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The Axiom of Choice in Topology

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1 Introduction and definitions In this paper we are concerned with soft applications of the axiom of choice (AC) in general topology. We define 16 properties which hold in ZF for each T_2 space, if and only if AC is true, and we investigate what implications between these axioms are provable without AC (in the presence of AC there is nothing to prove). Our results are summarized in two diagrams. In Figure 1 the 61 valid implications are listed. Counterexamples in three models prove 188 of the possible implications to depend on the axiom of choice, as is shown in Figure 2. Some problems remain open.

Our positive results are proved in $ZF°$, Zermelo-Fraenkel set theory without the axioms AC and foundation. Our counterexamples are constructed in models of ZF ($=_{df}$ $ZF°$ + foundation). We shall use Levy's axiom MC of multiple choice: If F is a family of nonempty sets, there is a mapping f on F, such that $\phi \neq f(x) \subseteq x$ and $f(x)$ is finite for each x in F. Rubin's axiom PW asserts that the power set $\mathcal{P}(x)$ of each well-orderable set x is well-orderable. In $ZF°$, AC implies MC which implies PW, and there are permutation models which show that in $ZF°$ the implications cannot be reversed. But AC, MC, and PW are equivalent in ZF. Similar axioms are studied in [6].

If P and Q are topological properties, $A(P)$ is the assertion that each T_2 space is P and $A(P,Q)$ says that each T_2 space which is P also satisfies Q. While for the properties P defined below we do understand the position of $A(P)$ in the hierarchy of choice principles, the same questions for $A(P,Q)$ remain unanswered in many cases, although as we have already noticed $A(P,Q)$ depends on AC in general.

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X is a space, $\underline{X}$ its topology; $f'(X) = \{f(x): x \in X\}$. We abbreviate "well-ordered" and "well-orderable" by w.o. and Dedekind-finite by D-finite. "Space" always means T_2 space.

W1: $\underline{X}$ is w.o.
W2: X is w.o.
D1: X is covered by a w.o. family of closed, discrete sets.
D2: X is covered by a w.o. family of discrete sets.
B1: X has a w.o. base.
B2: There is a function $f: X \times W \to \underline{X}$ such that W is w.o. and $f'\{x\} \times W$ is a neighborhood base at x.
B3: There is a mapping $f: X \times W \to \underline{X}$ such that W is w.o. and $\{x\} = \cap f'(\{x\} \times W)$.
L1: Each open cover of X has a w.o. subcover.
L2: Each open cover of X has a w.o. refinement.
S: X has a dense, w.o. subset.
C: Each family $O \subseteq \underline{X}$ of pairwise disjoint sets is w.o.
A1: If $O \subseteq \underline{X}$ covers X, there is a mapping $f: X \to O$ such that $x \in f(x)$.
A2: If $O \subseteq \underline{X}$ covers X, there is a mapping $f: X \to \underline{X}$ such that $x \in f(x)$ and $f'X$ refines O.
H1: X is hereditarily *A1*.
H2: X is hereditarily *A2*.
F: X is a continuous finite-to-one image of an *A1* space.

B1, *L1*, *S*, and *C* naturally appear in the study of cardinal functions in the sense of Juhász in $ZF°$. The introduction of *L2* is motivated by a result, due to Jech (Pennsylvania), that **R** is not Lindelöf, but **R** obviously is *L2*.

Topologically most interesting is *A2*, which is both a weakening of metacompact and w.o. local weight. The following theorem was proved by Fritsch (München): A *CW*-complex is paracompact if and only if it is *A2*. But it should be noted that none of the properties defined in the section imply any classical topological notion. *D2* is a weakening of *W2*. Rubin (Purdue) proved $A(D2) \Rightarrow PW$. Another natural consequence of *W2* is $K: X = \cup W$ where W is a w.o. family of compact sets. We do not study K, since it has no applications. For the same reason we consider *B2*, *B3* instead of *B2'*: Each point has a w.o. local base and *B3'*: Each singleton is an intersection of a w.o. family of open sets. We mention some further properties. B_{3a}: There is a mapping $f: X \times W \to \underline{X}$, W w.o., such that $\{x\} = \bigcap_{w \in W} f(x,w)^-$. K': $X = \cup W$, where W is a w.o. family of compact sets, such that each compact set is contained in some $V \in W$.

