# A Note on R_Matrices 

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Consider the following two DeMorgan monoids, in the sense of [1], for the system $R$ of relevant implication, with lattice operations $\&$ and $v$ defined according to the following Hasse diagrams, and $\rightarrow$ tables defined by the following matrices (with designated elements starred). ${ }^{1}$


$$
\rightarrow \quad T \quad t \quad f \quad F
$$

$$
{ }^{*} T \quad T F F F
$$

$$
{ }^{*} t \quad T \quad t \quad f \quad F
$$

$$
f \quad T \quad F \quad t \quad F
$$

$$
F \begin{array}{lllll}
F & T & T & T
\end{array}
$$


$\rightarrow \quad T t f \quad F$
*T T F F $F$
${ }^{*} f \quad T \quad F \quad t \quad F$
${ }^{*} t \quad T \quad t \quad f \quad F$
$\begin{array}{lllll}F & T & T & T & T\end{array}$

Since $\sim a$ is definable in every case as $a \rightarrow f$ (and should be typographically obvious anyway), and other DeMorgan monoid operations are definable from the above (e.g., fusion (or cotenability) $a \circ b$ as $\sim(a \rightarrow \sim b)$ ), with $t$ as monoid

[^0]identity, we get a complete set of matrices for $R$ in either case, or, saying the same thing from another viewpoint, we have defined in each case a DeMorgan monoid.

The matrix on the left I shall call the 4-diamond; that on the right, the 4 -chain. (The 4 -diamond is especially venerable; it is due to Church, who in [4] invented relevant implication.) What strikes the eye, however, is something else; namely, insofar as their $\rightarrow$ tables are concerned, the 4-diamond and the 4 -chain are the same. If we wrote down the tables for the other intensional operations, such as DeMorgan negation $\sim$, fusion ${ }^{\circ}$, fission + , and coimplication $\leftrightarrow$, these would also turn out to be the same in the two structures.

Clearly, there is something going on here of almost metaphysical import. For how can a chain be a diamond? (The converse question is easily answered, for, as is well known, a diamond can be a chain when one has presented it to the wrong lady.) But there are two answers to our question.

First, the 4 -chain and the 4 -diamond are certainly not isomorphic DeMorgan monoids. But, for the purposes of relevant logic, they are to some extent isomorphic-namely, they are isomorphic in what one might wish to identify as their intensional parts. Nor is either of these monoids even a homomorphic copy of the other, for lattice operations \& and $v$, associated with the corresponding connectives of $R$, are not preserved. But the transformation $h$ suggested by our notation is nonetheless an order homomorphism, preserving the partial order $\leqslant$ in the usual sense.

Second, let us forget about DeMorgan monoids and simply think about the 4 -chain and the 4 -diamond as matrices, concentrating on the $\rightarrow$ tables. From this viewpoint, we have passed from the 4 -diamond to the 4 -chain in a single move, simply taking $f$ as a further "designated" value. My main purpose here will be to examine the consequences of such a move, with emphasis on the actual use of matrices and their associated algebras to give us a better picture of systems like $R_{\rightarrow}$.

Let us consider a practical problem: for example, finding a matrix $M$ to refute a given nontheorem $A$ of $R_{\rightarrow}$. We should like to accomplish this task as economically as possible. Let us understand economy in the sense that we should like to find a matrix which is as small as possible, and with as few undesignated values as possible, which will do the job. The two points are connected; the more values we designate, the more often we shall, in general, have designated both $a \rightarrow b$ and $b \rightarrow a$, whence, as a general principle for dealing with $R \rightarrow$ matrices of an appropriate sort, we may identify $a$ and $b$, shrinking the matrix. Or we may simplify the order, as we did in passing from the 4 -diamond to the 4 -chain.

In fact, for $R_{\rightarrow}$, every nontheorem may be maximally refuted, in the sense that, where $A$ is a nontheorem of $R_{\gtrdot}$, there is a finite matrix $M$, and an interpretation of $R_{\rightarrow}$ in $M$, on which $A$ takes a value $u$ which is naturally understood as the greatest undesignated value in $M$. Moreover, $M$ is the smallest matrix, satisfying the axioms and rules of $R_{\rightarrow}$, in which $A$ is refutable. So, if we are looking for a matrix that will do in $A$, we might as well look for $M$. This property of maximal refutation continues to hold for the intensional fragment $R \subsetneq$ of $R$ (which Dr. M. McRobbie has taught me to call $R_{i}$, as I shall henceforth), as well as for all extensions, in the same connectives, of these sys-
tems. It holds moreover for some well-known further systems, such as the intuitionist logic $J$ and the classical logic $K$. But there are other well-known systems for which, in a suitable sense, the maximal refutation property holds; for example, the semirelevant extension $R M$ of $R$. But the property cannot be asserted for the full system $R$, or for other relevant logics closely related to $R$. First, it is not yet known whether $R$ has the finite model property, which we have built into our characterization of maximal refutability. Moreover, even if we were to change the characterization, or prove that $R$ does have a finite model property, there are complications induced by the $\& I$ rule. But the intensional part of $R$ is itself of interest, both because it is in $R_{i}$ that Church's original insights into relevant implication take their clearest form, and also because it is their theory of implication which forms the most distinctive and novel aspect of the relevant logics. So we shall concentrate in this note on proving a maximal refutation lemma for $R_{\rightarrow}$, extending it as applicable to further systems.

1 We begin with some familiar terminology. By a matrix for a formal language $L$, we mean a structure $M=\langle M, O, D\rangle$, where $M$ is a nonempty set, $O$ is a set of operations on $M$ similar to the connectives of $L$, and $D$ is a subset of (designated elements of) $M$. (We abuse language to the extent of often using ' $M$ ' for both the structure and its base set.) Here, we always take $\rightarrow$ as one of the connectives of $L$, as an implication connective appropriate to $L$; accordingly, each matrix $M$ will also have a binary implication operation $\rightarrow$ defined upon it. And implication, in turn, gives rise to a binary relation $\leqslant$, to be identified as matrix entailment, as follows, for all $a, b$ in $M$.
$D \leqslant a \leqslant b$ iff $a \rightarrow b \in D$.
Matrix entailment fathers in turn a binary relation $\simeq$ of matrix equivalence, which is nothing but two-sided entailment.
$D \simeq a \simeq b$ iff both $a \leqslant b$ and $b \leqslant a$.
A matrix is reduced, for our purposes, just in case no distinct elements of $M$ are equivalent; i.e.,
(1) $\quad M$ is reduced iff, for all $a, b \in M, a \simeq b$ iff $a=b$.

We shall prefer, where possible, reduced matrices. And we shall prefer even more those matrices that respect modus ponens, which we shall call closed. In the $\leqslant$ notation, with $\Rightarrow$ as metalogical if,
(2) $M$ is closed iff, for all $a, b \in M, a \leqslant b \Rightarrow . a \in D \Rightarrow b \in D$.

Thus the closed matrices are just those in which $D$ is "closed up" under the matrix entailment $\leqslant$.

In general, let us take a subset $G \subseteq M$ to be a matrix filter iff $G$ is closed up, and a matrix ideal iff $G$ is closed down; i.e.,
(3) $G$ is a matrix ideal iff, for all $a, b \in M, a \leqslant b \Rightarrow b \in G \Rightarrow a \in G$,
with the dual condition, generalizing (2), for filters, of which $D$ is required
to be one in closed matrices. Letting, now and henceforth, $U$ be the set $M-D$ of undesignated elements of $M$, trivially equivalent to (2) is the requirement that $U$ be an ideal. Each element $x \in M$ determines a corresponding principal ideal and principal filter, which we shall symbolize respectively by $[\leqslant x]$ and $[x \leqslant]$, so that $y \in[\leqslant x]$ iff $y \leqslant x$, and $y \in[x \leqslant]$ iff $x \leqslant y$. Where $G \subseteq M$, we shall also say that a member $x$ of $G$ is maximal in $G$ iff it precedes in $G$ only its matrix equivalents; i.e., $x$ is maximal in $G$ iff, $x \in G$ and, for all $y \in G$, $x \leqslant y \Rightarrow x \simeq y$. And a maximal element $x$ in $G$ is $G$-greatest iff, moreover, for all $y$ in $G, y \leqslant x$. Evidently given $G$ is an ideal, $x$ is $G$-greatest iff $G=[\leqslant x]$. In a reduced matrix, there is clearly at most one $G$-greatest element for any $G \subseteq M$. On the obvious duals, we may speak also of minimal and least elements of $G$.

What we wish to prove is, I trust, now shaping up: namely, that in $R_{\rightarrow}$ and similar systems, there is for each nontheorem $A$ a suitable finite, closed, reduced matrix $M$, such that $M$ is as small as possible and $A$ is refutable at a $U$-greatest element of $M$. This will count as a "best refutation" of $A$; it is the one that we should look for if we were, say, programming a computer to refute nontheorems of $R_{\rightarrow}$ (with, perhaps, another computer busy proving theorems). But let us return, briefly, to general considerations. We may take a theory $T$ on a sentential language $L$ simply as a subset of $L$; and we take $T$ to be closed just in case it is closed under modus ponens, again disinteresting ourselves in any theories that are not so closed. Then $T$ itself may be thought of as a closed matrix-namely, the so-called Lindenbaum matrix $\langle L, O, T\rangle$, where $L$ is taken as the set of all formulas, $O$ as the set of connectives and constants of the language, and $T$ as the set of theorems. (Note that, in this special case, we reverse our usual ambiguity, referring to a Lindenbaum matrix by the name ' $T$ ' of its set of theorems-i.e., of its designated elements, not all its elements.) Note that only in exceptional circumstances will a Lindenbaum matrix be reduced; but there are instances; the substantial achievement of Martin's [6] was to show precisely that the Lindenbaum matrix of the system $P-W$ ( $T_{\rightarrow}-W$ in [1]) is reduced, disposing affirmatively of a long nagging open problem ([1], p. 95).

