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The Class of Neat-Reducts of Cylindric Algebras is Not a Variety But is Closed w.r.t. HP

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Problem 2.11 formulated on p. 464 of [11] asks whether the class $\mathsf{Nr}_\alpha CA_\beta$ of all α-dimensional neat-reducts of β-dimensional cylindric algebras is closed under the formation of homomorphic images and subalgebras or not (for ordinals $\alpha < \beta$).

Theorem 1 below formulates the answer for every pair $\alpha < \beta$ of ordinals. At the end of this note some corollaries are formulated. Finally, a conjecture is stated which seems to be provable by the constructions used in the proofs of Theorem 1 here and II.8.6 in [12], p. 266.

Independently of us, Roger Maddux [13]-[14] obtained a partial solution of Problem 2.11 of [11] proving that $\mathsf{S}\,\mathsf{Nr}_3 CA_\beta \neq \mathsf{Nr}_3 CA_\beta$ if $\beta \geqslant 5$.

We shall use the notations of the monograph [11]; e.g., 1.5.8 refers to "item" 1.5.8 of [11], and p. 489 refers to page 489 of [11]. We shall often refer to items in the textbook [12] on cylindric set algebras (i.e., on representable CAs). Since [12] consists of two parts, we shall refer to an item n in the First Part by I.n and to an item n in the Second Part by II.n; e.g., I.1.1 is on p. 4, and II.1.1 is on p. 145.

Theorem 1 *For arbitrary ordinals $\alpha < \beta$, (i) and (ii) below hold.*

(i) $\mathsf{H}\,\mathsf{Nr}_\alpha CA_\beta = \mathsf{Nr}_\alpha CA_\beta$.
(ii) $\mathsf{S}\,\mathsf{Nr}_\alpha CA_\beta \neq \mathsf{Nr}_\alpha CA_\beta$ *if and only if* $1 < \alpha$.

Proof: In the proof we shall extensively use the notations of [11] without reference or any kind of warning; e.g., "s^i_j", "${}_\alpha s(i,j)$", "Cl_Γ". All these are collected at the end of [11] under the title "Index and symbols," p. 489.

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Proof of "if part" of (ii): Let $1 < \alpha < \beta$ be arbitrary. We have to prove $\mathbf{S}\,\mathbf{Nr}_\alpha CA_\beta \neq \mathbf{Nr}_\alpha CA_\beta$.

Notation: $\tau(x) =_{df} (s_1^0 c_1 x \cdot s_0^1 c_0 x)$.

Statement 1 *Let $\mathfrak{M} \in \mathbf{Nr}_\alpha CA_\beta$ be arbitrary. Let $X \subseteq M$ be such that $(\forall x \in X)\tau(x) \in At\,\mathfrak{M}$. Suppose sup X exists in $\mathfrak{M}$. Then sup $\{\tau(x): x \in X\}$ also exists in $\mathfrak{M}$.*

Proof of Statement 1: We shall need the following lemmas.

Lemma 1 *Let $\alpha \in \Gamma \subseteq \beta$ and $i,j \in \beta \sim \Gamma$. Let $\mathfrak{A} \in CA_\beta$. Then ${}_\alpha s(i,j)$ is a complete one-to-one endomorphism of $\mathfrak{Cl}_\Gamma\mathfrak{A}$. That is, ${}_\alpha s(i,j) \in Ism(\mathfrak{Cl}_\Gamma\mathfrak{A}, \mathfrak{Cl}_\Gamma\mathfrak{A})$. Further, ${}_\alpha s(i,j)$ is an automorphism if $|\Gamma| > 1$.*

Proof: We may assume $i \neq j$ since $Cl_\Gamma\mathfrak{A}\,1[{}_\alpha s(i,i)] \subseteq Id$ by 1.5.13(iii) and 1.5.8(i). ${}_\alpha s(i,j)$ is a complete endomorphism of $\mathfrak{Bl}\mathfrak{A}$, by 1.5.16. To prove that it is one-to-one on $Cl_\Gamma\mathfrak{A}$, it is enough to show $x > 0 \Rightarrow {}_\alpha s(i,j)(c_\alpha x) > 0$. By definition,

$$ {}_\alpha s(i,j)\,c_\alpha x = s_i^\alpha s_j^i s_\alpha^j c_\alpha x = c_\alpha(d_{\alpha i}\cdot c_i(d_{ij}\cdot c_j(d_{j\alpha}\cdot c_\alpha x))). $$

By 1.3.8, $0 < x \Rightarrow 0 < d_{kl}\cdot c_l x$ for every $k,l \in \beta$. Thus,

$$ 0 < x \Rightarrow 0 < {}_\alpha s(i,j)\,c_\alpha x\,; $$

i.e., ${}_\alpha s(i,j)$ is one-to-one on $Cl_\Gamma\mathfrak{A}$. It remains to check that $x \in Cl_\Gamma\mathfrak{A} \Rightarrow {}_\alpha s(i,j)x \in Cl_\Gamma\mathfrak{A}$. This follows from 1.6.13 which implies that

$$ \Delta({}_\alpha s(i,j)x) \subseteq \Delta x \cup \{i,j\}. $$

The automorphism statement follows from 1.5.17 (cf. also top of p. 195).