2 *Equivalents to AC* We start with the observation that if $P' \Rightarrow P$, $Q \Rightarrow Q'$ $A(P,Q) \Rightarrow A(P',Q')$. If P is hereditary, $A(P,Ai) \Leftrightarrow A(P,Hi)$. *PW* implies $A(P,Q)$ for $P \in \{W1, W2, B1, S\}$, since P-spaces are *W1* in $ZF° + PW$ (c.f. [2]). Moreover we note that AC^{wo} (*AC* for w.o. families) implies $A(L2, L1)$.[1]

2.1 Lemma *In $ZF°$ the implications of Figure 1 hold.*

Proof: Let X be *W1*. Since X is T_1, $\{X \setminus \{x\} : x \in X\}$ is w.o. and so X is *W2*.

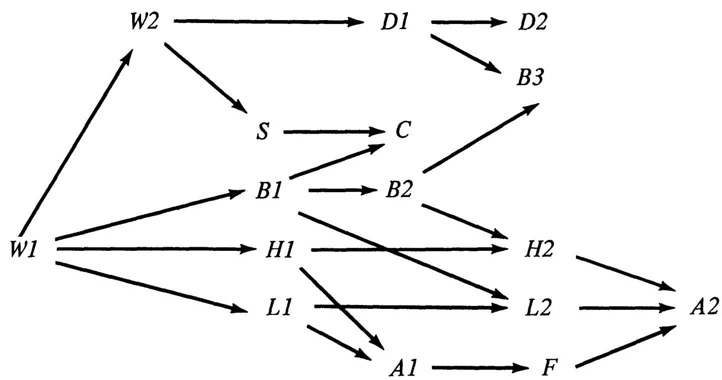

Figure 1. Implications.

B1 and *L1* are clear and we shall prove below $A(L1, A1)$ thus yielding *H1*. $A(W2, S)$ and $A(W2, D1)$ are obvious. So are $A(D1, D2)$, $A(L1, L2)$, $A(Hi, Ai)$, $A(A1, F)$, $A(A1, A2)$, $A(H1, H2)$, and–using T_1–$A(B2, B3)$.

If X is *D1*, there is a w.o. covering W by closed, discrete sets. Since $w\backslash\{x\}$ is closed for each $w \in W$, $f\colon X \times W \to \underline{X}$,

$$f(x,w) = X\backslash(w\backslash\{x\})$$

is a *B3*-mapping.

Let X be *B1* and W be a w.o. base. We define a *B2*-mapping through $f(x,w) = X$ if $x \notin w$ and $f(x,w) = w$ otherwise. If $O \subseteq \underline{X}$ covers X,

$$P = \{w \in W : w \subseteq o \in O\}$$

is a w.o. refinement; i.e. *L2*. Fix a well-ordering $<$ of W. If A is a family of nonempty, pairwise disjoint open sets,

$$f(a) = min\ \{w \in (W,<) : \phi \neq w \subseteq a\}$$

is one-to-one, whence A is w.o. So $A(B1, C)$ and similarly $A(S, C)$.

Let $f\colon X \times W \to \underline{X}$, where W is w.o., be a function assured by *B2*. If O is an open cover, $w(x) = min\ \{w \in W\colon f(x,w) \subseteq Q \in O\}$ is well defined, since $f'\{x\} \times W$ is a neighborhood base at x, whence $f(x,w(x))$ is an *A2*-mapping. $A(B2, H2)$ follows.

If X is *L1* and O is an open cover of X, a w.o. subcover defines a mapping as required for *A1*. If X is *L2*, we similarly may conclude that X is *A2*.