We now revert to the familiar. Let $M=\langle M, O, D\rangle$ and $M^{\prime}=\left\langle M^{\prime}, O^{\prime}, D^{\prime}\right\rangle$ be matrices for some sentential language $L$. A homomorphism from $M$ to $M^{\prime}$ is a homomorphism in the algebraic sense, preserving the operations in $O$. A matrix homomorphism $h$ is a homomorphism that preserves also $D$, in the sense that $a \in D \Rightarrow h a \in D^{\prime}$. Note that every matrix homomorphism must preserve matrix entailment also, since $a \leqslant b$ iff $a \rightarrow b \in D$, whence $h(a \rightarrow b)=$ $h a \rightarrow h b \in D^{\prime}$, so that $h a \leqslant h b$ in $M^{\prime}$ if $a \leqslant b$ in $M$.

Let $T$ now be a theory, identified with its Lindenbaum matrix $\langle L, O, T\rangle$, and let $M$ be a matrix for $L$. Then $M$ is a $T$-matrix just in case every homomorphism from $T$ to $M$ is a matrix homomorphism, or, as it is more ordinarily put, all theorems of $T$ are "true" (or "designated") in $M$ on all interpretations. Since our concern is with closed theories and closed matrices, we may assume that the rule of modus ponens is built in on both sides.

I conclude this section with a slight digression. Relevant logics have been equipped, through the work of Dunn and others, with an "algebraic analysis", to which we referred at the outset. Since the relation between the "algebras"
of relevant logics and their "matrices" is familiar to workers in the field, as part of the (conscious or unconscious) "folk wisdom" of the subject, it may be useful to spell out here exactly what that relation is. Briefly, the "algebras" in which relevant logicians have taken a special interest may themselves be identified as special kinds of matrices for relevant theories, tied rather closely to a specific choice of primitive vocabulary. What is special about these matrices is that they are taken as both closed and reduced (with, when \& is present in the vocabulary, the usual further requirement that the adjunction rule $\& I$ also be respected, in the sense that $D$ shall be closed under \&). What is specific about them is that the sentential constant $t$ and the fusion operation ${ }^{\circ}$ play important roles "algebraically", so that, when we count the relevant algebras as matrices for logics, we must be sure to count $t$ and ${ }^{\circ}$ as among the (primitive or defined) logical particles of the relevant logic in question.

Let us give some examples. The Church monoids of [1] (p. 376) are just the closed reduced $R_{\rightarrow}^{\prime}$ matrices (where $R_{\rightarrow}^{\prime}$ is just the result of adding $t$ and ${ }^{\circ}$, with their governing principles) to $R_{\rightarrow}$. In the same sense, Dunn monoids are closed, reduced $R+$ matrices, and DeMorgan monoids are closed, reduced $R$ matrices (with $\circ$ and $t$ in the vocabulary and with $D$ respecting the $\& I$ rule in both cases). Providing relevant logics in general with Dunn-style algebras was a project long stymied, on the point that the crucial fusion connective - is indefinable in $E$ and other weak sisters of $R$. But Routley and I in [7] found a way to overcome this problem, whence there are now $E+$ algebras that are the closed, reduced $E+$ matrices, $T$ algebras that are the closed, reduced $T$ matrices (where $T$ is the "ticket entailment" system of [1]), and so forth in general, on close analogy to the corresponding algebraic situation for $R$ and its fragments.

What is it that produces these close ties between relevant algebras, which one thinks of as having smooth properties and an interesting mathematical character, and the mere matrices for relevant theories, which one tends to think of as simply devices, of a more or less ad hoc character? The chief clues lie in Belnap's [3] and Dunn's [5], and they build on important connections between the matrix entailment relation $\leqslant$ (definable on any matrix with a suitable $\rightarrow$, or, for that matter, an unsuitable $\rightarrow$ ) and the set $D$ of matrix "truths". For, where $T$ is a logic, the "matrix truths" according to a $T$-matrix $M$ are of course parasitic upon the logical truths according to $T$. So, where $T$ has a decent $\rightarrow$, matrix $\leqslant$ begins to take on some familiar properties in any $T$-matrix. For example, if $p \rightarrow p$ is a theorem of $T, T$-matrix $\leqslant$ must be reflexive. If $q \rightarrow r \rightarrow . p \rightarrow q \rightarrow . p \rightarrow r$ is a theorem of $T, \leqslant$ must also be transitive, in any closed $T$-matrix. And so forth.

We have been viewing $\leqslant$ as deriving its $T$-matrix properties from the $\rightarrow$ of $T$, and, thereby, through the elements of $M$ that are required to be designated. For, matrixwise, $D$ determines $\leqslant$. But the converse point of view is also possible, and it is in fact characteristic of the "algebraic" approach, which starts out with a nice relation $\leqslant$ on $M$ (e.g., a partial order) and which, in effect, cooks up $D$ from $\leqslant$. And it is here, in fact, that the (Ackermann) sentential constant $t$ has been most helpful. For the properties of $t$ (in $R$, say) suffice precisely to make it $D$-least in any closed $R$ matrix, whence $D$ becomes the principal filter [ $t \leqslant$ ]. If, accordingly, we begin with $t$ and $\leqslant$, we may simply
define $D$ as $[t \leqslant$ ], which, in effect, is what Dunn does in [5]. And the crucial link between these two ways of thinking is secured by the principle $t \leqslant a \rightarrow b$ iff $a \leqslant b$, which, when one reflects on it, is exactly right, on the intuition that it is true that $a$ implies $b$ iff, in fact, $a$ implies $b$. (This intuition is central, and, though worked out in the form now being considered in [5], is more appropriately to be credited to [3].)

There is one more vital point, which Dunn has stressed in conversation. It lies in the fact that relevant algebras are to be taken as reduced matrices; i.e., if $a \simeq b$ in a matrix that counts also as an algebra, $a$ and $b$ shall be the same element. (Put otherwise, the matrix entailment $\leqslant$ shall be antisymmetric, which, on the minimal conditions already noted that make it reflexive and transitive, assure that $\leqslant$ will be a partial order in relevant algebras.) But reduction, unlike the other properties that decent logics require of their closed matrices, does not happen automatically. To be sure, we can cause it to happen, on the following familiar plan. If $\leqslant$ is reflexive and transitive, $\simeq$ is an equivalence relation, partitioning the members of $M$ into distinct equivalence classes. By identifying the members of the classes, we may pass to a "quotient matrix" $M / \simeq$, which will be reduced.

Well, we may be able to play this trick. First, let us look at its logical significance, on the principle that, however dimly, the properties that our favored matrices come to enjoy will be parasitic on their parent logics. The logical principle underlying the reduction step just traced is "replacement of equivalents". And the idea, perhaps, is that if $A \rightarrow B$ and $B \rightarrow A$ are both true, relative to some interpretation, then, from the point of view of that interpretation, $A$ and $B$ are also indistinguishable; they express, it might be said, the same proposition, on the interpretation. And this thought does call for reduction, for if we think of matrices, or algebras, as offering propositional models of our logics (as [3] might be held to suggest), it is an offense against Ockham to have, in the model, distinct entities with the same propositional content.

Let us, however, return to the logical point, since the place of the replacement principle in relevant logics is itself subtle. Perhaps the most straightforward form of this principle is just

$$
\begin{equation*}
(A \leftrightarrow B) \rightarrow . C(A) \leftrightarrow C(B), \tag{4}
\end{equation*}
$$

where $C(B)$ results from $C(A)$ by one or more replacements of occurrences of $A$ with $B$. But (4) holds in $R$ and other relevant logics only in intensional contexts-that is, where $C(A)$ is built up from $A$ and propositional variables using only intensional connectives, and not using, in particular, the truthfunctional connectives \& and v. But weaker forms of (4) hold quite generally; for example,
(5) $\quad(A \leftrightarrow B) \& t \rightarrow . C(A) \leftrightarrow C(B)$,
is a theorem scheme of $R$, licensing in particular the usual replacement rules for this system and its extensions.