Lemma 2 *Let $1 < \alpha < \beta$ be arbitrary. Let $\mathfrak{P} \in CA_\beta$ and $\mathfrak{M} =_{df} \mathfrak{Nr}_\alpha\mathfrak{P}$. Let $x \in M$ be such that $\tau(x) \in At\,\mathfrak{M}$. Then*

$$ {}_\alpha s(0,1)^{\mathfrak{P}} x = \tau(x) = \tau^{\mathfrak{P}}(x) = \tau^{\mathfrak{M}}(x). $$

Proof: Let $x \in M$ be such that $\tau(x) \in At\,\mathfrak{M}$. ${}_\alpha s(0,1)^{\mathfrak{P}} x$ is meaningful by $\alpha < \beta$ and $\mathfrak{P} \in CA_\beta$.

(*) ${}_\alpha s(0,1)$ is a complete one-to-one endomorphism on $M = Cl_{(\beta\sim\alpha)}\mathfrak{P}$, by Lemma 1.

$$ {}_\alpha s(0,1)x \leqslant {}_\alpha s(0,1)c_1 x = s_0^\alpha s_1^0 s_\alpha^1 c_1 x = s_0^\alpha s_1^0 c_1 x = s_0^\alpha s_1^0 c_\alpha c_1 x $$
$$ = s_0^\alpha c_\alpha s_1^0 c_1 x = c_\alpha s_1^0 c_1 x = s_1^0 c_1 x, $$

by (*), 1.5.8, and $x = c_\alpha x$.

Similarly, ${}_\alpha s(0,1)x \leqslant s_0^1 c_0 x$ by (*), 1.5.8, $x = c_\alpha x$, and 1.5.10(ii) (hint: see the proof of II.8.6.2.1(iii) in [12], p. 271). Thus, ${}_\alpha s(0,1)x \leqslant \tau(x)$ (for every $x \in Cl_{\{\alpha\}}\mathfrak{P}$). We have $x > 0$ since $\tau(x)$ is an atom. By (*) we have that ${}_\alpha s(0,1)x > 0$ and ${}_\alpha s(0,1)x \in M$. Since $\tau(x)$ is an atom of $\mathfrak{M}$, this means ${}_\alpha s(0,1)x = \tau(x)$.

Now we prove Statement 1. Let $1 < \alpha < \beta$. Let $\mathfrak{M} \in \mathbf{Nr}_\alpha CA_\beta$. Let $X \subseteq M$ be such that $(\forall x \in X)\tau(x) \in At\,\mathfrak{M}$. Suppose that *sup* X exists in $\mathfrak{M}$. We have to prove that *sup* $\{\tau(x): x \in X\}$ also exists in $\mathfrak{M}$.

Let $\mathfrak{M} = \mathfrak{Nr}_\alpha \mathfrak{P}'$ for some $\mathfrak{P}' \in CA_\beta$. Let $\mathfrak{P} =_{df} \mathfrak{Sg}^{(\mathfrak{P}')}M$. Then $\mathfrak{M} = \mathfrak{Nr}_\alpha \mathfrak{P}$ and M generates $\mathfrak{P}$.

Let $z =_{df} sup\ X$ in $\mathfrak{M}$. First we prove that

(1) $z = sup\ X$ in $\mathfrak{P}$, too.

Suppose that y is an upper bound of X in $\mathfrak{P}$. Since $\mathfrak{P}$ is generated by $Cl_{(\alpha\sim\beta)}\mathfrak{P}$, we have that $(\Delta y \sim \alpha) =_{df} \Delta$ is finite. Then $c^{\partial}_{(\Delta)}y \in Cl_{(\alpha\sim\beta)}\mathfrak{P} = M$ and $(\forall x \in X)$ $[x = c^{\partial}_{(\Delta)}x \leqslant c^{\partial}_{(\Delta)}y \leqslant y]$. Since $c^{\partial}_{(\Delta)}y$ is an upper bound of X in $\mathfrak{M}$, we have $z \leqslant c^{\partial}_{(\Delta)}y \leqslant y$. Thus $z = sup\ X$ in $\mathfrak{P}$ as well as in $\mathfrak{M}$.

By 1.5.16(i), (1) above implies

$$_\alpha s(0,1)z = sup\ \{_\alpha s(0,1)x : x \in X\} \text{ in } \mathfrak{P}.$$

By Lemma 1 we have $_\alpha s(0,1)z \in M$ and also $(\forall x \in X)_\alpha s(0,1)x \in M$. Therefore,

$$_\alpha s(0,1)z = sup\ \{_\alpha s(0,1)x : x \in X\} \text{ in } \mathfrak{M}.$$

By Lemma 2, $_\alpha s(0,1)z = sup\ \{\tau^{\mathfrak{M}}(x) : x \in X\}$ in $\mathfrak{M}$.

Statement 2 *For every* $1 < \alpha < \beta$ *there exist* $\mathfrak{A} \in \mathbf{S}\,\mathbf{Nr}_\alpha CA_\beta$ *and* $X \subseteq A$ *such that* $(\forall x \in X)\tau(x) \in At\mathfrak{A}$, *sup X exists and sup* $\{\tau(x) : x \in X\}$ *does not exist in* $\mathfrak{A}$.