It remains to prove $A(F, A2)$: Let $f\colon Y \to X$ be continuous, onto and finite-to-one, where Y is *A1* and let O cover X, $O \subseteq \underline{X}$. Then $f^{-1}(O)$ is an open cover of Y, and $g\colon Y \to f^{-1}(O)$ an *A1*-mapping; $g(y) = f^{-1}(V)$ for some $V \in O$ such that $f(y) \in V$. As f is onto, $f \circ f^{-1}(V) = V$, and so because $f^{-1}(x)$ is finite, $G(x) = \{f \circ g(y) : y \in f^{-1}(x)\}$ is a finite subset of O, such that $x \in \cap G(x) \in X$. Therefore $h(x) = \cap G(x)$ is an *A2*-mapping.

We note for later use that in $ZF° + AC_{fin}$ (AC for families of finite sets) $A(F, A1)$ holds. Moreover, orderable covers of F-spaces have *A1*-mappings.

2.2 Lemma *In ZF°, $A(P, W1) \Leftrightarrow PW$ for $P \in \{W2, B1, S\}$ and $A(C, W1) \Rightarrow PW$.*

Proof: In view of the remarks at the beginning of this chapter, it suffices to prove $A(P, W1) \Rightarrow PW$. Let X be w.o. and let $\underline{X}$ be the discrete topology. Then X is $W2, B1, S$, and C, and if X is $W1$, $\underline{X} = P(X)$ is w.o.

2.3 Lemma *In ZF°, $A(P, Q) \Leftrightarrow AC$, when $P \in \{D1, D2, B2, B3, H2, A2\}$, $Q \in \{W1, W2, B1, L1, L2, S, C\}$.*

Proof: Let X be a set, $\underline{X} = P(X)$. $(X,\underline{X})$ is P and if $(X,\underline{X})$ is Q, X is w.o., thus yielding the well-ordering theorem.

2.4 Lemma *In ZF°, $A(P,Q) \Leftrightarrow AC$, where $P \in \{L1, L2, H2, A1, F, A2\}$, $Q \in \{W1, W2, B1, S, C\}$.*

Proof: Let X be a set with the discrete topology and $X^+ = X \cup \{X\}$ its one-point-compactification. Then X^+ is $L1$ and since each subspace is Y^+ or discrete, X^+ is $H2$. If X^+ is C, $A = \{\{x\} : x \in X\}$ is w.o. and so X is w.o. This and 2.1 conclude the proof.

2.5 Lemma *In ZF°, $A(P,Q) \Leftrightarrow AC$, where $P \in \{D1, D2, B2, B3, H2, A2\}$, $Q \in \{H1, A1\}$.*

Proof: Let F be a family of nonempty sets. We construct a choice function c. We set

$$X = F \cup (\cup F)$$

and give it the discrete topology. X is P. If X is $A1$, the cover

$$O = \{\{x, V\} : x \in V \in F\}$$

has an $A1$-mapping f. We set

$$\{c(V)\} = f(V) \setminus \{V\}.$$

Then $c(V) \in V$.

2.6 Corollary *In ZF°, $A(P,H1) \Leftrightarrow AC$, $P \in \{L1, L2, A1, F\}$.*

Proof: Let X be the space of 2.5. Then X^+ is P and if X^+ is $H1$, X is $A1$ and F has a choice function.

MW is the assertion that each set is covered by w.o. family of finite sets. As was shown in [9], MW is equivalent to MC in ZF°.

2.7 Lemma *In $ZF^\circ + MC$, $A(P)$ holds, $P \in \{D1, D2, B2, B3, H2, A2\}$.*

Proof: In virtue of 2.1 we need only prove $D1$ and $B2$. $D1$ is immediate from T_1 and MW. $B2$ follows from an application of MW to $\underline{X}$, where $\underline{X} = \cup W$, and W is a w.o. family of finite sets ($E \cap F = \phi$ when $E \neq F$). If $E \in W$, $x \in \cup E$, we set

$$f(x,E) = \cap \{V \in E : x \in V\}$$

and $f(x,E) = X$ otherwise. If $x \in O \in Z$, there is an $E \in W$ containing O, whence $x \in f(x,E) \subseteq O$, proving $B2$.

