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Correction of the Semantics for S4.03 and a Note on Literal Disjunctive Symmetry

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The term *disjunctive symmetry,* designating that property possessed by a Kripke-model $\langle W, R, v\rangle$ just in case for each $x \in W$

(i) $(\exists y)[xRy \,\&\, (x')(y')[(xRx' \,\&\, yRy') \supset (x'Rx \vee y'Rx')]]$,

was defined in Georgacarakos's [2] where it was argued that *S4.03*—that is, *S4*(**I1**), the system obtained by adding each substitution instance of

I1 $L(Lp \rightarrow q) \vee (LMLq \rightarrow p)$

to *S4*—is characterized by the class of disjunctively symmetrical *S4*-models.

At first sight, this characterization seems altogether fitting: **I1** is a weakened version of

F $L(Lp \rightarrow q) \vee (MLq \rightarrow p)$,

the proper axiom for *S4.3.2*; and the condition used to define disjunctive symmetry is, similarly, a weakening of

(ii) $(x)(y)[(xRy \,\&\, xRz) \supset (zRy \vee yRx)]$,

which specifies a class of *S4*-models known to characterize *S4.3.2* (see, e.g., [4], Lemma 7.10). It turns out, however, that the appearance of having simultaneously relaxed semantic and syntactic constraints is deceptive. For, although *S4.03* is a proper subsystem of *S4.3.2* (see [1]), the system characterized by the class of disjunctively symmetrical *S4*-models is not; rather, it is *identical* with *S4.3.2,* a fact that will be proved shortly (Theorem 1).

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From all this we must conclude that there is an error in the proof of the main result of [2] and that a new semantics for *S4.03* is needed.[1] Theorem 2, below, accomplishes this latter task. The paper ends with a brief look at a family of (mostly) new extensions of *S4*, each of which is characterized by a class of *S4*-models whose members are disjunctively symmetrical in a somewhat more literal sense of the term than that mentioned above.

Considerable use will be made of post-Henkin style completeness proofs and of filtration theory as developed by Segerberg in [4]. The reader is presumed to be familiar with the terminology, methods, and results of this reference.

Theorem 1 *S4.3.2 is characterized by the class of disjunctively symmetrical S4-models.*

Proof:

Soundness. Suppose that some instance

$$L(L\alpha \rightarrow \beta) \vee (ML\beta \rightarrow \alpha)$$

of **F** is false at a point x in an *S4*-model $\langle W, R, v\rangle$ satisfying (i). Then

(a) $L(L\alpha \rightarrow \beta)$ is false at x

and

(b) $ML\beta$ is true at x

while

(c) α is false at x.

By (a), there is a point x_1' such that xRx_1' and

(d) $L\alpha$ is true at x_1'

and

(e) β is false at x_1'.

Moreover, by (b), there is a point x_2' such that xRx_2' and

(f) $L\beta$ is true at x_2'.

Since $\langle W, R, v\rangle$ satisfies (i), there is a point y such that xRy and

(g) $(x')(y')[(xRx' \;\&\; yRy') \supset (x'Rx \vee y'Rx')]$.

Now, given (c) and (d), $x_1' \not R x$; so, by (g),

(h) $(y')(yRy' \supset y'Rx_1')$.

Similarly, (e) and (f) require that $x_2' \not R x$; so, by (g) again,

(j) $(y')(yRy' \supset y'Rx_2')$.

Taken together, (h) and (e) imply that

(k) $ML\beta$ is false at y.

On the other hand, (j) and (f) imply that

(l) $LML\beta$ is true at y.

This, however, contradicts (k).

Completeness. The canonical model for *S4.3.2*, $K_{S4.3.2}$, is known to satisfy (ii) (see, e.g., [4], Lemma 7.10), and (i) is readily deducible from (ii) in the presence of reflexivity. $K_{S4.3.2}$ is therefore disjunctively symmetrical.

Theorem 2 *S4.03 is characterized by the class of S4-models satisfying*

(iii) $(x)(y)[(xRy \supset yRx) \vee (\exists z)[xRz \;\&\; (z')(zRz' \supset z'Ry)]]$.

Proof:

Soundness. Suppose that some instance

$$L(L\alpha \to \beta) \vee (LML\beta \to \alpha)$$

of **I1** fails at a point x in an *S4*-model $\langle W, R, v\rangle$ satisfying (iii). Then

(m) $L(L\alpha \to \beta)$ is false at x

and

(n) $LML\beta$ is true at x

but

(o) α is false at x.

By (m), there is a point y such that xRy and

(p) $L\alpha$ is true at y

while

(q) β is false at y.