Proof of Statement 2: Let $1 < \alpha$ be arbitrary. Ws_α denotes the class of all weak cylindric set algebras (of dimension α) as defined in Def. I.1.1(vi) of [12], p. 5. The *full* Ws_α with unit V is $\langle Sb\ V; \cap, \dots \rangle$, i.e., full means that the universe is the power set of the unit, see Def. I.1.1(iii) of [12], p. 5. The unit of a CA_α is its greatest element (see p. 162).

$\mathbf{Q}$ denotes the set of rational numbers. We shall construct a Ws_α with base $\mathbf{Q}$. Let $\bar{0} =_{df} \langle 0 : i < \alpha \rangle$. $\mathfrak{C}$ denotes the full Ws_α *with unit* ${}^\alpha\mathbf{Q}^{(\bar{0})}$, see Def. I.1.1 of [12], p. 5.

$$a =_{df} \left\{ s \in {}^\alpha\mathbf{Q}^{(\bar{0})} : s_0 + 1 = \sum_{0 \neq i < \alpha} s_i \right\}.$$

$$a_s =_{df} \{s\}, \text{ for every } s \in a.$$

$$\mathfrak{A} =_{df} \mathfrak{Sg}^{(\mathfrak{C})}\{a, a_s : s \in a\}.$$

Now we show that this $\mathfrak{A}$ has the desired property with $X = \{a_s : s \in a\}$. Clearly, $a = sup\ X$ in $\mathfrak{A}$. Since $\mathfrak{A} \in Ws_\alpha$, we have $\mathfrak{A} \in \mathbf{S}\,\mathbf{Nr}_\alpha CA_\beta$ for every $\beta \geqslant \alpha$, see p. 268 and 2.6.26.

It is easy to see that for every $s \in a$ we have

$$\tau(a_s) = \{\langle s_1, s_0, s_i \rangle_{1<i<\alpha}\}.$$

Of course, $\tau(a_s) \in At\mathfrak{A}$ for every $s \in a$. Let

$$b = \bigcup \{\tau(a_s) : s \in a\}$$
$$= \left\{ s \in {}^\alpha\mathbf{Q}^{(\bar{0})} : s_1 + 1 = \sum_{1 \neq i < \alpha} s_i \right\}.$$

Now we show that $sup\ \{\tau(a_s) : s \in a\}$ does not exist in $\mathfrak{A}$. First we observe that

$$\{s\} \in A \text{ for all } s \in {}^\alpha\mathbf{Q}^{(\bar{0})}$$

since

$$\{s\} = c_1\left\{\left\langle s_0, \left(s_0 + 1 - \sum_{1<i<\alpha} s_i\right), s_i\right\rangle_{1<i<\alpha}\right\} \cdot c_0\left\{\left\langle\left(\sum_{0\neq i<\alpha} s_i\right) - 1, s_i\right\rangle_{1\leq i<\alpha}\right\}.$$

Now suppose $y = sup\,\{\tau(a_s):s \in a\}$. Let $z \in {}^{\alpha}\mathbf{Q}^{(\bar{0})} \sim b$. Then $\tau(a_s) \leqslant -\{z\}$ for every $s \in a$, thus $y \leqslant -\{z\}$. Hence $y \leqslant b$. Since $y \geqslant b$ trivially holds, we have $y = b$.

We have seen that if $y = sup\,\{\tau(a_s):s \in a\}$ exists in $\mathfrak{A}$, then $y = b$. Thus, it suffices in order to show that $sup\,\{\tau(a_s):s \in a\}$ does not exist in $\mathfrak{A}$ to show $b \notin A$.

We show $b \notin A$ by elimination of cylindrifications.

Definitions

$$Pol =_{df} \left\{\left\{s \in {}^{\alpha}\mathbf{Q}^{(\bar{0})}: t + \sum_{i<\alpha} (r_i s_i) = 0\right\} : \{t, r_i : i < \alpha\} \subseteq \mathbf{Q}\right\}.$$

$Pol^{<} =_{df} \{p \in Pol: (\exists i < \alpha) c_i p = p\}$.

Remark:

$\{a, b, 1, d_{ij}: i,j \in \alpha\} \subseteq Pol$,
$a, b \notin Pol^{<}$, $1 \in Pol^{<}$, and
$[\{d_{ij}: i \neq j,\ i,j < \alpha\} \subseteq Pol^{<}$ iff $\alpha \geqslant 3]$.

$$G =_{df} \{a, -a, p, -p, c_{(\Gamma)}\{\bar{0}\}, -c_{(\Gamma)}\{\bar{0}\}: p \in Pol^{<} \cup \{d_{01}\}; \Gamma \in Sb_{\omega}\alpha,\ 0 \in \Gamma\}.$$

$$G^* =_{df} \left\{\prod_{i<n} g_i : g_i \in G\right\},$$

$$G^{**} =_{df} \left\{\sum_{i<n} g_i : g_i \in G^*\right\}.$$

Clearly, $G^{**} \supseteq \{a, a_s : s \in a\}$.

We shall show that G^{**} is closed under cylindrifications, i.e., that G^{**} is a subuniverse of $\mathfrak{C}$. We shall also show $b \notin G^{**}$. These statements show that $b \notin A$, as wished.

First we prove $b \notin G^{**}$. Suppose $b = \sum_{i<n} g_i$ and $\{g_i : i < n\} \subseteq G^*$. Then $g_i \subseteq b$ for every $i < n$. Let

$P(0) =_{df} \{p \in Pol^{<}: c_0 p \neq p\} \cup \{d_{01}\}$.

