346

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Inaccessible Worlds

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This note presents some considerations on the logic of inaccessibility from the point of view of the Kripke semantics for modal logic. That is, we are interested in the logical properties of the usual language of propositional modal logic (say with $\sim$, $\wedge$, and $\Box$ as primitive) as enriched by a new intensional primitive $\blacksquare$ with the semantical clause:

[$\blacksquare$] For any model $\langle W, R, V\rangle$ and any point $x \in W$, $\langle W, R, V\rangle \models_x \blacksquare\alpha$ iff for all $y \in W$ such that not Rxy, $\langle W, R, V\rangle \models_y \alpha$.

The other connectives receive the clauses familiar from the standard definition of truth (at a point in a model) in Kripke semantics.[1] We write $\Diamond(\blacklozenge)$ for the dual of $\Box(\blacksquare)$. In the enriched language, thus understood, many classes of frames are expressible which are not expressible in the customary language. (We speak of a formula's expressing a class of frames when the formula is valid on all and only frames in the class, where a formula is valid on a frame $\langle W, R\rangle$ just in case it is true at every point in every model $\langle W, R, V\rangle$ on that frame.) Here are five simple examples:

1. The class of irreflexive frames, expressed by: $\blacksquare p \rightarrow p$
2. The class of asymmetric frames, expressed by: $p \rightarrow \Box\blacklozenge p$
3. The class of intransitive frames, expressed by: $\blacksquare p \rightarrow \Box\Box p$
4. The class of strongly connected frames (i.e., frames $\langle W, R\rangle$ such that for all $x, y \in W$ either Rxy or Ryx), expressed by: $p \rightarrow \blacksquare\Diamond p$
5. The class of universal frames (i.e., frames $\langle W, R\rangle$ with $R = W \times W$), expressed by: $\blacksquare(p \wedge \sim p)$.

Apart from any purely technical interest such examples may have, some of them are of obvious relevance when the unenriched language is thought of in tense-logical terms (thinking of $\Box$ as Prior's G, [5]). Further applications are suggested by the fact that we may sometimes wish to consider an operator O

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with the reading: $O\alpha$ is true iff α is true *only* at accessible (*sc.* R-related) worlds, since $O\alpha$ may be defined as $\blacksquare{\sim}\alpha$. For example, in deontic logic, where $\Box$ and $\Diamond$ are thought of as expressing obligatoriness and permissibility, respectively, one may wish to consider the unorthodox notion of strong permissibility, $\boldsymbol{P}$, defined thus: $\boldsymbol{P}\alpha =_{df} \blacksquare{\sim}\alpha$, since all frames will then validate the equivalence

$$(\boldsymbol{P}\alpha \wedge \boldsymbol{P}\beta) \longleftrightarrow \boldsymbol{P}(\alpha \vee \beta)$$

much loved by the proponents of 'free-choice' permission.[2] Our attention in what follows, however, will not be on selling any particular application of the machinery but on isolating axiomatically the logic most appropriately regarded as a minimal system in the enriched language.

For the smoother application of the familiar concepts and established results of intensional logic, we pursue this project within the framework of bimodal logic [5], of which the following is a general characterization. The language of truth-functional logic is extended by two primitive intensional operators, $\Box_1$ and $\Box_2$ and the semantics is cast in terms of frames $\langle W, R_1, R_2\rangle$, the definition of truth at a point in a model proceeding *via* the usual clauses together with:

$$\langle W, R_1, R_2, V\rangle \models_x \Box_i\alpha \text{ iff for all } y \text{ such that } R_ixy, \langle W, R_1, R_2, V\rangle \models_y \alpha,$$
$$i = 1, 2.$$

By K^2 we mean the system axiomatized by the three schemata:

(AO) α where α is any substitution instance in the present language of a truth-functional tautology

and

(Ai) $(i = 1, 2)\Box_i(\alpha \to \beta) \to (\Box_i\,\alpha \to \Box_i\beta)$,

together with modus ponens and the two rules $\vdash\alpha$ to $\vdash\Box_i\alpha$ as rules of proof. K^2 is determined by the class of all frames $\langle W, R_1, R_2\rangle$ and among its more interesting extensions are systems determined by classes of frames characterized in terms of the relation between R_1 and R_2. The best known case (due to Lemmon [5]) is the class of frames in which R_2 is the *converse* of R_1, of special significance as a minimal tense logic, which determines that extension of K^2 which has for its proper axioms instances of the schema:

$$\alpha \to \Box_i\Diamond_j\alpha \text{ for } i \neq j;\ i, j \in \{1, 2\}.$$

More recently, several logicians have investigated the constraint that R_2 should be the ancestral of the relation R_1, and found the extension of K^2 determined by the class of frames in which that constraint is met [6]. For present purposes, of course, what we are interested in is the class of all frames $\langle W, R_1, R_2\rangle$ in which R_2 is the complement of the relation R_1; that is, those frames in which for all $x, y \in W$, R_1xy iff not R_2xy. We call such frames *complementary*, and to emphasize our interest in them we shall write R^+ and R^- for R_1 and R_2, and revert, for vividness, to our original notation of '$\Box$' and '$\blacksquare$' in place of '$\Box_1$' and '$\Box_2$'.

Transposed into the setting of bimodal logic, then, our quest is for the system determined by the class of all complementary frames. It is clear that this system will be a proper extension of K^2. Here for example, are some nontheorems of K^2 which are unfalsifiable at any point in a model on a complementary frame:

(a) $\Diamond T \vee \blacklozenge T$ (where 'T' abbreviates some tautology)
(b) $(\Box p \to p) \vee (\blacksquare q \to q)$
(c) $(\Diamond\blacklozenge p \vee \blacklozenge\Diamond p) \to (\Diamond p \vee \blacklozenge p)$.

So what we want is a set of axioms from which such formulas as these are deducible but from which no formula not valid on *every* complementary frame is deducible.

The solution to our problem turns out to be quite simple: we need only add as axioms to K^2 instances of a single schema. To state the schema we use '$\boldsymbol{W}$' to stand for strings of any (including zero) length of occurrences of the weaker operators '$\Diamond$' and '$\blacklozenge$', and similarly '$\boldsymbol{S}$' as schematic over strings of '$\Box$' and '$\blacksquare$', the 'strong' modal operators.[3] The schema then reads:

$$(*) \qquad \boldsymbol{W}(\Box\alpha \wedge \blacksquare\beta) \to \boldsymbol{S}(\alpha \vee \beta).$$

We refer to the system axiomatized by adding all instances of $(*)$ to the basis described above for K^2 as $K^2 + (*)$. That this system is sound with respect to the class of all complementary frames reduces to the claim that $(*)$ is valid on every such frame, and the correctness of this claim may be seen as follows. Suppose some formula of the form $(*)$ is false at a point x in a model on a complementary frame. Thus for this model—call it $\langle W, R^+, R^-, V\rangle$ or $\mathfrak{M}$ for short—we have:

$$\mathfrak{M} \models_x \boldsymbol{W}(\Box\alpha \wedge \blacksquare\beta) \quad \text{but } \mathfrak{M} \not\models_x \boldsymbol{S}(\alpha \vee \beta).$$

From the first of these facts, we conclude that x is the first element of a sequence of members of W each related to its predecessor either by R^+ or by R^-, whose last member is a point y such that $\mathfrak{M} \models_y \Box\alpha \wedge \blacksquare\beta$. (If we write $|\boldsymbol{W}|$ to denote the length of the prefix $\boldsymbol{W}$, then we may say more specifically that this sequence is of length $|\boldsymbol{W}|$, understanding that in the case where $|\boldsymbol{W}| = 0$, the point y is just x itself.) Similarly, from the second fact, we conclude that x begins an $(R^+ \cup R^-)$-chain of length $|\boldsymbol{S}|$, whose last member is a point z such that $\mathfrak{M} \not\models_z \alpha \vee \beta$. Now, since the frame of this model is complementary, either R^+yz or R^-yz. But the first is ruled out by our having $\mathfrak{M} \models_y \Box\alpha$ (since $\mathfrak{M} \models_y \Box\alpha \wedge \blacksquare\beta$) together with $\mathfrak{M} \not\models_z \alpha$ (since $\mathfrak{M} \not\models_z \alpha \vee \beta$), and the second by our having $\mathfrak{M} \models_y \blacksquare\beta$ (since $\mathfrak{M} \models_y \Box\alpha \wedge \blacksquare\beta$) together with $\mathfrak{M} \not\models_z \beta$ (since $\mathfrak{M} \not\models_z \alpha \vee \beta$). So it is not possible for an instance of $(*)$ to be false in any model on a complementary frame.