It might be argued that, in view of (5), relevant logics have the "usual" replacement properties. But, strictly speaking, this is not correct, a matter affecting the principal result of this note. For what, mathematically, allows
us to "collapse" an abstract algebra "modulo" some equivalence $\simeq$ ? Central is the requirement that the equivalence in question shall be a congruence with respect to all relevant operations: i.e., where $\circ$ is any $n$-ary operation of the algebra, the requirement is

$$
\begin{equation*}
a_{1} \simeq b_{1} \& \ldots \& a_{n} \simeq b_{n} \Rightarrow \circ\left(a_{1}, \ldots, a_{n}\right) \simeq \circ\left(b_{1}, \ldots, b_{n}\right) \tag{6}
\end{equation*}
$$

for all elements $a_{1}, \ldots, a_{n}, b_{1}, \ldots, b_{n}$ of the algebra in question.
Next let us note that, if (4) is a theorem scheme of a logic $T$, we may reasonably expect that the "congruence principle" (6) will hold for all closed $T$-matrices (after a little further manipulation relating conditionals and biconditionals, on plausible assumptions). But we cannot so easily come to this conclusion if replacement only holds in the form (5). For consider the particular case of a closed $R$-matrix in which $b \simeq c$, and let us ask ourselves, as an application of (6), whether $a \circ b \simeq a \circ c$. The answer is, "Yes". For, by definition, $b \rightarrow c$ and $c \rightarrow b$ are both designated. In $R, p \rightarrow q \rightarrow . q \rightarrow p \rightarrow . p \leftrightarrow q$ is a theorem, whence $b \leftrightarrow c$ is also designated, by the $\rightarrow E$ principle (2) for closed matrices. But, since $\circ$ is an intensional connective, the appropriate instance $p \leftrightarrow q \rightarrow . r \circ p \leftrightarrow . r \circ q$ of (4) is an $R$-theorem, whence, appealing again to (2), $a \circ b \leftrightarrow a \circ c$ must be designated also, after which, appealing still to nothing but $R$-theorems and (2), it readily follows that $a \circ b \simeq a \circ c$. Other cases are similar, for all intensional connectives of $R$ (i.e., the connectives of $R_{i}$ ). So, $\simeq$ is a congruence on any closed $R_{i}$-matrix. Accordingly, given a closed $R_{i}$-matrix, we may straightforwardly reduce it, identifying congruent elements and getting a quotient matrix $M / \simeq=\langle M / \simeq, O / \simeq, D / \simeq\rangle$ defining the operations in $O / \simeq$ by "representatives", and noting in particular that no confusion resulted in the definition of $D / \simeq$, since, again applying (2), if $a \simeq b$ in a closed matrix then $a \in D$ iff $b \in D$.

Let us now ask the same question where the matrix operation in question is truth-functional $\&$, assuming $b \simeq c$ in a closed $R$-matrix, and wondering whether $a \& b \simeq a \& c$. The above argument breaks down on the point that the appropriate instance of (4), namely $p \leftrightarrow q \rightarrow . r \& p \leftrightarrow . r \& q$, cannot be an $R$-theorem, since \& is truth-functional. And what this means, really, is that understanding the closure of an $R$-matrix with reference to modus ponens alone is insufficient to assure the congruence properties necessary for reduction.

There is a remedy for this breakdown of reduction, actually pursued in [5], and which readers of [1] might take to be the natural one. For $R$ and other relevant logics are formulated not with one rule but with two, adding $\& I$ to the customary $\rightarrow E$ rule. And we have already seen that the "algebras" of relevant logics respect $\& I$ as much as they respect $\rightarrow E$; the appropriate condition to add on matrices is, of course, just

$$
\begin{equation*}
a \in D \text { and } b \in D \Rightarrow a \& b \in D \tag{7}
\end{equation*}
$$

Let us call a matrix which satisfies both (2) and (7) strongly closed, since such a matrix would ordinarily be considered a strong matrix for the relevant logics. Attending to strongly closed matrices overcomes our congruence problems, since it enables us to use (5) (or an approximation thereof, even in the case where $t$ is not an explicit primitive) where (4) was used before, whence,
e.g., the matrix $\simeq$ is readily shown to be a congruence, with respect to all operations, whenever $M$ is a strongly closed $R$ matrix (or $R+$ matrix, $E$ matrix, etc.). Under such circumstances we can pass to quotients; we may also express our previous observation more exactly as "The DeMorgan monoids are exactly the reduced, strongly closed $R$-matrices, where $R$ is formulated with $t$ explicit", an observation to be extended mutatis mutandis to other algebras and systems.

Note, incidentally, the great advantages that are conferred by sticking to reducible matrices, and then reducing them. In the first place, an unreduced reducible matrix (i.e., one on which $\simeq$ is a nontrivial congruence) is in important ways superfluous; for any purpose for which we are likely to want a matrix-e.g., to reject nontheorems efficiently-is better served by a reduced matrix than by an unreduced one of which it is the quotient. For, as one sees immediately, any interpretation in the unreduced matrix is directly mirrored in its quotient, refuting and validating the same formulas. So if we had, for example, a computer program to churn out useful strongly closed $R$-matrices, we should want it to avoid the unreduced such matrices completely, lest the product be mainly a lot of waste paper. Moreover, a really important property of the matrices that have turned out to be "relevant algebras" is that they are so readily visualizable, via partial orders and Hasse diagrams (such as strew the pages of [1]), enabling one to take in at a glance the important structural features of a given model of a relevant logic.

However, all these advantages notwithstanding, there is something disturbing about the effects in $R$ of the $\& I$ rule. It is, after all, a mere truth-functional rule, not supported by any more convincing thesis of the system $R$ itself than $A \& B \rightarrow . A \& B$. (See [2] for discussion of this point.) More to the immediate point, mixing the rules $\& I, \rightarrow E$ in the deduction of theorems means that we lose some control of the deductive process; this is true even though, on a slight reformulation of $R$, all applications of the $\& I$ rule may precede any applications of $\rightarrow E$ (a point which I owe in part to Belnap). This complicates, for example, the decision problem. And even more to the immediate point, it apparently complicates also the problem of finding an efficient refutation of a given nontheorem, on which topic we are presently dwelling. So there may yet be a future, in metalogical investigations into relevant logics, for matrices that are closed but not strongly closed, despite the partial loss of congruence properties. For $R_{i}$ and its fragments and subsystems, the question does not arise, since here we have only $\rightarrow E$ to worry about anyway, while, as noted, reduction via matrix equivalence is always possible.

2 Having disposed of the preliminaries and accompanying observations, we can now get on with the main business of this note: efficient refutation of nontheorems of $R_{\rightarrow}$ and its kin. The method is that suggested by the passage from the 4 -diamond to the 4 -chain. The key is contained in the following lemma, which we state and prove first for $R_{\rightarrow}$, noting thereafter the $R_{\rightarrow}$ properties on which it depends, that we may draw the appropriate generalizations.

Finite matrix shrinking lemma for $R_{\rightarrow} \quad$ Let $M=\langle M, O, D\rangle$ be a closed, finite $R_{\rightarrow}$ matrix. Set $U=M-D$, and let $a \in U$. Then there is a closed, finite reduced
$R_{\rightarrow}$ matrix $M^{\prime}=\left\langle M^{\prime}, O^{\prime}, D^{\prime}\right\rangle$, with the following properties: (i) $M^{\prime}$ is a matrixhomomorphic image of $M$, in the sense that there exists a matrix homomorphism h from $M$ onto $M^{\prime}$; (ii) Setting $U^{\prime}=M^{\prime}-D^{\prime}$, ha $\in U^{\prime}$; more than that, $U^{\prime}$ is the principal ideal $[\leqslant h a]$ in $M^{\prime}$, so that, since $M^{\prime}$ is reduced, ha is the unique greatest undesignated element of $M^{\prime}$; (iii) $M^{\prime}$ can be shrunk no further without designating $a$; i.e., where $h^{\prime}$ is a matrix homomorphism (but not an isomorphism) onto a finite closed $R_{\rightarrow}$ matrix $M^{\prime \prime}=\left\langle M^{\prime \prime}, O^{\prime \prime}, D^{\prime \prime}\right\rangle$, then $h^{\prime}(h a) \in D^{\prime \prime}$.

Proof: We shall construct $M^{\prime}$ from $M$. Our construction splits into two stages. First, we construct a finite sequence of matrices $M=M_{0}, \ldots, M_{n}$, keeping the same base set $M$ and simply adding more designated elements, to form the increasing sequence $D=D_{0} \subseteq \ldots \subseteq D_{n}$, taking care not to put in any $D_{i}$ the element $a$ which we wish to keep undesignated. When this stage is completed, we then simply reduce $M_{n}$ modulo its matrix equivalence $\simeq$, forming $M^{\prime}$. We then must show that $M^{\prime}$ has the properties that we have claimed for it.

In the first place, it is trivial, for any $T$-matrix whatsoever, that it remains a $T$-matrix on designating more elements. For if all the theorems of $T$ are confined on interpretation to some subset $D$ of $M$, of course they are also confined to any superset $D^{\prime}$ of $D$. So our construction can only be interesting if it shows the proper respect for modus ponens by taking us from one closed matrix to another. And it is interesting that $R \rightarrow$ seems to have just about the right supply of theorems for the purpose, as we shall see.

Suppose then that we have arrived at stage $i$ of the suggested construction, at which we have a closed $R_{\rightarrow}$ matrix $M_{i}=\left\langle M, O, D_{i}\right\rangle$, where $M, O$, are as in our original matrix $M_{0}$, our original $D \subseteq D_{i}$, while the particular element $a$ that we are keeping undesignated fails to belong to $D_{i}$. The matrix equivalence $\simeq$ is defined on $M_{i}$, partitioning $M$ into equivalence classes. We keep track of these classes by using ' $B$ ' for the class which contains a particular element $b$ of $M$, etc.; in particular, $A$ is $\left\{c: c \simeq a\right.$ in $\left.M_{i}\right\}$, for our permanent "bad guy" $a$.

It may be that $a$ is already a greatest undesignated element of $M_{i}$; in that case, go on to stage 2 . Otherwise, I assert, there is some maximally undesignated element $b$ of $M$, such that $b \notin A$. For, since $M$ is finite, and since $U_{i}=$ $M-D_{i}$ is nonempty (containing at least $a$ ), and since $\leqslant$ is transitive ( $R_{\rightarrow}$ property) and reflexive (ditto), there exist, evidently, maximal elements of $U_{i}$. Indeed, for each $c$ in $U_{i}$, there is some $U_{i}$-maximal $b$ such that $c \leqslant b$. And if all the $U_{i}$-maximal $b$ belong to $A, a$ is already $U_{i}$-greatest.