$G_1 =_{df} \{g \in G^*: g \subseteq a\}$
$G_2 =_{df} \{g \in G^*: g \nsubseteq a$ and $g \subseteq p$ for some $p \in P(0)\}$
$G_3 =_{df} \{p_1 \cdot \ldots \cdot p_k : k \in \omega, \{p_1, \ldots, p_k\} \subseteq G \sim (\{a\} \cup P(0))\}$.

Clearly, $G_1 \cup G_2 \cup G_3 = G^*$.

Some facts:

(1) If $g \in G_1$ and $g \subseteq b$ then $g \subseteq d_{01}$.
(2) If $g \in G_3$ and $g \subseteq b$ then $g = 0$.

(2) can be seen as follows: Let $g = p_1 \cdot \ldots \cdot p_k$ where $\{p_1, \ldots, p_k\} \subseteq G \sim (\{a\} \cup P(0))$. Suppose $g \neq 0$. We show that $g \nsubseteq b$.

Let $z \in g$ be arbitrary. Define

$$[p] =_{df} \begin{cases} \left\{\dfrac{1}{r_0}\left(-t - \sum\limits_{0 \neq i < \alpha} r_i z_i\right)\right\} & \text{if } p = -\left\{s \in {}^{\alpha}Q^{(\bar{0})} : t + \sum\limits_{i<\alpha} r_i s_i = 0\right\} \text{ and } r_0 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $r \in \mathbf{Q} \sim \left(\left(\bigcup\limits_{1 \leqslant j \leqslant k} [p_k]\right) \cup [-b]\right)$ be arbitrary. Now

$$z_r^0 =_{df} ((z \sim \{\langle 0, z_0\rangle\}) \cup \{\langle 0, r\rangle\}) \in g \sim b.$$

(Note that if $c_{(\Gamma)}\{\bar{0}\} \in G$ or $-c_{(\Gamma)}\{\bar{0}\} \in G$ then $0 \in \Gamma$.)

Let

$$\mathfrak{G} = \sum_{i<n_1} g_i^1 + \sum_{i<n_2} g_i^2 + \sum_{i<n_3} g_i^3,$$

where $\{g_i^j : i < n_j\} \subseteq G_j$ and $g_i^j \subseteq b$ for every $i < n_j$, $j = 1, 2, 3$.

We show that $\mathfrak{G} \neq b$. By (1) and (2) above, $\mathfrak{G} \subseteq \sum\limits_{i<n} p_i$ for some $\{p_i : i < n\} \subseteq P(0)$. Now we show $b \nsubseteq \Sigma E$ for every finite $E \subseteq Pol(0)$. Let $L =_{df} \{p \in Pol^{<} : c_0 p \neq p\}$. Note that if $\alpha > 2$ then $P(0) = L$, and $P(0) = L \cup \{d_{01}\}$ otherwise. If $\alpha = 2$ then $b \subseteq -d_{01}$, otherwise $P(0) = L$; hence it is enough to prove $b \nsubseteq \Sigma E$ for every finite $E \subseteq L$, which is true because of the following.

It can be seen, by linear algebraic arguments, that for every ordinal α, for every $n \in \omega$ and for every system

$$t_0 + \sum_{i<\alpha} (r_{0i} x_i) = 0$$
$$\vdots$$
$$t_n + \sum_{i<\alpha} (r_{ni} x_i) = 0$$

of equations such that $\forall j \leqslant n\ (\exists i < \alpha) r_{ji} = 0$ and $r_{j0} \neq 0$ the equation $\sum\limits_{i<\alpha} x_i = 2x_1 + 1$ has a solution s in the weak space ${}^{\alpha}\mathbf{Q}^{(\bar{0})}$ such that, for every $j \leqslant n$, s is *not* a solution of $t_j + \sum\limits_{i<\alpha} (r_{ji} x_i) = 0$. (This is true for finite α as well as for infinite α.)

This proves $\mathfrak{G} \neq b$. Thus, $b \notin G^{**}$ is proved for every $1 < \alpha$.

Next we show that G^{**} is closed under cylindrifications. Let $j \in \alpha$ and $g \in G^*$ be arbitrary. We have to show $c_j g \in G^{**}$. (This is enough since c_j is additive.) We may suppose that

$$g = e \cdot p_1 \cdot \ldots \cdot p_n \cdot -P_1 \cdot \ldots \cdot -P_m \cdot y \cdot -c_{(\Gamma_1)}\{\bar{0}\} \cdot \ldots \cdot -c_{(\Gamma_N)}\{\bar{0}\}$$

where

$e \in \{a, -a, 1\}$
$n, m, N \in \omega$

$p_i, P_i \in Pol^< \cup \{d_{01}\} \subseteq Pol$, $c_j p_i \neq p_i$, $c_j P_i \neq P_i$
$y \in \{c_{(\Delta)}\{\bar{0}\}, 1 : \Delta \in Sb_\omega \alpha, 0 \in \Delta, j \notin \Delta\}$
$\{\Gamma_1, \ldots, \Gamma_N\} \subseteq \{x \in Sb_\omega \alpha : j \notin x, 0 \in x\}$.

