is one, that S has the finite matrix property. Somewhere, there must be a better result than this, making use of exactly the R_{+} properties, and applied to some class of extensions of R_{+} . But I shall not pursue it any further here (though [8], perhaps, is the source of some clues).

Finally, I turn to the problem of applying these results to the system R itself. Here, the first problem is that, in contrast to the systems J and K, there is no useful formulation of R, for our purposes, with $\rightarrow E$ as sole rule. And the second problem, already noted, is that the congruence property (8) fails for R. The final problem, also already noted, is that no finite matrix theorem has been proved for R; indeed, the system may be undecidable, in which case there is no such (useful) theorem. So, even if we bring the finite matrices for R under control, it is yet uncertain how much we have accomplished.

The central problem, if we try to prove our theorem for R and not just for R_{\rightarrow} or R_i , lies in caring for the adjunction rule. This is not a problem for RM, for the following reason:

RM Observation: Every nontheorem A of RM is maximally refutable. In fact, if A is a nontheorem of RM, there is, in the sense of [1], a finite Sugihara matrix S_n (where n is the number of matrix elements) such that: (a) A takes the value -1 on some interpretation in S_n , and (b) for all m < n, A is valid in S_m . (Since all the S_n are finite, reduced, strongly closed RM matrices, the refutation in question is in an evident sense a *best* refutation.)

Verification: If A is a nontheorem of RM, it is refutable in some finite Sugihara matrix S_p , at some value -q. Applying the proof technique of our principal lemma, show that all the matrix elements >-q may be added as new designated values. (The important point for immediate purposes is that our new D will be closed under &I also, since S_p is a chain.) The matrix equivalence induced by the new D is a congruence, whence we may pass to the quotient S_n . And it is easy to see that S_n is also a Sugihara matrix, which has resulted from S_p simply by identifying all elements b in S_p such that -q < b < q. Accordingly, on the natural matrix homomorphism h from S_p to S_n , all the elements between -q and q have been taken into 0, whence q becomes the new +1, under h, and -q becomes the new -1. So, if A is refutable in any Sugihara matrix S_p at a value <-1, there is always a better (i.e., smaller) matrix in which A is refutable at -1. So the smallest S_n in which A is refutable at all will refute A at -1, completing the verification of our observation.

The proof technique underlying the above observation has its amusing aspects. First, as noted, the intensional part RM_i of RM also has a dual *minimal refutability* property; every nontheorem is refutable at the *least* element of some Sugihara matrix S_p . Combining this with the above observation, we get the result that Parks reported in [12]: i.e., every nontheorem of RM_i , being both minimally and maximally refutable, is already refutable in the 3-point Sugihara matrix S3. There is also a connection with the work on extensions of RM that Dunn sets out in [1]. For pick the smallest n such that A is refutable (at -1) in S_n , and consider the result RM + A of adding A as a new axiom scheme. S_n , of course, cannot be a matrix for RM + A. But S_{n-1} is, naturally, an RM + A matrix, and, as Dunn shows by an elegant argument, it is in fact a (finite) characteristic matrix for RM + A.

We may, in fact, recast Dunn's argument in the following form, trading in some of its algebraic features for syntactical ones. Let B be a nontheorem of RM + A, where A is a nontheorem of RM. Let α be the set of all instances of A in the vocabulary of B (i.e., only sentential variables that occur in Boccur in any member of α). We note first that there is some finite conjunction & α of members of α such that, for every A' in α , & $\alpha \rightarrow A'$ is a theorem of RM. For, given the results that I reported in [1], it is evident (as Dunn has also observed) that there are only finitely many nonequivalent formulas of RM, in all, built out of the same sentential variables as B (since S_{2m} is characteristic for the *m*-variable fragment of RM); a fortiori, there are only finitely many nonequivalent instances of A in this vocabulary. So any formula C in the Bvocabulary is a theorem of RM + A iff it is deducible in RM, using the axioms and rules of RM, from the formula & α . Similarly, we may find a formula &RM that will stand in for the conjunction of all the theorems of RM in the B-vocabulary. (In fact, the conjunction of the $p \rightarrow p$ (the well-known "t-surrogate"), where p occurs in B, will serve for &RM.)

Then, simply applying the appropriate deduction theorem, for all C in the B-vocabulary, C will be a theorem of RM + A iff & α & & $RM \rightarrow C$ is already a theorem of RM itself. We have chosen a B which is a nontheorem of RM + A, whence by the completeness proof for RM, there is a finite Sugihara matrix S_p in which & α & & $RM \rightarrow B$ may be refuted at the value -1, by our RM observation. Indeed, there is a smallest such S_p . In order for this to happen, on inspection of the "truth-tables" for finite Sugihara matrices, each of & α , &*RM* must take one of the values 0, +1, while *B* must have been assigned -1 in S_p.

