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Some Preservation Results for Classical and Intuitionistic Satisfiability in Kripke Models

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Translations of classical into intuitionistic formal systems, as defined by Gödel and others (for a survey see [2], Section 81; [4] p. 41; or [6]), provide among other things a method for determining which classically valid formulas are intuitionistically valid. All of the translations share the following property: if T is an intuitionistic theory and T^c its classical counterpart (obtained, e.g., by adding the law of excluded middle) and if φ is a formula in an appropriate language (built up with connectives $\vee$, $\wedge$, $\rightarrow$, $\neg$, $\exists$, $\forall$) and φ' its translation then:

$$T^c \vdash \varphi \leftrightarrow \varphi' \text{ and}$$
$$T^c \vdash \varphi \text{ iff } T \vdash \varphi'.$$

The simplest (to describe) translation consists in attaching a double negation to each subformula. As a result we get an imbedding of the classical theory into the "negative fragment" of the intuitionistic theory that consists of formulas constructed without $\vee$ and $\exists$ from decidable (or doubly negated) atomic formulas. It may appear then as though the differences between classical and intuitionistic systems are due to disjunction and the existential quantifier. From the classical point of view, $\vee$ and $\exists$ could be regarded as new connectives (while classical disjunction is defined in terms of negation and conjunction and the existential quantifier is defined in terms of the universal quantifier). Such explanations are actually given in many popular accounts of intuitionism. Given Kripke models, however, the definition of forcing suggests that the real culprits are implication (as well as negation as its special case) and the universal quantifier. This also seems to be in better accord with the intuitionist interpretations of logical connectives. The results which will be presented here support this view.

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It will be shown that forcing coincides with classical satisfiability exactly (up to intuitionistic equivalence) on positive formulas (containing only $\vee$, $\wedge$, and $\exists$). It will also be shown that the formulas, forced by a node of a Kripke model whenever they are satisfied in the classical structure associated with that node, are exactly those which are classically equivalent to a positive formula and intuitionistically implied by it.

1 Preliminaries By a Kripke model we mean a structure $\langle (T, 0, \leqslant); \mathfrak{A}_t: t \in T\rangle$ where $(T, 0, \leqslant)$ is a partially ordered set with the least element 0 and $\mathfrak{A}_t$ for $t \in T$ are classical structures (of the same type) having the property: $t \leqslant s$ implies $\mathfrak{A}_t \subseteq^+ \mathfrak{A}_s$ ($\mathfrak{A}_t$ is a positive submodel of $\mathfrak{A}_s$, meaning $A_t \subseteq A_s$ and the interpretation of a relation symbol in $\mathfrak{A}_t$ is a subset of that in $\mathfrak{A}_s$). This definition differs inessentially from the original one [3] and is a simplification, which is possible since intuitionistic acceptability is not claimed. The forcing relation $t \Vdash \varphi[a_1, \ldots, a_n]$ is defined as usual. By $\mathfrak{A}_t \vDash \varphi[a_1, \ldots, a_n]$ we denote the classical satisfiability relation. $\vdash \varphi$ and $\vdash_c \varphi$ denote derivability in the intuitionistic and the classical predicate calculus, respectively.

Let P be the class of all formulas containing only the connectives $\vee$, $\wedge$, and $\exists$. Classically formulas from P are equivalent to Σ_1^0 formulas. We shall call formulas from P positive.

As a straightforward consequence of the definition of forcing we have the following:

Lemma 1 *If $\varphi(x_1, \ldots, x_n) \in P$ then for any Kripke model $\langle (T, 0, \leqslant); \mathfrak{A}_t: t \in T\rangle$ any $t \in T$ and any $a_1, \ldots, a_n \in A_t$,*

$$t \Vdash \varphi[a_1, \ldots, a_n] \text{ iff } \mathfrak{A}_t \vDash \varphi[a_1, \ldots, a_n].$$

Let P^* be the class of all formulas φ such that for some $\Psi \in P$, $\vdash_c \Psi \leftrightarrow \varphi$ and $\vdash \Psi \rightarrow \varphi$. Let $(*)_\varphi$ be the following property of a formula $\varphi(x_1, \ldots, x_n)$:

For any Kripke model $\langle (T, 0, \leqslant); \mathfrak{A}_t: t \in T\rangle$,

(*) any $t \in T$ and any $a_1, \ldots, a_n \in A_t$

$$\mathfrak{A}_t \vDash \varphi[a_1, \ldots, a_n] \text{ implies } t \Vdash \varphi[a_1, \ldots, a_n].$$

2 Results

Lemma 2 *$\varphi(x_1, \ldots, x_n) \in P^*$ implies $(*)_\varphi$.*

Proof: Let $\Psi \in P$ be such that $\vdash_c \Psi \leftrightarrow \varphi$ and $\vdash \Psi \rightarrow \varphi$. Then $\mathfrak{A}_t \vDash \varphi$ implies $\mathfrak{A}_t \vDash \Psi$. By Lemma 1, $t \Vdash \Psi$ so $t \Vdash \varphi$.