Distinction of cases

Notation: Let $p \in Pol$.

$$p(j|0) =_{df} c_j\{s \in p : s_j = 0\}.$$
$$(-p)(j|0) =_{df} -(p(j|0)).$$

Note that $p(j|0) \in Pol^<$ for every $p \in Pol$, since

$$p(j|0) = \left\{s \in {}^\alpha\mathbf{Q}^{(\bar{0})} : t + \sum_{j \neq i < \alpha} r_i s_i = 0\right\},$$

if $p = \left\{s \in {}^\alpha\mathbf{Q}^{(\bar{0})} : t + \sum_{i<\alpha} r_i s_i = 0\right\}$.

Case I. $y = c_{(\Delta)}\{\bar{0}\}$.

$$\begin{aligned} &c_j(e \cdot p_1 \cdot \ldots \cdot p_n \cdot -P_1 \cdot \ldots \cdot -P_m \cdot c_{(\Delta)}\{\bar{0}\} \cdot -c_{(\Gamma_1)}\{\bar{0}\} \cdot \ldots \cdot -c_{(\Gamma_N)}\{\bar{0}\}) \\ &\quad = e(j|0) \cdot p_1(j|0) \cdot \ldots \cdot p_n(j|0) \cdot -P_1(j|0) \cdot \ldots \cdot -P_m(j|0) \cdot c_j c_{(\Delta)}\{\bar{0}\} \\ &\qquad \cdot -c_j c_{(\Gamma_1)}\{\bar{0}\} \cdot \ldots \cdot -c_j c_{(\Gamma_N)}\{\bar{0}\}. \end{aligned}$$

Case II. $y = 1$.

$$\begin{aligned} &c_j(e \cdot p_1 \cdot \ldots \cdot p_n \cdot -P_1 \cdot \ldots \cdot -P_m \cdot -c_{(\Gamma_1)}\{\bar{0}\} \cdot \ldots \cdot -c_{(\Gamma_N)}\{\bar{0}\}) \\ &\quad = f(e) \cdot \prod_{k \leq n}\left(\left(\prod_{i \leq n} c_j(p_k \cdot p_i)\right) \cdot \left(\prod_{i \leq m} c_j(p_k \cdot -P_i)\right) \cdot \left(\prod_{i \leq N} c_j(p_k \cdot -c_{(\Gamma_i)}\{\bar{0}\})\right)\right), \end{aligned}$$

where

$$f(a) =_{df} \left(\prod_{k \leq n} c_j(a \cdot p_k)\right) \cdot \left(\prod_{i \leq m} c_j(a \cdot -P_i)\right) \cdot \left(\prod_{i \leq N} c_j(a \cdot -c_{(\Gamma_i)}\{\bar{0}\})\right),$$

$f(-a) =_{df} \prod_{k \leq n} c_j(p_k \cdot -a)$, and

$f(1) =_{df} 1$.

Note that for every $p, q \in Pol$ there are $p', q', p'', q'' \in Pol^<$ such that $c_j(p \cdot q) = p' \cdot q'$, $c_j(p \cdot -q) = p'' \cdot -q''$, and if $j \in \Delta p \sim \Gamma$ then

$$c_j(p \cdot -c_{(\Gamma)}\{\bar{0}\}) = -p(j|0) + p(j|0) \cdot -c_j c_{(\Gamma)}\{\bar{0}\}.$$

By this, Statement 2 is proved.

Statements 1 and 2 imply that $\mathbf{S}\,\mathbf{Nr}_\alpha CA_\beta \neq \mathbf{Nr}_\alpha CA_\beta$ for every $1 < \alpha < \beta$.

Proof of "only if part" of (ii): Let $\beta \geq 1$ be arbitrary. We have to prove $\mathbf{S}\,\mathbf{Nr}_i CA_\beta = \mathbf{Nr}_i CA_\beta$, for $i = 0,1$. $\mathbf{Nr}_0 CA_\beta = BA = CA_0$, by 2.6.30(iii) (cf. p. 171 too). Thus $\mathbf{S}\,\mathbf{Nr}_0 CA_\beta = \mathbf{Nr}_0 CA_\beta$.

Now we prove $\mathbf{Nr}_1 CA_\beta = CA_1$. Let $\mathfrak{B} \in CA_1$ be arbitrary. We prove $\mathfrak{B} \in \mathbf{Nr}_1 CA_\beta$. Since $CA_1 = \mathbf{SP}\, Cs_1$ (cf. p. 171),

$$\mathfrak{B} \cong \mathfrak{B}' \subseteq \underset{i\in I}{P}\, \mathfrak{B}_i$$

for some $\{\mathfrak{B}_i : i \in I\} \subseteq Cs_1$ and for some $\mathfrak{B}'$.

Every algebra $\mathfrak{B}_i \in Cs_1$ is isomorphic to a Cs_1 having only infinite or empty sets as elements. Thus we may suppose that $(\forall i \in I)(\forall b \in B_i)[b = 0 \vee |b| \geqslant \omega]$. For every $i \in I$, let $U_i =_{df} \cup B_i = 1_{\mathfrak{B}_i}$ and let $\mathfrak{P}_i$ denote the full Cs_β with base U_i. Let $\mathfrak{P} =_{df} \underset{i\in I}{P}\, \mathfrak{P}_i$.