So, on the assumption that we still have work to do, choose any $b$ that is $U_{i}$-maximal but not equivalent to $a$, and form $D_{i+1}$ by adding $b$ and all its $M_{i}$-equivalents to $D_{i}$; i.e., $D_{i+1}=D_{i} \cup B . M_{i+1}=\left\langle M, O, D_{i+1}\right\rangle$. On our trivial observation, $M_{i+1}$ remains an $R_{\rightarrow}$ matrix, and we need only show that it is closed; i.e., that $D_{i+1}$ remains a filter in $M_{i+1}$. So let us assume that $c$ and $c \rightarrow d$ both belong to $D_{i+1}$. We must show $d \in D_{i+1}$. There are four cases.

Case 1. $c \in D_{i}$ and $c \rightarrow d \in D_{i}$. In this case, since $M_{i}$ is closed on assumption, $d \in D_{i} \subseteq D_{i+1}$.
Case 2. $c \simeq b$ and $c \rightarrow d \in D_{i}$, where $\simeq$ is $M_{i}$ equivalence. This implies $b \leqslant c$ and $c \leqslant d$ in $M_{i}$, whence, since $\leqslant$ is transitive in the closed $R_{\rightarrow}$ matrix $M_{i}, b \leqslant d$
in $M_{i}$. But we chose $b$ as $U_{i}$-maximal, whence either $d \in D_{i}$ or $d \simeq b$. In either case, $d \in D_{i+1}$.
Case 3. $c \in D_{i}$ and $b \simeq c \rightarrow d$ in $M_{i}$. We observe that, in virtue of the $R_{\rightarrow}$ theorem $p \rightarrow . p \rightarrow q \rightarrow q, c \rightarrow d \rightarrow d \in D_{i}$ (since $D_{i}$ is a filter), whence, in $M_{i}$, we have both $b \leqslant c \rightarrow d$ and $c \rightarrow d \leqslant d$, whence, again by transitivity of $M_{i} \leqslant$ and $U_{i}$-maximality of $b, d \in D_{i+1}$ for the same reason as in the previous case.
Case 4. $c \simeq b$ and $b \simeq c \rightarrow d$. This is, apparently, the interesting case, but the most interesting thing about it is that it cannot arise, on account of the $R_{\rightarrow}$ contraction principle. For, again by definition of $\simeq$ and transitivity of $\leqslant$, we have $c \leqslant c \rightarrow d$ in $M_{i}$; i.e., $c \rightarrow . c \rightarrow d \in D_{i}$. We observe that, in virtue of the $R_{\rightarrow}$ theorem $(p \rightarrow . p \rightarrow q) \rightarrow . p \rightarrow q$, and the fact that $D_{i}$ is a filter, $c \rightarrow d \in D_{i}$; i.e., $c \leqslant d$ in $M_{i}$. But since $c \rightarrow d \in D_{i}$ and $c \rightarrow d \leqslant b$, the latter by the hypothesis of the case, $b \in D_{i}$, since $D_{i}$ is still a filter. But we chose $b$ as an undesignated element of $M_{i}$, whence, indeed, the case does not arise.

This exhausts the cases, and assures that we can continue enlarging our original $D$ until we reach an $M_{n}=\left\langle M, O, D_{n}\right\rangle$ in which $a$ is a greatest undesignated element. (Evidently we reach $M_{n}$ in a finite number of steps, since $M$ was finite to begin with, whence we shall eventually run out of further undesignated elements to add.) We now observe, for reasons noted in Section 1, that the $\simeq$ of $M_{n}$ is a congruence (with respect to the only operation of $R_{\rightarrow}$, namely $\rightarrow$ ), whence we may pass to the quotient matrix $M_{n} / \simeq=M^{\prime}=$ $\left\langle M^{\prime}, O^{\prime}, D^{\prime}\right\rangle$, also for reasons noted. We must now verify the various statements of our lemma concerning $M^{\prime}$.

First, let $h$ be the natural homomorphism from $M_{n}$ onto $M^{\prime}$. Since $M_{n}$ is closed, evidently $M^{\prime}$ is also closed; equally evidently, it is finite and reduced. Since, moreover, $h$ is a matrix homomorphism from $M_{n}$ to $M^{\prime}$, it is also a matrix homomorphism from our original matrix $M$ to $M^{\prime}$ (since, if $h$ preserves $D_{n}$, it preserves its subset $D$ ). This establishes (i) of the conclusion of our lemma.

To show (ii), we recall that $a$ was $U_{n}$-greatest, whence $h a$ will be $U^{\prime}$ greatest (and hence unique, since $M^{\prime}$ is reduced).

Finally, we consider (iii). We need to show that any proper homomorphic image $M^{\prime \prime}$ of $M^{\prime}$, where $M^{\prime \prime}$ is a closed $R_{\rightarrow}$ matrix and $h^{\prime}$ is a matrix homomorphism carrying $M^{\prime}$ onto $M^{\prime \prime}$, contains $h^{\prime}(h a)$ as a designated element. Let $D^{*}=h^{\prime-1}\left(D^{\prime \prime}\right)$; i.e., for all $b \in M^{\prime}, b \in D^{*}$ iff $h^{\prime} b \in D^{\prime \prime}$. Since matrix homomorphisms preserve designated elements, $D^{\prime} \subseteq D^{*}$. First, suppose $D^{\prime}=D^{*}$. Then, I assert, $M^{\prime \prime}$ is an isomorphic copy of $M^{\prime}$. To do this, it suffices to show that if $h^{\prime} b=h^{\prime} c$ in $M^{\prime \prime}, b=c$ in $M^{\prime}$ (for then $h^{\prime}$ will be a bijection from $M^{\prime}$ to $M^{\prime \prime}$, preserving operations in $O$ and, by supposition, preserving both $D^{\prime}$ and its complement). For suppose $h^{\prime} b=h^{\prime} c$. Then, in view of the $R$ theorem $p \rightarrow p, h^{\prime} b \rightarrow$ $h^{\prime} c=h^{\prime}(b \rightarrow c) \in D^{\prime \prime}$, whence $b \rightarrow c \in D^{*}$, which is $D^{\prime}$ by supposition. So $b \leqslant c$ in $M^{\prime}$, and, by parity of reasoning, $c \leqslant b$; i.e., $b \simeq c$ in $M^{\prime}$ and, since $M^{\prime}$ is reduced, $b=c$. So, in this case, $M^{\prime \prime}$ is not a proper homomorphic image of $M^{\prime}$, but an isomorphic copy. For the other case, suppose that $D^{\prime}$ is a proper subset of $D^{*}$. First, we show that $D^{*}$ is a filter in $M^{\prime}$. For suppose $b \in D^{*}$ and $b \leqslant c$. Then $h^{\prime} b \leqslant h^{\prime} c$ in $M^{\prime \prime}$ (as we observed in Section 1), whence, since $M^{\prime \prime}$ is closed, $h^{\prime} c \in D^{\prime \prime}$ and so $c \in D^{*}$. So $D^{*}$ is a filter, which, as a proper superset of $D^{\prime}$, contains some undesignated element $b$ of $M^{\prime}$. But $h a$ is the greatest undesig-
nated element of $M^{\prime}$, whence $b \leqslant h a$, whence, by filterhood of $D^{*}$, $h a \in D^{*}$, whence, by definition, $h^{\prime}(h a) \in D^{\prime \prime}$, completing the proof of the finite matrix shrinking lemma for $R_{\rightarrow}$.

The idea of our shrinking lemma is that, given any refutation of a nontheorem $A$ of $R_{\rightarrow}$ at an undesignated element $a$ of a closed $R_{\rightarrow}$ matrix $M$, we may always shrink $M$ so that $a$ becomes the greatest undesignated element of the shrunken matrix, in effect. Note that every application of the shrinking lemma either increases the number of designated elements, or identifies previously distinct elements; the passage from the 4 -diamond to the 4 -chain, taking the former as the $M$ of the lemma and the latter as $M^{\prime}$, with $F$ the element to be kept undesignated, is an application of the first kind. Such applications always change the matrix partial order $\leqslant$, in the "flattening" way observed in the passage from the 4 -diamond to the 4 -chain. For observe that, in a closed $R_{\rightarrow}$ matrix, $b \rightarrow b \leqslant b$ holds iff $b$ is designated, whence increasing the number of designated elements also makes the relation $\leqslant$ hold more often.

The proof of the shrinking lemma raises a number of related questions. Those to which I know the answers turn out negative, which may furnish a clue to the remainder. Our sample case, after all, terminated with a chain. Could it be, accordingly, that, given any closed $R_{\rightarrow}$ matrix, there is some way to keep adding new designated elements, reducing as we go, until the result is a chain? No, it could not be. For consider the matrix $M_{0}$ of [1], taken simply as an $R \rightarrow$ matrix. $M_{0}$ has a least element -3 , which will remain undesignated in all closed homomorphic images of $M_{0}$ but the trivial one. Moreover, we have for the distinct elements $+1,+2$ of $M_{0},+1 \rightarrow+2=+2 \rightarrow+1=-3$. So, shrink as we will to closed nontrivial $R_{\rightarrow}$ matrices, +1 and +2 will remain incomparable under matrix $\leqslant$, whence $M_{0}$ cannot be shrunk to a chain.