It will now suffice for Dunn's result, in its present recension, if we can show p < n, where S_n is the smallest Sugihara matrix in which A itself is refutable. (For if a formula is refutable in any Sugihara matrix, it is also refutable in any larger one; whence, since B is an arbitrary nontheorem of RM + A, if p < n then B will be refutable in S_{n-1} in particular, clinching the claim that S_{n-1} is characteristic for RM + A.)

To nail down p < n, it will suffice to show that A is valid in S_p . (For A is not valid in any Sugihara matrix from S_n on.) Suppose, to the contrary, that A is not valid. Then there is some refuting interpretation I of A in S_p , different from the interpretation I', perhaps, that we just picked to refute & α & &RM $\rightarrow B$. But we shall show that, under these circumstances, & α could not have been true on I', either.

First, we observe that, for each element b of S_p , there is some sentential variable q of B such that I'(q) = b or $I'(\sim q) = b$. (For matrix values assigned to neither sentential variables or their negates are assigned, on a Sugihara interpretation, to no formula, whence we could have got a smaller Sugihara matrix than S_p which would have done the same job by dropping the superfluous elements. But S_p is the smallest matrix that will do the job.) Accordingly, for our refuting interpretation I, and each sentential variable r of A, there is a sentential variable q of B such that either I(r) = I'(q) or $I(r) = I'(\sim q)$. Evidently, then, there is a substitution instance A' of A, in the vocabulary of B, such that I(A) = I'(A'). Since, on the (reductio) assumption presently in force, A is undesignated on I, A' is already undesignated on I'. But we chose & α so that it would *RM*-entail all instances of *A* in the *B* vocabulary, including A' in particular. Since Sugihara matrices respect RM-entailment this requires $I'(\&\alpha) < 0$ also, a possibility that we have already ruled out. Accordingly, A is valid in S_p , whence p < n and B is refutable in S_{n-1} . So S_{n-1} is characteristic for RM + A, as Dunn said that it would be.

We seem to have fallen a little short of the actual Dunn result, incidentally, which is that every extension of RM, closed under $\rightarrow E$, &I, and substitution, has a finite characteristic matrix. But this is only seeming. For let S be any proper extension of RM, and let S_n be the smallest finite Sugihara matrix in which at least one theorem of S is rejected. (For, since S is a proper extension of RM, it contains at least one nontheorem of RM, which is rejectable in some finite Sugihara matrix.) Then, I assert, S_{n-1} is characteristic for S (or, more accurately, Dunn asserts). For let A be a theorem of S refutable in S_n . Then, by what we have already shown, S_{n-1} is already characteristic for RM + A, since, by leastness of n, neither A nor any other member of S is rejectable in S_{n-1} . And so, in fact, S and RM + A coincide, as is now clear. (In all of this, incidentally, we have been tacitly avoiding the trivial extension of RM, of which everything is a theorem. But this extension also falls under the general rubric; its characteristic matrix consists of just 0, and that designated, trivially validating all formulas. Letting this matrix be S_1 , and noting that S_2 is truthtables, S_1 is characteristic for the theory that we get if we are silly to add a truth-functionally invalid axiom scheme to RM, as is widely known.)

That was quite a digression, but it makes a nice application of our princi-

pal lemma, when transmuted into our "RM observation". Alas, the application is deeply RM-specific, and we return to our problems with R. But, in the RMcase, we got over our &I problems by utilizing special properties of chains, at least in the Sugihara case. For R, however, we have an outright counterexample to the finite matrix shrinking lemma, if we try to put "R matrix" for "R, matrix" and "strongly closed matrix" for "closed matrix" everywhere in that lemma.

For consider the case of the 4-diamond, now taken as a DeMorgan monoid (with lattice connectives) and henceforth dubbed 4D. Were the modified shrinking lemma true, there would be a homomorphic image 4D' or 4D, under a matrix homomorphism h, such that hF is the greatest undesignated element of 4D', and 4D' is reduced and strongly closed. This is the situation that we were in before, when we passed from 4D to the 4-chain, but then we were only trying to preserve \leq and the intensional operations.