In [5] we defined the following class of formulas S_ω and showed that $\varphi \in S_\omega$ implies $(*)_\varphi$.

Let $S_0 = P$ and, if S_n is already defined, let S_{n+1} contain S_n and for all $\varphi, \Psi \in S_n$ the following:

(i) $\neg\neg\varphi$ (and consequently also $\neg\forall x\neg\varphi$ and $\neg(\varphi \rightarrow \neg\Psi)$)
(ii) $\neg\varphi \rightarrow \Psi$
(iii) $\varphi \vee \Psi$, $\varphi \wedge \Psi$, $\exists x\varphi$.

It is not difficult to show that $S_\omega \subseteq P^*$ (in fact, that will follow from Theorem 1). Unfortunately, not every formula from P^* is (intuitionistically) equivalent to a formula from S_ω (e.g., $(\varphi \to (\neg\Psi \vee \neg\chi)) \to \xi$, where φ, Ψ, χ, and ξ are positive).

We shall need the following classical result:

Lemma 3 (classical) *A sentence is equivalent to a positive sentence iff it is preserved under positive extensions of models.*

Proof: One direction is an easy induction on the complexity of positive sentences. To prove that a sentence φ preserved under positive extensions is equivalent to a positive sentence it is enough to show that $\mathfrak{A} \models \varphi$ and for all $\Psi \in P$, ($\mathfrak{A} \models \Psi$ implies $\mathfrak{B} \models \Psi$), implies $\mathfrak{B} \models \varphi$ (see [1], Lemma 3.2.1 and Corollary 3.2.5). But, from: for all $\Psi \in P$ ($\mathfrak{A} \models \Psi$ implies $\mathfrak{B} \models \Psi$), it follows that $\mathfrak{A}$ is a positive submodel of an elementary extension of $\mathfrak{B}$. So is $\mathfrak{A} \models \varphi$ then $\mathfrak{B} \models \varphi$ (even if φ contains names for some elements of A).

Remark: In intuitionistic predicate calculus, free variables are treated as parameters, i.e., as individual constants.

Lemma 4 *If* $(*)_\varphi$ *and for some* $\Psi \in P$, $\vdash_c \Psi \leftrightarrow \varphi$ *then* $\vdash \Psi \to \varphi$.

Proof: Suppose $t \Vdash \Psi$. Then $\mathfrak{A}_t \models \Psi$, so $\mathfrak{A}_t \models \varphi$. Since $(*)_\varphi$, we have $t \Vdash \varphi$.

We can prove now the main result of this paper.

Theorem 1 $\varphi \in P^*$ *iff* $(*)_\varphi$.

Proof (classical): One direction was proved by Lemma 2. So suppose now that $(*)_\varphi$ and $\varphi \notin P^*$. By Lemma 4 this means that φ is not classically equivalent to any positive formula. If $\varphi = \varphi(x_1, \ldots, x_n)$, we can treat $x_1, \ldots, x_n$ as individual constants, so by Lemma 3 φ is not preserved under positive extensions. Let $\mathfrak{A}$ and $\mathfrak{B}$ be classical structures and $a_1, \ldots, a_n \in A$ such that $\mathfrak{A} \subseteq^+ \mathfrak{B}$, $\mathfrak{A} \models \varphi[a_1, \ldots, a_n]$ and $\mathfrak{B} \not\models \varphi[a_1, \ldots, a_n]$. We can construct then a Kripke model $\langle(\{0,1\}, 0, \leqslant); \mathfrak{A}_0, \mathfrak{A}_1\rangle$ where $\mathfrak{A}_0 = \mathfrak{A}$, $\mathfrak{A}_1 = \mathfrak{B}$ and $0 \leqslant 1$. Since 1 is a terminal node of this model, forcing at 1 coincides with satisfiability in $\mathfrak{B}$ (i.e., $1 \Vdash \xi$ iff $\mathfrak{B} \models \xi$) so $1 \nVdash \varphi[a_1, \ldots, a_n]$ which is a contradiction.

It is easy now to prove the converse to Lemma 1.

Theorem 2 *A formula is intuitionistically equivalent to a positive formula if and only if in any Kripke model*

$$\mathfrak{A}_t \models \varphi[a_1, \ldots, a_n] \text{ iff } t \Vdash \varphi[a_1, \ldots, a_n].$$

Proof: One direction is Lemma 1. For the other direction, assume that in any Kripke model $\mathfrak{A}_t \models \varphi[a_1, \ldots, a_n]$ iff $t \Vdash \varphi[a_1, \ldots, a_n]$. Then $(*)_\varphi$, so $\varphi \in P^*$, i.e., for some $\Psi \in P$, $\vdash_c \Psi \leftrightarrow \varphi$ and $\vdash \Psi \to \varphi$. We have to show only $\vdash \varphi \to \Psi$. Suppose $t \Vdash \varphi$. Then $\mathfrak{A}_t \models \varphi$ and by the classical equivalence $\mathfrak{A}_t \models \Psi$. But Ψ is positive, so $t \Vdash \Psi$.

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