At this point, some empirical evidence intrudes. John Slaney has been writing programs that find $R$ matrices by the bucketful. And he reports that, even among rather large matrices, the number of chains is somewhat staggering; e.g., about a third of the $10 \times 10$ matrices are chains, by far the largest contribution to the supply of DeMorgan monoids from underlying DeMorgan lattices. We might interpret this phenomenon to mean that, even if the shrinking process does not invariably produce chains, it very often does. And some of the most interesting and useful matrices are chains-e.g., the Sugihara matrices of [1], which we shall discuss further below. And, while specialization to chains is not generally attractive for $R$, since \& and v must also be catered for, one might wonder whether, in $R_{\rightarrow}$ itself, every nontheorem is refutable in some chain. Many nontheorems are; all 1-variable nontheorems of $R_{\rightarrow}$, for example, are refutable in the 4 -chain. To get a counterexample to the hypothesis that every nontheorem of $R_{\rightarrow}$ is chain-refutable, consider the Church disjunction $A \oplus B={ }_{d f} A \rightarrow B \rightarrow . B \rightarrow A \rightarrow A$, introduced in [4]. Note that $(p \rightarrow q) \oplus(q \rightarrow p)$ is valid in every closed $R \rightarrow$ matrix for which matrix $\leqslant$ is a total order. For one of $p \rightarrow q, q \rightarrow p$ must be designated in such a matrix, whence, since Church disjunctions are relevantly implied by their disjuncts, the displayed disjunction must be chain-valid. But it is refutable in $M_{0}$ on assigning +1 to $p$ and +2 to $q$.

Our lemma shows that every nontheorem of $R_{\rightarrow}$ is, in a certain sense, maximally refutable. A related question is, "Is every nontheorem minimally
refutable?", in the sense that every nontheorem can be refuted at the least value in some closed $R_{\rightarrow}$ matrix. An equivalent question is, "Can every nontheorem be rejected in a closed $R_{\rightarrow}$ matrix with a lone undesignated element?" For what it is worth, Slaney's evidence shows that such matrices are uncommonly common, while, again, all 1 -variable nontheorems have this property. While, maybe, one expects that the answer is negative, the question is interesting. Consider the case of Peirce's law, $p \rightarrow q \rightarrow p \rightarrow p$. This isn't valid, either, in the intuitionist logic $J$, which is a supersystem of $R_{\rightarrow}$, so that we can try looking at a $J$ refutation. Here's one, in the familiar matrix $J 3$.

$\rightarrow$ is defined familiarly on $J 3$ by $a \rightarrow b=T$ if $a \leqslant b, T \rightarrow b=b$ always, and, otherwise, $a \rightarrow b=F . T$ is the only designated value. As a $J_{\rightarrow}$ matrix, $J 3$ is a fortiori a closed, reduced $R_{\rightarrow}$ matrix.

Peirce's law is refutable in $J 3$, but on only one assignment. For $N \rightarrow F \rightarrow$ $N \rightarrow N=N$, yielding a maximal (and maximally efficient) refutation in terms of our shrinking lemma. Question: can we manipulate $J 3$ further so as to refute Peirce's law at $F$ ? Answer: No, because any further manipulation, of the above sort, must designate $N$ also, given the argument for (iii) of our lemma, whence there will be no refutation whatsoever. Moreover, in no reduced closed $J$ matrix $M$ whatsoever is Peirce's law refutable, if there is only one undesignated element; for the only such $M$ is truth-tables, and Peirce's law is a truthtable tautology.

However, this does not settle the matter for $R_{\rightarrow}$. In fact, there are lots of $R_{\rightarrow}$ matrices which are not $J$ matrices. The most justly famous of them all, the 3-point Sobociński (or Sugihara) matrix $S 3$, has a Hasse diagram just like $J 3$, but so defines $\rightarrow$ that $N \rightarrow N=N$, but, otherwise, $a \rightarrow b=T$ if $a \leqslant b$ and $a \rightarrow b=F$ if $a>b$, designating both $T$ and $N$. This time, $N \rightarrow F \rightarrow N \rightarrow N=F$, whence Peirce's law is $R$-refutable at a lone $F$ after all. In fact, since $S 3$ is characteristic for the intensional part $R M_{i}$ of $R M$, as Parks and I (independently) showed, those nontheorems of $R_{\rightarrow}$ that fail also in $R M_{\rightarrow}$ are certainly refutable at $F$, including, naturally, all classical nontautologies. So it would be interesting to know whether this can always be done, sharpening for $R_{\rightarrow}$ the main result here.

Mention of $J_{\rightarrow}$ matrices prompts the following observation:
Fact: A closed reduced $R_{\rightarrow}$ matrix is a $J_{\rightarrow}$ matrix iff it has exactly one designated value. The same holds, mutatis mutandis, for $R+$ matrices.

Reason: Suppose that $M$ is a closed reduced $R_{\rightarrow}$ matrix with but one designated value $d$. It will suffice to validate the paradox $A \rightarrow B \rightarrow B$, since, added to $R_{\rightarrow}$, this produces $J_{\rightarrow}$. For this it suffices that, for each $a$ in $M, a \leqslant d$. In fact, $a \leqslant(a \rightarrow . a \rightarrow a) \rightarrow . a \rightarrow a=d$, by the identity, permutation, and contraction principles of $R$ and the fact that $d$ is the only undesignated element. Conversely, the noisome paradox $B \rightarrow . A \rightarrow B$ assures, as is well-known, that closed, reduced $J_{\rightarrow}$ matrices will have but one designated value, which, a fortiori, makes them $R_{\rightarrow}$ matrices with this property.

Further Fact: A closed reduced $R_{i}$ matrix with exactly one designated value is a Boolean algebra, validating exactly the classical tautologies. The same holds, mutatis mutandis, for $R$ matrices.

Further Reason: For the same reason as above, the noisome paradox $A \rightarrow$. $B \rightarrow A$ is valid in any $R_{i}$ matrix with exactly one designated value, after which all classical tautologies in the $R_{i}$ vocabulary must be valid (in view of the standard properties of the DeMorgan negation of $R$ ). But $\rightarrow, \sim$ is a sufficient basis for classical logic, whose closed, reduced matrices are exactly the Boolean algebras. Enough said.

Dunn has suggested in conversation that the above fact and further fact may be known (mentioning Pahi in particular as an author who has thought about these things). I wouldn't be surprised. But it is at least interesting that, for a closed reduced matrix to be properly relevant, it is both sufficient and necessary that more than one proposition should be considered true. For the root of both the elegance and the absurdity of the conventional logical wisdom may well be located in the assumption that all Truths are One. And it is interesting to find technical reflection of this assumption in the fact that positive $R$ matrices suffer intuitionist breakdown, and full $R$ matrices suffer classical breakdown, when it is made. (In the mutatis mutandis clauses of our fact and further fact, incidentally, one should take, I suppose, the $R+$ and $R$ matrices in question to be strongly closed, in which case, if there is only one designated value and the matrix is reduced, \& and $\circ$ will coincide, so that the extensional case reduces to the intensional one.)

3 In this section, we generalize our principal lemma of the last section and draw appropriate conclusions. For the proof of this lemma is quite general, and depends only in a few places on particular properties of $R_{\rightarrow}$, whence it may also be asserted of any supersystems of $R_{\rightarrow}$ that retain these properties, with or without additional connectives, or constants. The theorems of $R_{\rightarrow}$ to which we appealed were the following:

| B axiom | $q \rightarrow r \rightarrow . p \rightarrow q \rightarrow . p \rightarrow r$ |
| :--- | :--- |
| CI axiom | $p \rightarrow . p \rightarrow q \rightarrow q$ |
| W axiom | $(p \rightarrow . p \rightarrow q) \rightarrow . p \rightarrow q$ |
| I axiom | $p \rightarrow p$. |

In addition, we appealed implicitly to the fact that the set of theorems of $R_{\rightarrow}$ is closed under substitution for sentential variables (since this is built in, more or less, to the matrix approach) and under modus ponens (since this is what gives us a preference for closed matrices). These were the only assumptions used in the part of our proof in which, given an undesignated element $a$ of a closed matrix $\langle M, O, D\rangle$, with $M$ finite, we were able to go on adding new designated elements until $a$ was a greatest undesignated element, in the closed matrix $\left\langle M, O, D_{n}\right\rangle$. Since, in fact, the axioms to which we appealed are exactly sufficient for $R_{\rightarrow}$, that part of the proof will go through for any supersystem of $R_{\rightarrow}$ whatsoever, with whatever connectives one pleases (in addition to $\rightarrow$ ). So

Corollary 1 Let $S$ be any system among whose theorems are the B, CI, $W$, and $I$ axioms above. Let $M=\langle M, O, D\rangle$ be a closed, finite $S$ matrix. Set $M-D=U$, and let a $\in U$. Then there is a closed $S$ matrix $M_{n}=\left\langle M, O, D_{n}\right\rangle$ such that $D \subseteq D_{n}$ and $U_{n}=M-D_{n}=[\leqslant a]$.