2.8 Lemma *In ZF°, $MC \Leftrightarrow A(P,Q)$, where $P \in \{D2, L1, L2, H2, A1, F, A2\}$, $Q \in \{D1, B2, B3\}$.*

Proof: "$\Rightarrow$" follows from 2.7. Let X be an infinite set with the discrete topology and X^+ its one-point-compactification. We show that X satisfies MW if X^+ is Q. Clearly X^+ is P (for $H2$ c.f. 2.4).

If W is a w.o. family of open sets such that $\{X\} = \cap W$, then

$$X = \cup \{X \backslash V : V \in W\}$$

and each set $X\backslash V$ is finite. Hence $A(P, B3) \Rightarrow MW$.

If X^+ is $D1$, there is a w.o. family W of closed, discrete sets covering X^+. Since X^+ is compact, each $D \in W$ is finite, proving MW.

2.9 Lemma *In ZF°, $A(D2, Q) \Leftrightarrow MC$, $Q \in \{H2, A2\}$.*

Proof: Let G be a family of nonempty sets. We construct an MC function. Obviously we may assume that each $g \in G$ is infinite, and that $g \neq g'$ implies $g \cap g' = \phi$ (otherwise consider $G' = \{g \times \{g\} : g \in G\}$). Let

$$g^+ = g \cup \{g\}$$

be the Alexandroff compactification of the discrete topology on g and

$$X = \cup \{g^+ : g \in G\}$$

be the topological sum which is locally compact. G and $\cup G$ are discrete and $X = \cup \{G, \cup G\}$, so X is $D2$.

$O = \{g^+ \backslash \{p\} : p \in g \in G\}$ is an open cover of X. If f is an $A2$-mapping for O, then: (i) $g \in f(g) \in \underline{X}$ and (ii) $f(g) \subseteq h^+ \backslash \{p\}$ for some $p \in h \in G$. It follows from (i) that $M(g) = g \backslash f(g)$ is finite. As $g \in f(g)$ and $h = g$ in (ii), $p \in M(g)$ and M is the desired MC-function.

2.10 Corollary *In ZF°, the following assertions hold*

(i) $A(P, H2) \Leftrightarrow MC$, $P \in \{L1, L2, A1, F, A2\}$
(ii) $A(P, F) \Rightarrow MC$, $P \in \{D1, D2, B2, B3, H2, A2\}$.

Proof: Let X be the space of 2.9, $\underline{X}$ its topology.
(i) Since X is locally compact, we may form the one-point extension X^+. It is $L1$ and if it is $H2$, G has an MC-function.
(ii) Consider the discrete space $(X, P(X))$ which is P. If it is F, then $(X, \underline{X})$ is F, since the identity is continuous, and therefore $(X, \underline{X})$ is $A2$ (2.1) and G has an MC-function.

It follows from the proof of 2.10 (ii) that MC is equivalent to: Continuous one-to-one images of $B2$ spaces are $A2$. A similar argument in 2.8 shows MC is equivalent to: Continuous one-to-one images of $B2$-spaces are $B3$. In contrast, continuous images of Li-spaces are Li and as in 2.1 continuous one-to-one images of $A1$ spaces are $A1$ in ZF°.

2.11 Lemma *In ZF°, $A(P, Q) \Rightarrow PW$,*

$$P \in \{B1, B2, B3, S, C, L2, H2, A2\}, Q \in \{W1, W2, D1, D2\}.$$

Proof: Parts of 2.11 are covered by the simple 2.2. Let X be w.o. and consider the product topology on 2^X. For $x \in X$ and $f \in 2^X$ we set $p_x(f) = f(x)$. A subbase for the topology is given by $\{p_x^{-1}\{0\}, p_x^{-1}\{1\} : x \in X\}$ which is w.o., whence 2^X is $B1$. D is the set of all $f \in 2^X$ such that $f^{-1}\{1\}$ is finite: D is dense and w.o. and so 2^X is S. Hence P holds. We next show that, if 2^X is $D2$, it is w.o., thus proving $A(P, Q) \Rightarrow PW$. If D is discrete and $d \in D$, there is a finite $E \subseteq X$ such that $\{d\} = \{f \in D : p_x(f) = p_x(d), x \in E\}$. Let $E(d, D)$ be the least such E in the lexicographic w.o. Assume that $2^X = \cup W$, W a w.o. family of discrete sets. For $f \in 2^X$ we let $D(f)$ be the least $D \in W$ containing f and $G(f) = (D(f), E(f, D(f)))$. G is one-to-one and so it induces a w.o. of 2^X.