Now we define a one-to-one homomorphism $h\colon \mathfrak{B}' \rightarrowtail \mathfrak{Nr}_1\mathfrak{P}$. Let $x \in B_i$. Then $\hat{x} =_{df} \{s \in {}^\beta U_i : s_0 \in x\}$, and $h(\langle x_i\rangle_{i\in I}) =_{df} \langle \hat{x}_i\rangle_{i\in I}$. Clearly, $h\colon \mathfrak{B}' \rightarrowtail \mathfrak{Nr}_1\mathfrak{P}$.

Let $X =_{df} h^*B' = \{h(x) : x \in B'\}$, and let $\mathfrak{A} =_{df} \mathfrak{Sg}^{(\mathfrak{P})}X$. Now $\mathfrak{B} \cong h^*\mathfrak{B}' \subseteq \mathfrak{Nr}_1\mathfrak{A}$. We show $X = Nr_1\mathfrak{A}$. $\mathfrak{A}$ is a monadic generated CA_β since $(\forall x \in X)$ $\Delta^{(\mathfrak{P})}(x) \subseteq 1$. Thus we can apply 2.2.24 which states that

$$Nr_1\mathfrak{A} = Cl_{(\beta\sim 1)}\mathfrak{A} = Sg^{(\mathfrak{Bl}\,\mathfrak{A})}(X \cup C),$$

where

$$C = \left\{c_{(\kappa)}\left(\bar{d}(\kappa \times \kappa)\cdot \prod_{\lambda<\kappa} s^0_\lambda x\right) : \kappa < (\beta + 1) \cap \omega,\ x \in X\right\}.$$

(This is true because X is closed w.r.t. the Boolean operations of $\mathfrak{A}$ and s^0_λ is a Boolean endomorphism by 1.5.3.)

Now we show $C \subseteq X$.

Notation: $\mathfrak{S}_\kappa(x) =_{df} c_{(\kappa)}\left(\bar{d}(\kappa \times \kappa)\cdot \prod_{\lambda<\kappa} s^0_\lambda x\right)$.

It is enough to show that $\mathfrak{S}_\kappa(x) = c_0 x$ for every κ, and $x \in X$, since X is closed w.r.t. c_0.

Coordinatewise: $\mathfrak{S}_\kappa(\langle \hat{x}_i\rangle_{i\in I}) = \langle \mathfrak{S}_\kappa(\hat{x}_i)\rangle_{i\in I}$.

Let $i \in I$ and let $0 \neq x_i \in B_i$. Then

$$s^0_\lambda(\hat{x}_i) = \{s \in {}^\beta U_i : s_\lambda \in x_i\}.$$

Thus $\mathfrak{S}_\kappa(\hat{x}_i) = 1 = c_0\hat{x}_i$, since $|x_i| \geqslant \omega$. Further, $\mathfrak{S}_\kappa(\hat{0}) = 0 = c_0\hat{0}$. By these $C \subseteq X$ is proved.

Now, $Nr_1\mathfrak{A} = X$, since X is closed w.r.t. the Boolean operations of $\mathfrak{A}$, i.e., $\mathfrak{B} \cong h^*\mathfrak{B}' = \mathfrak{Nr}_1\mathfrak{A}$, which implies $\mathfrak{B} \in \mathbf{Nr}_1 CA_\beta$. By this, $CA_1 = \mathbf{Nr}_1 CA_\beta = \mathbf{S}\,\mathbf{Nr}_1 CA_\beta$ is proved.

Proof of (i): Let $\alpha \leqslant \beta$ be arbitrary ordinals. We have to prove $\mathbf{H}\,\mathbf{Nr}_\alpha CA_\beta = \mathbf{Nr}_\alpha CA_\beta$. Let $\mathfrak{A} \in \mathbf{Nr}_\alpha CA_\beta$. Then $\mathfrak{A}$ is the generating neat-reduct of some $\mathfrak{B} \in CA_\beta$, i.e., $\mathfrak{A} = \mathfrak{Nr}_\alpha\mathfrak{B}$ and A generates $\mathfrak{B}$. Let R be a congruence of $\mathfrak{A}$. We have to prove that $\mathfrak{A}/R \in \mathbf{Nr}_\alpha CA_\beta$. By 2.3.8 R has an extension R' to $\mathfrak{B}$ such that $R' \in Co\,\mathfrak{B}$ and $R' \cap (A \times A) = R$. Now $\mathfrak{A}/R \subseteq \mathfrak{Nr}_\alpha(\mathfrak{B}/R')$.