At any rate, suppose that there is such an h and such a 4D'. We note first that 4D' (and any nontrivial matrix homomorphic image of 4D which is closed and preserves D and \rightarrow) must continue to have 4 elements, in view of $hT \rightarrow hf =$ $hT \rightarrow ht = hf \rightarrow ht = ht \rightarrow hF = hf \rightarrow hF = hT \rightarrow hF = hF$. Moreover, since hpreserves \leq , hF remains least in 4D' also, whence, since it is also maximally undesignated, in 4D', hF must be the only undesignated element in 4D'. In particular, hf is designated, whence, in 4D', $ht \leq hf$. But this produces the 4-chain, as 4D', which, we have essentially shown, is the only nontrivial, proper reduced matrix homomorphic image of 4D, considering the former as an \rightarrow matrix only. If we must attend to & also, the matrix homomorphism snaps. For, in 4D', hf & ht = hF must continue to hold, whence the &I principle fails for 4D'.

We are back now where we started—reflecting on the relations between the 4-diamond 4D and the 4-chain 4C, considered as matrices for R. And it is now clear that we have not one choice but two for the & and v tables to be added to the intensional specifications on 4C, to make 4C an R-matrix. In the first place, we may use its ordering under its own matrix \leq to define & and v on 4C; this has the effect of making f & t = t, whence 4C is strongly closed. Or else we can continue to define & and v, even in 4C, in the 4D way, with f & t = F. Note that, on the latter plan, a & b remains a lower bound in 4C for a and b, but it is no longer a greatest lower bound, with respect to the 4C matrix ordering.

Where we started, we shall end, since I see no present hope of giving further content, for the system R itself, to the concept of "best" refutation which it has been the purpose of this note to set out. This is doubly ironic, since it is *exactly* the R_{\rightarrow} machinery on which we have depended to set out the concept. But not only does the key congruence condition (8) fail for R on the lattice connectives & and v, but adding the theorems necessary to nail down this condition produces intuitionist breakdown in the R^+ case, and classical breakdown in the R case. For consider the result of adding to R^+ the new axiom scheme $A \rightarrow B \rightarrow . B \rightarrow A \rightarrow . A \& C \rightarrow . B \& C$. In the presence of the Church constant T (see [10]) whose chief property is that everything implies it, this yields $T \rightarrow T \rightarrow . T \rightarrow . T \& C \rightarrow . T \& C$, which readily reduces to $T \rightarrow . C \rightarrow C$ and then to $D \rightarrow . C \rightarrow C$, which is what it takes to get J+ from R+. T itself is irrelevant to the argument (being always contextually replaceable), whence the smallest extension of R+ satisfying (8) is indeed J+; and of R, it is K. So the congruence condition, for the lattice connectives, looks, and is, irrelevant. Nor shall we think any further here about what is necessary to extend our result to implicational or other fragments of weaker neighbors of R, such as E_{+} .

One final point, however, is worth noting. What our principal result is, really, is an *excluded middle* theorem, on what Slaney calls the Principle of Relativity for a greatest falsehood f. Logic, we expect, should not be in the business of telling us *which* is the greatest falsehood. This is partly concealed, in relevant and other logics (e.g., K, as well as R and Curry's D), by favored interpretations on which a sentential constant f is *intended* as a greatest falsehood, and, happily, turns out to be so; 4D is such an interpretation of R. But there are other possible interpretations on which a shall be by the least truth t; and it is an odd situation, which I shall leave for the metaphysicians to ponder, when the standard truth is, on interpretation, even more false than the standard falsehood.

In fact, the word 'false', as applied to formal theories (and, in general, to logic) easily lends itself to equivocation. On the one hand, there is what a theory *asserts* to be false—say by having $A \rightarrow f$ or $\sim A$ as a theorem; on the other hand, there is what a theory simply *fails to assert*—by not having A as a theorem. Only in rare—though theoretically desirable—circumstances is the equivocation removed; for this happens when, and only when, the theory T is consistent and complete relative to what passes as its negation.

Now it is often thought, or felt, or at least devoutly hoped, that we have a semantical overview that disposes of these vexing questions, and which divides real sentences (as opposed to the mere scribblings of logicians) exclusively and exhaustively into true sheep and false goats. That's as may be; in practice, God is not at our elbow, giving us hints as to how he carves up the world. And so, relative to actual formal theories—and actual informal ones, for that matter—it is useful to characterize what is false *according to* a theory on grounds intrinsic to that theory, without appealing to anybody's sense that his axioms are God's axioms, which, as it happily turns out, correspond to Reality. So let us say that a sentence of a theory is *directly false*, according to that theory, if its negation shows up among the theorems: and that it is *indirectly false* when the sentence itself fails to make the Honour Roll of theorems, without regard to the appearance, or nonappearance, of its negation on that Roll.