The 'soundness' half of our claim that $K^2 + (*)$ is determined by the class of all complementary frames thus established, we turn to the proof of completeness. The argument we use is modeled after Cresswell's adaptation to modal logic of the method Henkin used to prove the completeness of first-order logic, rather than the more widely known adaptation of that method due to Scott and Makinson [1]. (In the former case maximal consistent sets of formulas are correlated with elements of the falsifying model but the correlation is not required to be one-one, so that there is more freedom in constructing the required accessibility relation than on the latter approach—as generally implemented—in which the maximal consistent sets are identified with the points of the model serving to falsify any given nontheorem. The 'canonical models' of the latter approach are not, as they stand, very helpful in delivering the completeness result we are after.) An overall description of the procedure is as follows. We begin with the negation of a nontheorem of $K^2 + (*)$, and blow it up to a maximal consistent set of formulas (taking maximal consistency here, and below, relative to $K^2 + (*)$). This set of formulas will be associated with a point, the 'root' of a tree we are going to construct. Its lower nodes will likewise be associated with maximal consistent sets of formulas in such a way that the resulting tree will be a model verifying the formula we started with at the root (or 'top node') of the tree. The schema $(*)$ will be appealed to in order to show that this can be done in such a way that the frame of the model in question is complementary. We now fill in some of the details, skimming over those familiar from general post-Henkin completeness proofs.

Let α be the formula whose negation is unprovable in $K^2 + (*)$, and Γ a maximal consistent set containing α. We call the top node of our tree $\langle\Gamma\rangle$, and in general will use finite sequences $\langle\Gamma, \beta_1, \ldots, \beta_n\rangle$ for formulas β_i, to label the nodes. These sequences can be construed in any convenient way (e.g., as partial functions from positive integers to formulas and sets thereof) which does not identify $\langle\Gamma\rangle$ with Γ. For such a sequence x, the sequence $x\hat{}\beta$ is the $n + 1$-termed sequence, taking x to be of length n, whose first n positions are filled as in x, and whose final term is the formula β. With each of the points x in the tree we associate a set of formulas $f(x)$. For the top node we stipulate that $f(\langle\Gamma\rangle) = \Gamma$. Now, for any formula of the form $\Diamond\beta$ in Γ, we adjoin as an immediate descendant below $\langle\Gamma\rangle$ a point $\langle\Gamma, \Diamond\beta\rangle$ and take $f(\Gamma, \Diamond\beta)$ to be some maximal consistent extension of the set $\{\beta\} \cup \{\gamma | \Box\gamma \in \Gamma\}$. Similarly, for any formula $\blacklozenge\beta$ in Γ, we subjoin the point $\langle\Gamma, \blacklozenge\beta\rangle$, with $f(\langle\Gamma, \blacklozenge\beta\rangle)$ a maximal consistent extension of $\{\beta\} \cup \{\gamma | \blacksquare\gamma \in \Gamma\}$. (The consistency of the sets in question, given the consistency of Γ, follows from facts about provability in K^2; the extendibility to maximal consistent sets from Lindenbaum's Lemma applied to the present language.) To points of the first kind, we stipulate that $\langle\Gamma\rangle$ is to bear the relation R^+, and to points of the second kind, R^-. We continue to build the tree downward, defining f as we go, in the same manner. For example, if $\Diamond\beta' \in f(\langle\Gamma, \blacklozenge\beta\rangle)$, we provide as an immeidate R^+-related descendant a new point $\langle\Gamma, \blacklozenge\beta, \Diamond\beta'\rangle$ with $f(\langle\Gamma, \blacklozenge\beta, \Diamond\beta\rangle)$ being some maximal consistent extension of $\{\beta'\} \cup \{\gamma | \Box\gamma \in f(\langle\Gamma, \blacklozenge\beta\rangle)\}$. And so on. (Typically, the whole tree thus constructed will have both infinite branches and infinitary branching. We are not here interested in finite models.) Calling the totality of nodes in the tree W, we define a model on the frame $\langle W, R^+, R^-\rangle$ by specifying that $V(p_i, x)$ for a propositional variable p_i and a point $x \in W$, is to be T iff $p_i \in f(x)$. By standard arguments, we may show:

(**) *For any formula* β, $\langle W, R^+, R^-, V\rangle \models_x \beta$ *iff* $\beta \in f(x)$, *for all* $x \in W$.