We shall show $\mathfrak{A}/R = \mathfrak{Nr}_\alpha(\mathfrak{B}/R')$. Let $b \in B$ be arbitrary. Suppose $(b/R') \in Nr_\alpha(\mathfrak{B}/R') = Cl_{(\alpha\sim\beta)}(\mathfrak{B}/R')$. It is enough to show that $((b/R') \cap A) \neq 0$ since then

$$\{\langle a/R, a/R'\rangle : a \in A\} : \mathfrak{A}/R \cong \mathfrak{Nr}_\alpha(\mathfrak{B}/R').$$

Since A generates B and $A = Cl_{(\beta\sim\alpha)}\mathfrak{B}$, we have that $(\Delta b \sim \alpha)$ is finite. Let $\Gamma =_{df} (\Delta b \sim \alpha)$. Now $\Delta(c_{(\Gamma)}b) \subseteq \alpha$ and thus $c_{(\Gamma)}b \in Cl_{(\beta\sim\alpha)}\mathfrak{B} = A$. Since $(b/R') \in Cl_{(\alpha\sim\beta)}(\mathfrak{B}/R')$ implies $c_{(\Gamma)}(b/R') = (b/R')$, we have $c_{(\Gamma)}b \in (b/R')$. Thus $c_{(\Gamma)}b \in (b/R') \cap A$. This proves $\mathbf{H}\{\mathfrak{A}\} \subseteq \mathbf{Nr}_\alpha CA_\beta$ which means $\mathbf{H}\,\mathbf{Nr}_\alpha CA_\beta = \mathbf{Nr}_\alpha CA_\beta$ since $\mathfrak{A} \in \mathbf{Nr}_\alpha CA_\beta$ was chosen arbitrarily.

By these, the theorem is proved.

Notation: $\mathbf{P^r}K$ denotes the class of all isomorphic images of all *reduced products* of elements of the class K (cf. 0.3.62(i)).

Corollary 2 $\mathbf{Up}\,\mathbf{Nr}_\alpha CA_\beta = \mathbf{P^r}\,\mathbf{Nr}_\alpha CA_\beta = \mathbf{Nr}_\alpha CA_\beta$, *for every* $\alpha \leqslant \beta$.

Proof: $\mathbf{P}\,\mathbf{Nr}_\alpha CA_\beta = \mathbf{Nr}_\alpha CA_\beta$, by 2.6.32(i). By definition, any reduced product of a family $\langle \mathfrak{A}_i : i \in I\rangle$ of algebras is a homomorphic image of their direct product. Thus for any class K of algebras, $\mathbf{P^r}K \subseteq \mathbf{HP}\,K$ (cf. 0.3.69(i)). Now, Theorem 1(i) completes the proof.

Corollary 3 $CA_1 = \mathbf{Nr}_1 CA_\beta$ *for every ordinal* $\beta \geqslant 1$.

Proof: This follows from the proof of the "only if" part of Theorem 1(ii).

Remarks: Theorem 1 is the solution of Problem 2.11, p. 464 (see Remark 2.6.33). Corollary 2 is an improvement of 2.6.32(i). Corollary 3 is an improvement of 2.6.39. Further, 2.6.46 can be improved by adding that

$$\mathfrak{Fr}_\delta CA_\alpha = \mathfrak{Fr}_\delta \mathbf{Nr}_\alpha CA_\beta \text{ iff } CA_\alpha = \mathbf{Nr}_\alpha CA_\beta,$$

for arbitrary δ and $\alpha \leqslant \beta$.

Corollary 4 *Let* $1 < \alpha < \beta$. *Then* $\mathbf{S}\,\mathbf{Nr}_\alpha CA_\beta \nsubseteq \mathbf{Uf}\,\mathbf{Up}\,\mathbf{Nr}_\alpha CA_\beta$.

Proof: In the proof of II.8.8 in [12], p. 275, a first-order formula ϕ was exhibited and it was proved that $\mathbf{Nr}_\alpha CA_\beta \models \phi$ but for the $\mathfrak{A} \in Ws_\alpha \cap \mathbf{S}\,\mathbf{Nr}_\alpha CA_\beta$ constructed in the proof of Statement 2 of the present paper $\mathfrak{A} \not\models \phi$ (this formula ϕ was denoted by $\forall y_0 \exists y_1 \psi(y_0, y_1)$ in [12]). This claim $\mathfrak{A} \not\models \phi$ was proved immediately below II.8.8.1 in [12].

Below we hint that the results of this paper are not only about algebraic logic but also have implications in logic, e.g., in abstract model theory (which is sometimes called soft model theory). Algebraic logic for abstract model theory was developed in [1], [9], [3-5]. The class Lr_α of *locally* finite *regular* cylindric set algebras was introduced in [2], [1], [15] to investigate connections between logic (first of all model-theory) and *CA*s. (In some of the quoted papers, the adjective "locally *i*-finite" was used instead of "locally finite regular". The class Lr_α was denoted by German L in [2] and by script Lv in [1]. See also the note above Section II.2 in [12], p. 153, in this connection.) It was shown in the quoted papers that Lr_ω coincides with the class of cylindric algebras corresponding to models of first-order logic (see, e.g., the function h: Models $\to Lr_\omega$ defined on p. 30 of [1] or Proposition 1(ii) on p. 564 of [15]). By applying the Cantor-Bernstein argument to Proposition 1(ii) of [15], we obtain an isomorphism h: Models $\rightarrowtail\!\!\!\twoheadrightarrow Lr_\omega$ definable in *ZFC* by an absolute formula without parameters. More information on the connections between *CA*s and logic can be found in [15] and [9], especially on pp. 564-572 and pp.

601-603 of [15]. It turns out there too, how and why cylindric set algebras constitute that part of algebraic logic which corresponds to the part of logic called model theory.

In II.8.9 of [12], $\mathbf{SP}\,Lr_\alpha \nsubseteq \mathbf{Nr}_\alpha CA_\beta$ for $\beta > \alpha > 1$ was proved to be a corollary of Theorem 1 of the present paper. In view of results in the above quoted works, this corollary has implications in logic. Note that $Lr_\alpha \subseteq Lf_\alpha$.