In the interesting cases, $S$ will be, as noted, an extension of $R_{\rightarrow}$, closed under modus ponens and substitution. Among such extensions are $R_{i}, R, R+$ (and the Boolean extensions $C R^{*}, C R$ thereof), $R M_{\rightarrow}, R M, J_{\rightarrow}, J, D, K$, and the various extensions of these systems. But the utility of Corollary 1 is considerably less than that of the lemma, since, although we get in general a closed $S$ matrix $M_{n}$, in which $a$ is maximally undesignated, we may not be permitted to reduce $M_{n}$ to get, usually, something simple. So let us examine the particular features of $R_{\rightarrow}$ that permit also the reduction step of the proof of our lemma. From the discussion of Section 1, we already know, pretty well, what they are. The key point is that the matrix equivalence $\simeq$, when we arrive at it, must be a congruence on $M_{n}$. A sufficient (and, in the usual cases, necessary) condition for this is that replacement of equivalents in the form (4) of Section 1 shall be guaranteed by a theorem scheme of the system $S$ in question. Let us restate this condition to avoid its apparent dependence on a $\leftrightarrow$ connective, calling it the congruence condition for the system $S$.
$p \rightarrow q \rightarrow . q \rightarrow p \rightarrow . A(p) \rightarrow A(q)$ is a theorem of $S$, for each formula $A(p)$ in the vocabulary of $S$ in which $p$ occurs, and for each result $A(q)$ of substituting $q$ for exactly one occurrence of $p$ in $A(p)$.
Congruence Fact: Let $S$ be any system satisfying, without restriction, the congruence condition (8), and let $M$ be any closed $S$-matrix. Let moreover the Transitivity Axiom B and the Reflexivity Axiom I above be theorems of $S$. Then the matrix equivalence $\simeq$ is a congruence on $M$, whence $M$ may be reduced modulo this congruence to get an equivalent closed $S$-matrix $M / \simeq$.

Reason: That $\simeq$ is a congruence means that (6) of Section 1 holds, for each operation $\circ$ of $M$, and that $\simeq$ itself is an equivalence relation. For the latter, we have already noted that Axioms $B$ and $I$ suffice, in closed matrices. And we merely illustrate the former, choosing $v$ as a binary operation and showing that, in the presence of the congruence condition, $a \simeq c$ and $b \simeq d$ suffice for $a \vee b \simeq c \vee d$, in the closed $S$-matrix $M$. In fact, $p \rightarrow q \rightarrow . q \rightarrow p \rightarrow .(p \vee r) \rightarrow$ ( $q \vee r$ ) is an instance of (8), choosing $A(p)$ as $p \vee r$, whence $a \rightarrow c \rightarrow . c \rightarrow a \rightarrow$. $(a \vee b) \rightarrow(c \vee b) \in D$, since $M$ is an $S$-matrix. Since the two antecedents belong to $D$, on assumption, and since $M$ is closed, $(a \vee b) \leqslant(c \vee b)$. Similarly, $(c \vee b) \leqslant(a \vee b)$. Since $p \rightarrow q \rightarrow . q \rightarrow p \rightarrow .(r \vee p) \rightarrow(r \vee q)$ is likewise an instance of (8), we can also show $c \vee b \simeq c \vee d$, whence, by transitivity of $\simeq a \vee b \simeq$ $c \vee d$, as claimed. So the situation is familiar enough, and any further verification is left to the reader.

Corollary 2 Let $S$ be any extension of $R_{\rightarrow}$, with perhaps additional connectives, for which the congruence condition (8) holds. Then the finite matrix shrinking lemma holds for $S$.

Proof of Corollary 2 is hardly needed, since, given Corollary 1 and the Congruence Fact, proof is exactly as of the lemma, just putting ' $S$ ' wherever
' $R_{\rightarrow}$ ' appears in the statement or proof of that lemma. Thus nontheorems are maximally refutable, in a finite reduced, closed matrix, in many famous and infamous systems; among them are $R_{\dashv}, R_{i}, R M_{i}$, and our old friends $D, J$, and $K$. For the old friends, the "finitude" part is established by well-known results; for $R_{\rightarrow}$ and $R_{i}$, it is established by little known results, namely:

Finitude Fact for $R_{\rightarrow}$ and $R_{i}: \quad R_{\rightarrow}$ and $R_{i}$ have the finite matrix property; i.e., every nontheorem of these systems is refutable in a closed, finite matrix for the system.

Proof of the finitude fact is rather intricate, being established in the draft manuscript [8] by adaptation of the Kripke proof-theoretic argument for the decidability of $R_{\rightarrow}$. So I state it here merely as a fact. But note that, given the finitude fact, we can find a maximally efficient finite refutation of a given nontheorem, simply by applying our lemma. And the result will always be that a nontheorem is refuted at a greatest undesignated element in a closed, reduced matrix, since, if it isn't, we can simply cut the matrix down still further. And, of course, some refutation will also be maximally efficient, since there will be a closed, reduced matrix of smallest finite size in which the maximal refutation takes place. Really efficient decision procedures, of course, may be another story; even for $J$, the extant decision procedures (of which I know, anyway) quickly become quite cumbersome; and the situation for $R_{i}$, say, is at writing even worse. But any improvements can themselves be improved by the method sketched here.

4 In this section, I turn to the limits of these methods. An immediate one lies in the method by which we proved our shrinking lemma. To get a best refutation, we added new designated elements in no particular order, save that we always chose maximally undesignated ones to add. We can, of course, try out all orders. But, as a practical matter, it would be good to have some criterion on which we can make a "best choice", if our methods are to be at all efficient.

A second limitation is that the above reasoning only applies to reasonably strong systems, and that it depends on both the CI principle (which fails in $E$ ) and on the $W$ principle (which fails in many systems, including, not surprisingly, $R-W$ ). These principles were needed to get us through the "cases" 3 and 4 respectively in the proof of the key lemma. Accordingly, it would be particularly interesting to know what form, if any, our lemma takes in weaker relevant logics (or irrelevant ones, for that matter).

A more important limitation is the dependence on finitude, in the matrices from which we start. In fact, we can state our lemma so that it avoids such dependence. Here is the idea. Instead of designating new matrix elements, one at a time, we may pick any matrix filter $F$, and designate $F \cup D$ at one fell swoop, provided that the filter in question is directed down; i.e., provided that any two elements of $F$ have a lower bound in $F$, on the matrix entailment ordering relation $\leqslant$ On the same $R_{\rightarrow}$ principles as before, this process takes us from closed matrices to closed matrices. So here is a plan to turn an arbitrary $R_{\rightarrow}$ matrix $M$ into one in which a given undesignated element $a$ is maximally
undesignated, reminiscent of how completeness proofs go. $D$ being fixed and set equal to $D_{0}$, well-order the elements of $M$. Let $a$ be set aside to be kept undesignated. Suppose that a matrix $M_{i}$ has been defined, on the same base set $M$, with matrix entailment $\leqslant_{i}$. Pick any element $b$ such that $b \$_{i} a$, and let $D_{i+1}=D_{i} \cup[b \leqslant]$. This time, the reader may verify that the process takes us from closed matrices to closed matrices, letting $D_{i+1}$ serve as the new set of designated elements, where we had an $R \rightarrow$ matrix to start with. Since we are not assuming finitude any more, every so often this process will lead us to a limit ordinal, and we will have to sum up, taking $D_{i}$ at the limit as the union of all its predecessors. For familiar reasons, such $D_{i}$ will still be closed, and will still lack $a$. At some point, we must run out of further undesignated elements to add, so that, at this last gasp, all undesignated elements will be $\leqslant a$, providing a maximal, but not necessarily finite, refutation of any nontheorem of $R_{\rightarrow}$ refutable at $a$. And this situation also can be generalized in the spirit of Corollaries 1 and 2, with the reducibility of our last $M_{i}$ governed by the universal theoremhood of the formulas (8).

We can, in particular, go through the process just sketched with respect to the Lindenbaum matrix of whatever extension $S$ of $R_{\rightarrow}$ we are considering. This time, however, the process of choosing the $b$ whose principal filters are to be added becomes a real headache. For we should like to make, at each stage, a finitizing choice, if we can; that is, we should like to arrive at a $D_{i}$ such that, if we reduced $M_{i}$ modulo a suitable congruence (which can be matrix equivalence when $S$ satisfies the congruence condition), the result would be a closed, finite matrix for $S$. It is not immediately apparent, however, how we can make such a choice, except through some other argument, if there is one, that $S$ has the finite matrix property. Somewhere, there must be a better result than this, making use of exactly the $R \rightarrow$ properties, and applied to some class of extensions of $R_{\rightrightarrows}$. But I shall not pursue it any further here (though [8], perhaps, is the source of some clues).

Finally, I turn to the problem of applying these results to the system $R$ itself. Here, the first problem is that, in contrast to the systems $J$ and $K$, there is no useful formulation of $R$, for our purposes, with $\rightarrow E$ as sole rule. And the second problem, already noted, is that the congruence property (8). fails for $R$. The final problem, also already noted, is that no finite matrix theorem has been proved for $R$; indeed, the system may be undecidable, in which case there is no such (useful) theorem. So, even if we bring the finite matrices for $R$ under control, it is yet uncertain how much we have accomplished.

The central problem, if we try to prove our theorem for $R$ and not just for $R_{\rightarrow}$ or $R_{i}$, lies in caring for the adjunction rule. This is not a problem for $R M$, for the following reason:

RM Observation: Every nontheorem $A$ of $R M$ is maximally refutable. In fact, if $A$ is a nontheorem of $R M$, there is, in the sense of [1], a finite Sugihara matrix $S_{n}$ (where $n$ is the number of matrix elements) such that: (a) $A$ takes the value -1 on some interpretation in $S_{n}$, and (b) for all $m<n, A$ is valid in $S_{m}$. (Since all the $S_{n}$ are finite, reduced, strongly closed $R M$ matrices, the refutation in question is in an evident sense a best refutation.)