Given that xRy and $y\not R x$ (by (o) and (p)), (iii) requires that there be a point z such that xRz and

(r) $(z')(zRz' \supset z'Ry)$.

By (n), $ML\beta$ is true at z; so there is a point z' such that zRz' and

(s) $L\beta$ is true at z'.

Together, (r) and (s) imply that β is true at y; but this contradicts (q).

Completeness. Suppose that γ is a nontheorem of *S4.03*. Then there is a point t in the canonical model for *S4.03*, $K_{S4.03}$, at which γ is false. Let Ψ be the smallest set containing γ that is closed under the formation of subformulas and modalities (Ψ will be finite, since *S4.03* is a normal extension of *S4*), and let $K' = \langle W', R', v'\rangle$ be a Lemmon-filtration of $K_{S4.03}$ through Ψ. K' will be finite, reflexive, and transitive; and γ will fail at $[t]$ in K'.

All that remains to be shown is that K' satisfies (iii). So suppose, for a *reductio*, that it does not. Then there are points $[x]$ and $[y]$ in W' such that $[x]R'[y]$ but

(t) $[y] \not R' [x]$.

Moreover,

(u) $([z])\{[x]R'[z] \supset (\exists[z'])([z]R'[z'] \;\&\; [z'] \not R' [y])\}$.

Given that K' is a Lemmon-filtration, (t) and the Filtration Theorem ([4], p. 66) imply that there is a formula $L\alpha \in \Psi$ such that

(v) $L\alpha$ is true at $[y]$

but

(w) $L\alpha$ is false at $[x]$.

Since K' is finite, $[x]$ bears R' to at most finitely many points in W'–say $[z_1], \ldots, [z_n]$; and by (u), for each $1 \leqslant i \leqslant n$, there is a point $[z_i']$ such that $[z_i]R'[z_i']$ and

(x) $[z_i'] \not R' [y]$.

Consequently, there are formulas $L\beta_1, \ldots, L\beta_n \in \Psi$ such that for each $1 \leqslant i \leqslant n$

(y) $L\beta_i$ is true at $[z_i']$

while

(z) $L\beta_i$ is false at $[y]$.

By Theorem 7.5 of [4], K' is a finest filtration, which implies that there is a point $u \in [x]$ and a point $w \in [y]$ such that $uR_{S4.03}w$. With $L\alpha \in \Psi$ and $u \in [x]$, (w) and the Filtration Theorem guarantee that

(a′) $L\alpha$ is false at u.

Similarly, since $L\alpha, L\beta_1, \ldots, L\beta_n \in \Psi$ and $w \in [y]$, (v), and (z) imply that

(b′) $L\alpha$ is true at w

and, for $1 \leqslant i \leqslant n$,

(c′) $L\beta_i$ is false at w.

From (b′) and (c′) we may conclude that $L\left(L\alpha \rightarrow \sum_i L\beta_i\right)$ is false at u and, therefore, that

(d′) $L\left(LL\alpha \rightarrow \sum_i \beta_i\right)$ is false at u.

Pick any point u' such that $uR_{S4.03}u'$. Then $[u]R'[u']$–that is, $[x]R'[u']$– which means $[u'] = [z_i]$, for some $1 \leqslant i \leqslant n$. Say $[u'] = [z_j]$. By (y), this implies that $ML\beta_j$ is true at $[u']$; and since $ML\beta_j \in \Psi$, $ML\beta_j$ is true at u', from which it follows that $ML\left(\sum_i L\beta_i\right)$ is true at u'. As u' was selected arbitrarily,

(e′) $LML\left(\sum_i L\beta_i\right)$ is true at u.

Finally, letting $\sigma = L\alpha$ and $\tau = \sum_i L\beta_i$, we have, by (a′), (d′), and (e′), that

$$L(L\sigma \rightarrow \tau) \vee (LML\tau \rightarrow \sigma) \text{ is false at } u$$

in $K_{S4.03}$, which is impossible.

Corollary 3 *S4.03 is decidable.*

Proof: The completeness portion of the proof of Theorem 2 shows that *S4.03* has the finite model property; and this, together with the finite axiomatizability of *S4.03*, guarantees decidability.

When taken together with the semantic characterizations of *S4.01, Z1,* and *K1* known in the literature, Theorem 2 also yields semantics for *S4.01*(**I1**), *Z1*(**I1**), and *K1*(**I1**)–systems introduced by Georgacarakos in [1], where they are called *S4.05, Z1.5*, and *K1.1.5*, respectively. In particular, defining an *S4.03-model* to be an *S4*-model satisfying (iii), we have

Corollary 4 (a) *S4.05 is characterized by the class of finite S4.03-models in which every proper final cluster is last;* (b) *K1.1.5 is characterized by the class of S4.03-models in which each point is contained in or precedes a simple final cluster; and* (c) *Z1.5 is characterized by the class of S4.03-models that satisfy*

(iv) $(x)(y)[(xRy \supset yRx) \vee (\exists z)[yRz \,\&\, (z')(zRz' \supset z' = z)]]$.