To see that the above corollary is not immediate by the proof of Theorem 1, we formulate Proposition 5 below.

Notation: Let κ be a cardinal and K a class of similar algebras. Then $\mathbf{P}^{\mathbf{r}}_\kappa K$ denotes the class of algebras isomorphic to κ-complete reduced products of members of K. Note that $\mathbf{P}^{\mathbf{r}}_\omega K = \mathbf{P}^{\mathbf{r}} K$.

Proposition 5 *Let $\alpha \geqslant \omega$. Then the Ws_α $\mathfrak{A}$ constructed in the present proof of Theorem 1(ii) is such that $\mathfrak{A} \notin \mathbf{SP}^{\mathbf{r}}_{\omega^+} Lf_\alpha$. Further, $(Ws_\alpha \cap Dc_\alpha) \nsubseteq \mathbf{SP}^{\mathbf{r}}_{\omega^+} Lf_\alpha$.*

Proof: Let q be the ω^+-ary quasiequation $(\wedge\{c_{2n}x = y : n \in \omega\}) \to x = y$. Clearly $Lf_\alpha \models q$ and hence by Corollary 4 of [7], p. 33, we have $\mathbf{SP}^{\mathbf{r}}_{\omega^+} Lf_\alpha \models q$. But $\mathfrak{A} \not\models q$ and $Ws_\alpha \cap Dc_\alpha \not\models q$.

The class $Ws_\alpha \subseteq CA_\alpha$ was introduced in the proof of Statement 2. The class Cs_α^{reg} of regular cylindric set algebras is investigated in [12]. By passing we note that $Lr_\alpha = Cs_\alpha^{reg} \cap Lf_\alpha$.

Corollary 6 *Let $\beta > \alpha > 1$. Then*

(i) $Cs_\alpha^{reg} \nsubseteq \mathbf{Nr}_\alpha CA_\beta$
(ii) $Ws_\alpha \nsubseteq \mathbf{Nr}_\alpha CA_\beta$.

Proof: By the proof of Corollary 4, the Ws_α $\mathfrak{A}$ constructed in the proof of Statement 2 in this paper is not in $\mathbf{Uf\,Up\,Nr}_\alpha CA_\beta$. Hence $Ws_\alpha \nsubseteq \mathbf{Uf\,Up\,Nr}_\alpha CA_\beta$, proving (i). I.7.13 of [12], p. 98, states $Ws_\alpha \subseteq \mathbf{I}Cs_\alpha^{reg}$. This proves (ii).

For more and stronger corollaries of the present proof of Theorem 1, see Section II.8 of [12], pp. 261-309.

The class Crs_α was defined in [12]. The class Crs_α^{rc} of rectangular Crs_αs was defined in [10], the following way.

Definition 1 Let $\mathfrak{A} \in Crs_\alpha$. We define $\mathfrak{A} \in Crs_\alpha^{rc}$ to hold iff $1^{\mathfrak{A}} = PF$ for some function F. Note that $PF = P_{i \in Do(F)} F_i$.

In [10] Crs_α^{rc}s were called "cylindric set algebras with diagonal" but that name was already reserved for other purposes in [12] and [11]. For any class K of algebras similar to CA_βs we let $\mathbf{Nr}_\alpha K$ be as defined in Definition 3(ii) of [15], p. 574, if $\alpha \leqslant \beta$.

Proposition 7 *Let $1 < \alpha \leqslant \beta$. Then $\mathbf{I}Crs_\alpha^{rc} = \mathbf{Nr}_\alpha \mathbf{I}Crs_\beta^{rc}$.*

The proof is the same as that of Proposition 5(iii) in [15].

The theory of CAs was generalized to a more universal algebraic framework in [3-5]. The basic concept there is a system of varieties definable by schemes. It appears that the present Theorem 1 does not generalize under the conditions given there. For example $\mathbf{I}Crs = \langle \mathbf{I}Crs_\alpha : \alpha \in Ord\rangle$ is a counterexample to the generalized version of (ii) of Theorem 1, since by Theorem 24 and Proposition 9 of [15], **I***Crs* is a system of varieties definable by schemes, but $\mathbf{I}Crs_\alpha = \mathbf{Nr}_\alpha \mathbf{I}Crs_\beta$ for all $\beta \geqslant \alpha > 1$ holds by Proposition 5(iii) of [15]. It would be interesting to know the necessary conditions for the present Theorem 1 to generalize.

Other kinds of universal algebraic generalizations of neat-reducts (i.e., of the operator $\mathbf{Nr}_\alpha$) are found in [6].

Conjecture 8 *Let $\beta > \alpha > 1$. Then we conjecture that $\mathbf{Nr}_\alpha CA_\beta$ is not elementary. That is, we conjecture $\mathbf{Uf}\ \mathbf{Nr}_\alpha CA_\beta \neq \mathbf{Nr}_\alpha CA_\beta$.*

For $\alpha = 2$ this conjecture is proved to hold in II.8.6 of [12], p. 266. Cf. Problem 2.2 of [11], p. 463, and Problem II.8.7 of [12].

Bibliographical remark: The reference [AN4] on p. 314 of [12] refers to the present paper.

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