So far so good; but of course the frame of this model is not complementary, since although the relations R^+ and R^- are mutually exclusive (disjoint), they are not jointly exhaustive (do not have $W \times W$ for their union), as no point is related to any points other than its immediate descendants by either of the relations R^+, R^-. We must now show that it is possible to extend R^+ and R^- so that they come to jointly exhaust $W \times W$, without jeopardizing the truth of (**). So take a pair of (not necessarily distinct) points y, z with neither R^+yz nor R^-yz. We claim it must be possible, consistently with (**), to add the pair $\langle y, z\rangle$ to R^+ or else to add it to R^-. For, if not, this can only be because for some formula β, $\Box\beta \in f(y)$ while $\beta \notin f(z)$, and also for some formula γ, $\blacksquare\gamma \in f(y)$ while $\gamma \notin f(z)$. But each of the points y and z is located finitely many steps down the tree from the top node $\langle\Gamma\rangle$. Say y is m steps down; more specifically that for some formulas $\beta_1, \ldots, \beta_m$ $y = \langle\Gamma, \boldsymbol{W}_1\beta_1, \ldots, \boldsymbol{W}_m\beta_m\rangle$, where each $\boldsymbol{W}_i$ is either $\Diamond$ or $\blacklozenge$. And let $z = \langle\Gamma, \boldsymbol{W}'_1\beta'_1, \ldots, \boldsymbol{W}'_n\beta'_n\rangle$ for some formulas $\beta'_1, \ldots, \beta'_n$, with the '$\boldsymbol{W}'_i$' notation similarly understood. Using '$\boldsymbol{S}_i$' for '$\Box$' if $\boldsymbol{W}'_i$ is '$\Diamond$', and for '$\blacksquare$' if $\boldsymbol{W}'_i$ is '$\blacklozenge$', consider the instance of (*):

$$\boldsymbol{W}_1 \ldots \boldsymbol{W}_m(\Box\beta \wedge \blacksquare\gamma) \rightarrow \boldsymbol{S}_1 \ldots \boldsymbol{S}_n(\beta \vee \gamma).$$

Ex hypothesi, the location of the point y in the tree and the truth of each of $\Box\beta$ and $\blacksquare\gamma$ at y (in $\langle W, R^+, R^-, V\rangle$) guarantee that the antecedent of this formula is true at $\langle\Gamma\rangle$, and so therefore that its consequent is. But that would mean we must have $\langle W, R^+, R^-, V\rangle \models_z \beta \vee \gamma$, whereas we have had to suppose that $\langle W, R^+, R^-, V\rangle \not\models_z \beta$ and $\langle W, R^+, R^-, V\rangle \not\models_z \gamma$ (since this follows from (*) and the assumption that β and γ fail to belong to $f(z)$).

So either it is possible to preserve (*) while adding $\langle y, z\rangle$ to R^+ or it is possible to preserve (*) while adding $\langle y, z\rangle$ to R^-. Thus we complete our specification of the falsifying model (for the originally supposed nontheorem $\sim\alpha$) by saying that one or other (but not both) of these extensions is made to R^+ or R^- for each unrelated pair $\langle y, z\rangle$. This makes the frame of the model a complementary frame, and because (**) was not interfered with, we may still conclude that the given nontheorem is false at the node $\langle\Gamma\rangle$ in that frame, according to the model indicated, and thus, that K^2 + (*) is complete with respect to the class of complementary frames.