After working with disjunctive symmetry as it is defined at the start of this paper, it is natural to wonder which extensions of *S4* are characterized by those classes of Kripke-models that are disjunctively symmetrical in the more literal sense of the phrase. Put more precisely, we want to know, for each $n \geqslant 1$, what system is characterized by the class of *S4*-models that satisfy

LDS$_n$ $$(x)(y_1)\ldots(y_n)\left[\left(\prod_i xRy_i \,\&\, \prod_{i<j} y_i \neq y_j\right) \supset \sum_i y_iRx\right].$$

An answer is easily obtained, and we shall state it without proof as

Theorem 5 *Let* **LDS$_n$** *be the formula*

$$p \vee L(Lp \rightarrow q_1) \vee \ldots \vee L\left[\left(Lp \,\&\, \prod_{1 \leqslant i<n} q_i\right) \rightarrow q_n\right].$$

Then, for each $n \geqslant 1$, S4(**LDS$_n$**) *is characterized by the class of S4-models satisfying* **LDS$_n$**.

S4(**LDS$_1$**) is obviously just *S5*. Not so obvious, perhaps, is the fact that *S4*(**LDS$_2$**) is also a system known in the literature, namely, *Z8*. This will be proved in several stages, beginning with

Lemma 6 *Each substitution instance of* **LDS$_2$** *is a theorem of Z8.*

Proof: Z8 is characterized by the class of *S4*-models that satisfy (ii) and (iv); so if some instance

$$\alpha \vee L(L\alpha \rightarrow \beta) \vee L[(L\alpha \,\&\, \beta) \rightarrow \gamma]$$

of **LDS$_2$** were a nontheorem of *Z8*, it would have to fail in an *S4*-model $\langle W, R, v\rangle$ satisfying both of those conditions. We assume, for a *reductio*, that it does. Then there is a point $x \in W$ such that (f′) α is false at x

and

(g′) $L(L\alpha \rightarrow \beta)$ is false at x

and

(h′) $L[(L\alpha \,\&\, \beta) \rightarrow \gamma]$ is false at x.

By (g′), there is a point y such that xRy and

(j′) $L\alpha$ is true at y

while

(k′) β is false at y.

Similarly, by (h′), there is a point z such that xRz and

(l′) $L\alpha \,\&\, \beta$ is true at z

while

(m′) γ is false at z.

Now (f′) and (j′) imply that

(n′) $y\not R x$

and (f′) and (1′) imply that

(o′) $z\not R x$.

Further, (o′) and condition (ii) yield

(p′) yRz.

By (n′) and condition (iv), y bears R to a 'terminal' point z'; and since yRz but $z \neq y$ (by (k′) and (l′)), z' must be distinct from y. This, in light of the fact that z' is terminal, means that

(q′) $z'\not R y$.

But xRy and xRz'; thus (q′) and (ii) give

$$yRx$$

contradicting (n′).

The straightforward semantic proof of the following lemma is left to the reader.

Lemma 7 *Each substitution instance of* **F** *is a theorem of S4*($\mathbf{LDS_2}$).

Lemma 8 *Each substitution instance of*

Z2 $L(LMp \rightarrow MLp) \vee L(Mq \rightarrow LMq)$

is a theorem of S4($\mathbf{LDS_2}$).

Proof: Suppose an instance

$$L(LM\alpha \rightarrow ML\alpha) \vee L(M\beta \rightarrow LM\beta)$$

of **Z2** were to fail in an *S4*-model $\langle W, R, v\rangle$ satisfying LDS_2. Then there would be a point $x \in W$ such that

(r′) $L(LM\alpha \rightarrow ML\alpha)$ is false at x

and

(s′) $L(M\beta \rightarrow LM\beta)$ is false at x.

By (r′), there is a point y such that xRy and

(t′) $LM\alpha$ is true at y

while

(u′) $ML\alpha$ is false at y.

Now (t′), in the presence of reflexivity, guarantees that y bears R to a point at which α is true; and if α is false at y, then that point must be distinct from y. Similarly, (u′) guarantees that y bears R to a point at which α is false; and if α is true at y, then that point is distinct from y. So we may conclude that there is a point z such that yRz and $y \neq z$. But this, together with LDS_2 and the fact that xRy, requires that yRx. Consequently,

(v′) $LM\alpha$ is true at x

and

(w′) $ML\alpha$ is false at x.