It has been an important point, in relevant semantical analysis (e.g., in [14]) to *distinguish* between direct and indirect falsehood, using the latter to offer a kind of semantical explication of the former. What our present result suggests, however, is that indirect falsehood can quite generally be transmuted into direct falsehood, just by switching what counts as 'f'. For, unless a theory is to be wholly trivial, *something* in it will fail to be asserted. And the import of our main result here, for R_i and the other theories for which it holds, is that any nontheorem may be taken as the 'f' of a suitable

theory which rejects that formula. Moreover, this choice of f is a strongly two-valued one; for refuting a given nontheorem B, on interpretation, at a greatest undesignated value means precisely that, in the regular theory consisting of just the formulas mapped into "true" values on this interpretation, we shall have $A \rightarrow B$ as a theorem if and only if A itself is not a theorem. Note also that the fate of the actual sentential constant f, if present, is not involved in our analysis, despite the fact that this constant was presumably intended to serve as a greatest falsehood. For our actual theories do not necessarily realize our intentions, and the f that we intended should be formally false may in fact stand high on the list of formal truths. In that case, we may well speak of f as both true and false, the latter in the direct sense, on what was called the "American plan"² of semantical analysis in [11]. But we may equally well say, in this situation, that the constant f is just not the real f of the theory, and that the "both true and false" talk, which sounds exciting, is in prosaic reality much less so. For, lurking behind our intuitions as to what *should* be true or false, according to the intentions that we set out to formalize, there is that which is true or false, according to what we (or our theories) actually say. And let us not get so hung up on our formal insights, and their delicious interrelationships, that we lose our common sense, and begin to babble what is absurd. No more than any other sort of logics have relevant logics made sense of what, in Reality, is both true and false. What they have made direct sense of is our commonsense conviction, which the paradoxes of implication belie, that our reason does not buckle under when confronted with contraditions—or, as we might wish to put it, with what is together asserted-true and asserted-false. But, on the latter point, as Slaney (and, before him, Johansson) correctly intuited, what is asserted-false is very much relative to what shall be counted as *formally false*, in the direct sense. A common criterion is, "A is formally false if it implies something repugnant". But this is clearly relative to what we find repugnant, and to what degree we find it repugnant. (It is obvious, of course, that we may season a theory with several f's, and several accompanying negations, depending on how bad we would feel if the f in question were provable. While it is not unreasonable to have a standard f-say, the f of R-we can prove that fwhenever we get into difficulties, so that it is good to have fallback, unprovable f's, which, like a quarantine during an epidemic, keep the diseased areas of our thinking from infecting its healthy parts. The runaway classical f, which says to us "If I get sick, I will cough in your face, and bring all of thought to ruin", is a menace to the public health of logic, and rightly belongs to the era in which it was invented-to the 19th century, when diseases of all sorts ran riot.) Meanwhile, the conclusion of this note is that under very general conditions, which R_{\rightarrow} suffices to guarantee, we can *pick* our f as any nontheorem B of logic, dividing the sheep and the goats around it. It will be interesting to know whether and how far this result extends to systems not covered by this note.

NOTES

^{1.} Although this paper does not, in general, depend upon [1], it will be helpful if the reader has access to that book. For one thing, I shall employ its notation and notational con-

ventions (with some adaptations too trivial, I trust, to require explicit mention). For another, I shall cite [1] as the most convenient source for results and discussion significant for this note. Because [1] is a conglomerate affair, to which many authors contributed, imputing a result to it does not necessarily—and, in this note, usually does not—impute that result to one of the main authors of [1]. More often, here, the results cited are in those sections of [1] written by Dunn, while the Sugihara matrix completeness proofs for RM, used in passing, are mine alone.

2. The plan is "American" because it was invented by Dunn, though Slaney has pointed out that it was to some (small) degree foreshadowed in previous work, e.g., in Rescher's [13], though Rescher's own interests and purposes were quite different (and, as Dunn has assured me in correspondence, furnished no "input" to his own thinking, in refutation of my speculation in [9] that there was perhaps some likelihood that such input had occurred). The contrasting "Australian" plan was invented by R. and V. Routley, and has formed the basis for most work in the Kripke-style semantical analysis of relevant logic, e.g., in [14]. I have used both plans in my own work (the American plan in [11]) and consider them technically equivalent. On the philosophical grounds set out in [9], however, I like the Australian plan better.

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