The fact that we have found K^2 + (*) to be the system determined by the class of complementary frames shows that we have succeeded in locating the correct logic for that class of frames in one sense of 'correct logic': the sense appropriate to what has been called the theory of *completeness,* as opposed to the theory of correspondence [7]. The relevant question from the latter point of view may be put with the aid of Lewis's helpful term 'range' [3]: the *range* of a system being the class of all those frames on which every theorem of the system is valid. What system, we may ask, has for its range the class of complementary frames? The answer is: no system. For suppose that class is the range of a system $\mathcal{L}$, and consider two complementary frames $\langle W, R^+, R^-\rangle$ and $\langle X, S^+, S^-\rangle$ with $W \cap X = \phi$; each of these frames, *ex hypothesi*, validates every theorem of $\mathcal{L}$. But then it follows from the Generation Theorem for bimodal logic that no theorem of $\mathcal{L}$ can fail to be valid on the frame $\langle W \cup X, R^+ \cup S^+, R^- \cup S^-\rangle$; so the range of $\mathcal{L}$ includes a frame which is patently not complementary [5]. Thus there is a striking contrast between the class of bimodal frames in which the two relations are converses—the class of importance in tense logic—and the class of bimodal frames in which the two relations are complementary. For in the former case the minimal system of tense logic, mentioned above, is not just determined by the class of frames concerned, but has that class for its range, while in the latter case only the determination result is available. We close by noting some repercussions of this fact.

In the first place, we observe that no formula expresses (in the sense of our opening paragraph) the class of complementary frames: for by the completeness result such a formula would have to be a theorem of $K^2 + (*)$, while we have seen that some noncomplementary frames validate every theorem of that system. In fact by standard arguments, one may go further than this and show that neither the class of frames in which R^+ and R^- are mutually exclusive, nor the class of frames in which these relations are jointly exhaustive, is expressible. Secondly, we note the muting effect of these facts on the original promise of using the extended language to express hitherto inexpressible classes of frames (Examples 1-5). One must retreat to the weaker position of saying that the formulas mentioned in those examples are valid on, among the complementary frames, all and only such frames as meet the cited conditions. This point appears to tell against the suitability of the original formulation of the truth conditions of '■'-formulas and in favour of the bimodal approach.[4] Finally, it is worth mentioning the noncumulativity (or nonadditivity, as it is often put) of completeness proofs achieved by the method employed here, in contrast to those based on canonical models without further constructions (such as Segerberg's bulldozing). For example, the class of frames $\langle W, R^+, R^-\rangle$ which are complementary and have R^+ reflexive can be shown, by a similar Cresswellian procedure, to determine $K^2 + (*) + \Box\alpha \rightarrow \alpha$. Likewise, *mutatis mutandis,* for the case of R^- reflexive (the case of Example 1). Clearly, however, no complementary frames can validate both schemata; yet, far from being determined by the empty class of frames, which implies inconsistency for the logic concerned, the system is consistent and has among its range all those far-from-complementary frames in which R^+ and R^- coincide as $W \times W$.

NOTES

1. See [1], [5] for background, terminology, and notation not explained here. We use $\vee$, $\rightarrow$, and $\longleftrightarrow$ for the defined truth-functional connectives, 'p', 'q', . . . for propositional variables, and 'α', 'β', . . . as metalinguistic variables ranging over formulas. The ■/♦ notation is due to my colleague T. E. Karmo, as are Examples 1-3 below, and therewith the stimulus for investigating this topic.

2. I am here mentioning, rather than endorsing, this suggestion, which may be extracted from some of von Wright's remarks in [8]; see also [4]. On the subject of 'only', let me mention that part of my interest in the subject of inaccessibility arose from pondering over a misprint in Leblanc's [2], where (at p. 232) the truth-conditions for formulas of the form $\Box\alpha$ are given in terms of the truth of α at all *and only* accessible worlds. This raises a question not to be pursued here: for which modal logics can a notion of validity which coincides with theoremhood be given based on this deviant truth-definition?

3. 'Strong' and 'weak' are useful labels not to be taken too literally in the present context since the logic on offer here proves neither $\Box p \rightarrow \Diamond p$ nor $\blacksquare p \rightarrow \blacklozenge p$.

4. At least for theoretical reasons: for on the original approach, the contrast between what I have just distinguished as stronger and weaker claims cannot be drawn.

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