By (s′), there is a point u such that xRu and

(x′) $M\beta$ is true at u

but

(y′) $LM\beta$ is false at u.

And, by (y′), there is a point v such that uRv and

(z′) $M\beta$ is false at v.

Since xRv, (v′) and (w′) imply that

(a″) $M\alpha$ is true at v

and

(b″) $L\alpha$ is false at v.

Employing an argument similar to the one used in establishing (v′) and (w′), we may infer from (a″) and (b″) that there is a point w such that vRw and $v \neq w$. Given LDS_2 and the fact that uRv, it follows that vRu and so, by (z′), that

(c″) $M\beta$ is false at u.

This, however, contradicts (x′).

Since *Z8* is *S4*(**F**, **Z2**), Lemmas 6, 7, and 8 suffice for

Theorem 9 $S4(\mathbf{LDS_2}) = Z8$.

Theorems 5 and 9, taken together, provide a new semantic characterization of *Z8*. At least one other result of this sort can also be obtained using Theorem 9. Defining an *S4.3.2-model* to be an *S4*-model satisfying (ii), we have

Corollary 10 *Z8 is characterized by the class of S4.3.2-models in which every proper cluster is first.*

Proof: The proof of soundness—that each instance of $\mathbf{LDS_2}$ is valid in each *S4.3.2*-model satisfying the stated condition—is left to the reader.

Completeness. Assume that γ is a nontheorem of *Z8*. Then γ fails at a point t in the canonical model K_{Z8} for *Z8*. Moreover, K_{Z8} satisfies (ii), since *Z8* is an extension of *S4.3.2*.

Let $K' = \langle W', R', v' \rangle$ be the model generated from K_{Z8} by t. Then K', too, satisfies (ii) and rejects γ at t (see [4], Theorem 3.10). Now suppose that K' fails to satisfy the stated condition. This can only mean that there is a proper cluster C in K' which is not first. Thus there is a point $z \in W'$ to which none of the points in C bears R'.

Let x and y be distinct points in C. Then

(d″) $x \not{R}' z$.

Since t generates K',

(e″) $tR'x$

(f″) $tR'y$

and

(g″) $tR'z$.

By (d″), (g″), and transitivity,

(h″) $x \not{R}' t$.

Therefore, there is a formula α such that

(j″) $L\alpha$ is true at x

while

(k″) α is false at t.

But x and y are in the same cluster; so

(l″) $L\alpha$ is true at y

as well; and, moreover, since $x \neq y$, there is a formula β_1 such that

(m″) β_1 is true at y

but

(n″) β_1 is false at x.

By (l″) and (m″),

(o″) $(L\alpha \,\&\, \beta_1) \rightarrow {\sim}\beta_1$ is false at y

and, by (j″) and (n″)

(p″) $L\alpha \rightarrow \beta_1$ is false at x.

Putting (k″), (o″), and (p″) together with (e″) and (f″), we may infer that an instance of $\mathbf{LDS}_2$, namely,

$$\alpha \vee L(L\alpha \rightarrow \beta_1) \vee L[L\alpha \,\&\, \beta_1) \rightarrow {\sim}\beta_1]$$

is false at t; but, given Theorem 9, this is impossible.

As for the remaining members of the $S4(\mathbf{LDS}_n)$ family, semantic considerations readily show that each, save $S4(\mathbf{LDS}_3)$, is a proper subsystem of *Z8* and its extensions, independent of the other well-known extensions of *S4* (for which, see the diagram on p. 574 of [3]) and of *S4.03, Z1.5,* and *K1.1.5.* The account of $S4(\mathbf{LDS}_3)$ differs only in that it is a proper extension of *S4.01.*

NOTE

1. The error occurs on p. 506 of [2]: $LML\gamma \in \Gamma_i$ is inferred from $ML\gamma \in \Gamma_j$, when the latter only warrants the conclusion that $ML\gamma \in \Gamma_i$.

REFERENCES

[1] Georgacarakos, G. N., "Additional extensions of S4," *Notre Dame Journal of Formal Logic,* vol. 18 (1977), pp. 477-488.

[2] Georgacarakos, G. N., "Semantics for S4.03," *Notre Dame Journal of Formal Logic,* vol. 18 (1977), pp. 504-506.

[3] Goldblatt, R. I., "A new extension of S4," *Notre Dame Journal of Formal Logic,* vol. 14 (1973), pp. 567-574.

[4] Segerberg, K., *An Essay in Classical Modal Logic,* Filosofiska Studier, Uppsala, Sweden, 1971.

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