Verification: If $A$ is a nontheorem of $R M$, it is refutable in some finite Sugihara matrix $S_{p}$, at some value $-q$. Applying the proof technique of our principal lemma, show that all the matrix elements $>-q$ may be added as new designated values. (The important point for immediate purposes is that our new $D$ will be closed under \&I also, since $S_{p}$ is a chain.) The matrix equivalence induced by the new $D$ is a congruence, whence we may pass to the quotient $S_{n}$. And it is easy to see that $S_{n}$ is also a Sugihara matrix, which has resulted from $S_{p}$ simply by identifying all elements $b$ in $S_{p}$ such that $-q<b<q$. Accordingly, on the natural matrix homomorphism $h$ from $S_{p}$ to $S_{n}$, all the elements between $-q$ and $q$ have been taken into 0 , whence $q$ becomes the new +1 , under $h$, and $-q$ becomes the new -1 . So, if $A$ is refutable in any Sugihara matrix $S_{p}$ at a value $<-1$, there is always a better (i.e., smaller) matrix in which $A$ is refutable at -1 . So the smallest $S_{n}$ in which $A$ is refutable at all will refute $A$ at -1 , completing the verification of our observation.

The proof technique underlying the above observation has its amusing aspects. First, as noted, the intensional part $R M_{i}$ of $R M$ also has a dual minimal refutability property; every nontheorem is refutable at the least element of some Sugihara matrix $S_{p}$. Combining this with the above observation, we get the result that Parks reported in [12]: i.e., every nontheorem of $R M_{i}$, being both minimally and maximally refutable, is already refutable in the 3-point Sugihara matrix $S 3$. There is also a connection with the work on extensions of $R M$ that Dunn sets out in [1]. For pick the smallest $n$ such that $A$ is refutable (at -1 ) in $S_{n}$, and consider the result $R M+A$ of adding $A$ as a new axiom scheme. $S_{n}$, of course, cannot be a matrix for $R M+A$. But $S_{n-1}$ is, naturally, an $R M+A$ matrix, and, as Dunn shows by an elegant argument, it is in fact a (finite) characteristic matrix for $R M+A$.

We may, in fact, recast Dunn's argument in the following form, trading in some of its algebraic features for syntactical ones. Let $B$ be a nontheorem of $R M+A$, where $A$ is a nontheorem of $R M$. Let $\alpha$ be the set of all instances of $A$ in the vocabulary of $B$ (i.e., only sentential variables that occur in $B$ occur in any member of $\alpha$ ). We note first that there is some finite conjunction $\& \alpha$ of members of $\alpha$ such that, for every $A^{\prime}$ in $\alpha, \& \alpha \rightarrow A^{\prime}$ is a theorem of $R M$. For, given the results that I reported in [1], it is evident (as Dunn has also observed) that there are only finitely many nonequivalent formulas of $R M$, in all, built out of the same sentential variables as $B$ (since $S_{2 m}$ is characteristic for the $m$-variable fragment of $R M$ ); a fortiori, there are only finitely many nonequivalent instances of $A$ in this vocabulary. So any formula $C$ in the $B$ vocabulary is a theorem of $R M+A$ iff it is deducible in $R M$, using the axioms and rules of $R M$, from the formula $\& \alpha$. Similarly, we may find a formula $\& R M$ that will stand in for the conjunction of all the theorems of $R M$ in the $B$-vocabulary. (In fact, the conjunction of the $p \rightarrow p$ (the well-known " $t$-surrogate"), where $p$ occurs in $B$, will serve for $\& R M$.)

Then, simply applying the appropriate deduction theorem, for all $C$ in the $B$-vocabulary, $C$ will be a theorem of $R M+A$ iff $\& \alpha \& \& R M \rightarrow C$ is already a theorem of $R M$ itself. We have chosen a $B$ which is a nontheorem of $R M+A$, whence by the completeness proof for $R M$, there is a finite Sugihara matrix $S_{p}$ in which $\& \alpha \& \& R M \rightarrow B$ may be refuted at the value -1 , by our $R M$ observation. Indeed, there is a smallest such $S_{p}$. In order for this to happen, on inspec-
tion of the "truth-tables" for finite Sugihara matrices, each of $\& \alpha, \& R M$ must take one of the values $0,+1$, while $B$ must have been assigned -1 in $S_{p}$.

It will now suffice for Dunn's result, in its present recension, if we can show $p<n$, where $S_{n}$ is the smallest Sugihara matrix in which $A$ itself is refutable. (For if a formula is refutable in any Sugihara matrix, it is also refutable in any larger one; whence, since $B$ is an arbitrary nontheorem of $R M+A$, if $p<n$ then $B$ will be refutable in $S_{n-1}$ in particular, clinching the claim that $S_{n-1}$ is characteristic for $R M+A$.)

To nail down $p<n$, it will suffice to show that $A$ is valid in $S_{p}$. (For $A$ is not valid in any Sugihara matrix from $S_{n}$ on.) Suppose, to the contrary, that $A$ is not valid. Then there is some refuting interpretation $I$ of $A$ in $S_{p}$, different from the interpretation $I^{\prime}$, perhaps, that we just picked to refute $\& \alpha \&$ $\& R M \rightarrow B$. But we shall show that, under these circumstances, $\& \alpha$ could not have been true on $I^{\prime}$, either.

First, we observe that, for each element $b$ of $S_{p}$, there is some sentential variable $q$ of $B$ such that $I^{\prime}(q)=b$ or $I^{\prime}(\sim q)=b$. (For matrix values assigned to neither sentential variables or their negates are assigned, on a Sugihara interpretation, to no formula, whence we could have got a smaller Sugihara matrix than $S_{p}$ which would have done the same job by dropping the superfluous elements. But $S_{p}$ is the smallest matrix that will do the job.) Accordingly, for our refuting interpretation $I$, and each sentential variable $r$ of $A$, there is a sentential variable $q$ of $B$ such that either $I(r)=I^{\prime}(q)$ or $I(r)=I^{\prime}(\sim q)$. Evidently, then, there is a substitution instance $A^{\prime}$ of $A$, in the vocabulary of $B$, such that $I(A)=I^{\prime}\left(A^{\prime}\right)$. Since, on the (reductio) assumption presently in force, $A$ is undesignated on $I, A^{\prime}$ is already undesignated on $I^{\prime}$. But we chose $\& \alpha$ so that it would $R M$-entail all instances of $A$ in the $B$ vocabulary, including $A^{\prime}$ in particular. Since Sugihara matrices respect $R M$-entailment this requires $I^{\prime}(\& \alpha)<0$ also, a possibility that we have already ruled out. Accordingly, $A$ is valid in $S_{p}$, whence $p<n$ and $B$ is refutable in $S_{n-1}$. So $S_{n-1}$ is characteristic for $R M+A$, as Dunn said that it would be.

We seem to have fallen a little short of the actual Dunn result, incidentally, which is that every extension of $R M$, closed under $\rightarrow E, \& I$, and substitution, has a finite characteristic matrix. But this is only seeming. For let $S$ be any proper extension of $R M$, and let $S_{n}$ be the smallest finite Sugihara matrix in which at least one theorem of $S$ is rejected. (For, since $S$ is a proper extension of $R M$, it contains at least one nontheorem of $R M$, which is rejectable in some finite Sugihara matrix.) Then, I assert, $S_{n-1}$ is characteristic for $S$ (or, more accurately, Dunn asserts). For let $A$ be a theorem of $S$ refutable in $S_{n}$. Then, by what we have already shown, $S_{n-1}$ is already characteristic for $R M+A$, since, by leastness of $n$, neither $A$ nor any other member of $S$ is rejectable in $S_{n-1}$. And so, in fact, $S$ and $R M+A$ coincide, as is now clear. (In all of this, incidentally, we have been tacitly avoiding the trivial extension of $R M$, of which everything is a theorem. But this extension also falls under the general rubric; its characteristic matrix consists of just 0 , and that designated, trivially validating all formulas. Letting this matrix be $S_{1}$, and noting that $S_{2}$ is truthtables, $S_{1}$ is characteristic for the theory that we get if we are silly to add a truth-functionally invalid axiom scheme to $R M$, as is widely known.)

That was quite a digression, but it makes a nice application of our princi-
pal lemma, when transmuted into our " $R M$ observation". Alas, the application is deeply $R M$-specific, and we return to our problems with $R$. But, in the $R M$ case, we got over our $\& I$ problems by utilizing special properties of chains, at least in the Sugihara case. For $R$, however, we have an outright counterexample to the finite matrix shrinking lemma, if we try to put " $R$ matrix" for " $R_{\rightarrow}$ matrix" and "strongly closed matrix" for "closed matrix" everywhere in that lemma.

For consider the case of the 4-diamond, now taken as a DeMorgan monoid (with lattice connectives) and henceforth dubbed $4 D$. Were the modified shrinking lemma true, there would be a homomorphic image $4 D^{\prime}$ or $4 D$, under a matrix homomorphism $h$, such that $h F$ is the greatest undesignated element of $4 D^{\prime}$, and $4 D^{\prime}$ is reduced and strongly closed. This is the situation that we were in before, when we passed from $4 D$ to the 4 -chain, but then we were only trying to preserve $\leqslant$ and the intensional operations.

At any rate, suppose that there is such an $h$ and such a $4 D^{\prime}$. We note first that $4 D^{\prime}$ (and any nontrivial matrix homomorphic image of $4 D$ which is closed and preserves $D$ and $\rightarrow$ ) must continue to have 4 elements, in view of $h T \rightarrow h f=$ $h T \rightarrow h t=h f \rightarrow h t=h t \rightarrow h F=h f \rightarrow h F=h T \rightarrow h F=h F$. Moreover, since $h$ preserves $\leqslant, h F$ remains least in $4 D^{\prime}$ also, whence, since it is also maximally undesignated, in $4 D^{\prime}, h F$ must be the only undesignated element in $4 D^{\prime}$. In particular, $h f$ is designated, whence, in $4 D^{\prime}, h t \leqslant h f$. But this produces the 4 -chain, as $4 D^{\prime}$, which, we have essentially shown, is the only nontrivial, proper reduced matrix homomorphic image of $4 D$, considering the former as an $\rightarrow$ matrix only. If we must attend to \& also, the matrix homomorphism snaps. For, in $4 D^{\prime}, h f \& h t=h F$ must continue to hold, whence the $\& I$ principle fails for $4 D^{\prime}$.

We are back now where we started-reflecting on the relations between the 4 -diamond $4 D$ and the 4 -chain $4 C$, considered as matrices for $R$. And it is now clear that we have not one choice but two for the \& and $v$ tables to be added to the intensional specifications on $4 C$, to make $4 C$ an $R$-matrix. In the first place, we may use its ordering under its own matrix $\leqslant$ to define \& and $v$ on $4 C$; this has the effect of making $f \& t=t$, whence $4 C$ is strongly closed. Or else we can continue to define $\&$ and $v$, even in $4 C$, in the $4 D$ way, with $f \& t=F$. Note that, on the latter plan, $a \& b$ remains a lower bound in $4 C$ for $a$ and $b$, but it is no longer a greatest lower bound, with respect to the $4 C$ matrix ordering.

Where we started, we shall end, since I see no present hope of giving further content, for the system $R$ itself, to the concept of "best" refutation which it has been the purpose of this note to set out. This is doubly ironic, since it is exactly the $R \rightarrow$ machinery on which we have depended to set out the concept. But not only does the key congruence condition (8) fail for $R$ on the lattice connectives $\&$ and $v$, but adding the theorems necessary to nail down this condition produces intuitionist breakdown in the $R+$ case, and classical breakdown in the $R$ case. For consider the result of adding to $R+$ the new axiom scheme $A \rightarrow B \rightarrow . B \rightarrow A \rightarrow . A \& C \rightarrow . B \& C$. In the presence of the Church constant $T$ (see [10]) whose chief property is that everything implies it, this yields $T \rightarrow T \rightarrow . T \rightarrow T \rightarrow . T \& C \rightarrow . T \& C$, which readily reduces to $T \rightarrow . C \rightarrow C$ and then to $D \rightarrow . C \rightarrow C$, which is what it takes to get
$J+$ from $R+. T$ itself is irrelevant to the argument (being always contextually replaceable), whence the smallest extension of $R+$ satisfying (8) is indeed $J+$; and of $R$, it is $K$. So the congruence condition, for the lattice connectives, looks, and is, irrelevant. Nor shall we think any further here about what is necessary to extend our result to implicational or other fragments of weaker neighbors of $R$, such as $E_{\rightarrow}$.

One final point, however, is worth noting. What our principal result is, really, is an excluded middle theorem, on what Slaney calls the Principle of Relativity for a greatest falsehood $f$. Logic, we expect, should not be in the business of telling us which is the greatest falsehood. This is partly concealed, in relevant and other logics (e.g., $K$, as well as $R$ and Curry's $D$ ), by favored interpretations on which a sentential constant $f$ is intended as a greatest falsehood, and, happily, turns out to be so; $4 D$ is such an interpretation of $R$. But there are other possible interpretations on which the situation simply turns confusing; e.g., $4 C$, in which, although $f$ may have been intended as a greatest falsehood, it is in fact entailed by the least truth $t$; and it is an odd situation, which I shall leave for the metaphysicians to ponder, when the standard truth is, on interpretation, even more false than the standard falsehood.

In fact, the word 'false', as applied to formal theories (and, in general, to logic) easily lends itself to equivocation. On the one hand, there is what a theory asserts to be false-say by having $A \rightarrow f$ or $\sim A$ as a theorem; on the other hand, there is what a theory simply fails to assert-by not having $A$ as a theorem. Only in rare-though theoretically desirable-circumstances is the equivocation removed; for this happens when, and only when, the theory $T$ is consistent and complete relative to what passes as its negation.

Now it is often thought, or felt, or at least devoutly hoped, that we have a semantical overview that disposes of these vexing questions, and which divides real sentences (as opposed to the mere scribblings of logicians) exclusively and exhaustively into true sheep and false goats. That's as may be; in practice, God is not at our elbow, giving us hints as to how he carves up the world. And so, relative to actual formal theories-and actual informal ones, for that matter-it is useful to characterize what is false according to a theory on grounds intrinsic to that theory, without appealing to anybody's sense that his axioms are God's axioms, which, as it happily turns out, correspond to Reality. So let us say that a sentence of a theory is directly false, according to that theory, if its negation shows up among the theorems: and that it is indirectly false when the sentence itself fails to make the Honour Roll of theorems, without regard to the appearance, or nonappearance, of its negation on that Roll.

It has been an important point, in relevant semantical analysis (e.g., in [14]) to distinguish between direct and indirect falsehood, using the latter to offer a kind of semantical explication of the former. What our present result suggests, however, is that indirect falsehood can quite generally be transmuted into direct falsehood, just by switching what counts as ' $f$ '. For, unless a theory is to be wholly trivial, something in it will fail to be asserted. And the import of our main result here, for $R_{i}$ and the other theories for which it holds, is that any nontheorem may be taken as the ' $f$ ' of a suitable
theory which rejects that formula. Moreover, this choice of $f$ is a strongly two-valued one; for refuting a given nontheorem $B$, on interpretation, at a greatest undesignated value means precisely that, in the regular theory consisting of just the formulas mapped into "true" values on this interpretation, we shall have $A \rightarrow B$ as a theorem if and only if $A$ itself is not a theorem. Note also that the fate of the actual sentential constant $f$, if present, is not involved in our analysis, despite the fact that this constant was presumably intended to serve as a greatest falsehood. For our actual theories do not necessarily realize our intentions, and the $f$ that we intended should be formally false may in fact stand high on the list of formal truths. In that case, we may well speak of $f$ as both true and false, the latter in the direct sense, on what was called the "American plan" ${ }^{2}$ of semantical analysis in [11]. But we may equally well say, in this situation, that the constant $f$ is just not the real $f$ of the theory, and that the "both true and false" talk, which sounds exciting, is in prosaic reality much less so. For, lurking behind our intuitions as to what should be true or false, according to the intentions that we set out to formalize, there is that which is true or false, according to what we (or our theories) actually say. And let us not get so hung up on our formal insights, and their delicious interrelationships, that we lose our common sense, and begin to babble what is absurd. No more than any other sort of logics have relevant logics made sense of what, in Reality, is both true and false. What they have made direct sense of is our commonsense conviction, which the paradoxes of implication belie, that our reason does not buckle under when confronted with contraditions-or, as we might wish to put it, with what is together asserted-true and asserted-false. But, on the latter point, as Slaney (and, before him, Johansson) correctly intuited, what is asserted-false is very much relative to what shall be counted as formally false, in the direct sense. A common criterion is, " $A$ is formally false if it implies something repugnant". But this is clearly relative to what we find repugnant, and to what degree we find it repugnant. (It is obvious, of course, that we may season a theory with several $f$ 's, and several accompanying negations, depending on how bad we would feel if the $f$ in question were provable. While it is not unreasonable to have a standard $f$-say, the $f$ of $R$-we can prove that $f$ whenever we get into difficulties, so that it is good to have fallback, unprovable $f$ 's, which, like a quarantine during an epidemic, keep the diseased areas of our thinking from infecting its healthy parts. The runaway classical $f$, which says to us "If I get sick, I will cough in your face, and bring all of thought to ruin", is a menace to the public health of logic, and rightly belongs to the era in which it was invented-to the 19th century, when diseases of all sorts ran riot.) Meanwhile, the conclusion of this note is that under very general conditions, which $R_{\rightarrow}$ suffices to guarantee, we can pick our $f$ as any nontheorem $B$ of logic, dividing the sheep and the goats around it. It will be interesting to know whether and how far this result extends to systems not covered by this note.

## NOTES

1. Although this paper does not, in general, depend upon [1], it will be helpful if the reader has access to that book. For one thing, I shall employ its notation and notational con-
ventions (with some adaptations too trivial, I trust, to require explicit mention). For another, I shall cite [1] as the most convenient source for results and discussion significant for this note. Because [1] is a conglomerate affair, to which many authors contributed, imputing a result to it does not necessarily-and, in this note, usually does not-impute that result to one of the main authors of [1]. More often, here, the results cited are in those sections of [1] written by Dunn, while the Sugihara matrix completeness proofs for $R M$, used in passing, are mine alone.
2. The plan is "American" because it was invented by Dunn, though Slaney has pointed out that it was to some (small) degree foreshadowed in previous work, e.g., in Rescher's [13], though Rescher's own interests and purposes were quite different (and, as Dunn has assured me in correspondence, furnished no "input" to his own thinking, in refutation of my speculation in [9] that there was perhaps some likelihood that such input had occurred). The contrasting "Australian" plan was invented by R. and V. Routley, and has formed the basis for most work in the Kripke-style semantical analysis of relevant logic, e.g., in [14]. I have used both plans in my own work (the American plan in [11]) and consider them technically equivalent. On the philosophical grounds set out in [9], however, I like the Australian plan better.

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