Considering Figure 2, we observe that 132 statements $A(P, Q)$ are equivalent to AC in ZF and 61 statements are provable in $ZF°$. The remaining 63 assertions are investigated in Section 5. We next summarize our results for $A(P)$.

2.12 Theorem

(i) *In* ZF, $A(P) \Leftrightarrow AC$, $P \in$ Fig. 1
(ii) *In* $ZF°$, $A(P) \Leftrightarrow AC$, *for* $P \in \{W1, W2, B1, L1, L2, S, C, H1, A1\}$
(iii) *In* $ZF°$, $A(P) \Leftrightarrow MC$, *for* $P \in \{D1, B2, B3, H2, A2\}$
(iv) *In* $ZF°$, $AC \Rightarrow A(F) \Rightarrow MC \Rightarrow A(D2) \Rightarrow PW$.

The next two sections deal with $D2$ and F.

3 The position of A(D2) It follows from the results of the preceding chapter that in $ZF°$, $MC \Rightarrow A(D2) \Rightarrow PW$. We show that in $ZF°$ $A(D2)$ does not

P \ Q	W1	W2	D1	D2	B1	B2	B3	L1	L2	S	C	H1	H2	A1	F	A2
W1	→	→	→	→	→	→	→	→	→	→	→	→	→	→	→	→
W2	PW	→	→	→	?	?	→	9^2	?	→	→	9	?	9	9	?
D1	AC	AC	→	→	AC	8	→	AC	AC	AC	AC	AC	8	AC	MC	8
D2	AC	AC	MC	→	AC	MC	MC	AC	AC	AC	AC	AC	MC	AC	MC	MC
B1	PW	PW	PW	PW	→	→	→	6	→	6	→	6	→	6	6	→
B2	AC	AC	PW	PW	AC	→	→	AC	AC	AC	AC	AC	→	AC	MC	→
B3	AC	AC	PW	PW	AC	7	→	AC	AC	AC	AC	AC	7	AC	MC	7
L1	AC	AC	MC	5	AC	MC	MC	→	→	AC	AC	AC	MC	→	→	→
L2	AC	AC	MC	PW	AC	MC	MC	6	→	AC	AC	AC	MC	6	6	→
S	PW	PW	PW	PW	7	7	?	7	7	→	→	7	7	7	7	7
C	PW	PW	PW	PW	4	4	4	4	4	4	→	4	4	4	4	4
H1	1	1	2	?	1	2	2	1	1	1	1	→	→	→	→	→
H2	AC	AC	MC	PW	AC	MC	MC	AC	AC	AC	AC	AC	→	AC	MC	→
A1	AC	AC	MC	5	AC	MC	MC	1	1	AC	AC	AC	MC	→	→	→
F	AC	AC	MC	5	AC	MC	MC	1	1	AC	AC	AC	MC	3	→	→
A2	AC	AC	MC	PW	AC	MC	MC	AC	AC	AC	AC	AC	MC	AC	MC	→

Figure 2. $A(P,Q)$ in $ZF°$.

imply MC and that $A(D2)$ is not provable in $ZF^\circ + PW$. We work in standard permutation models of ZF° in this section.

M_1 is the Fraenkel-Halpern model, determined by a set U_1 of urelements (which is countable in the real world), the group G_1 of all bijective mappings on U_1 and the ideal I_1 of all finite subsets of U_1. M_2 is the Mostowski model, where U_2 is ordered by $<$ like $\mathbf{Q}$, G_2 is the group of order-preserving permutations and I_2 is the ideal of finite sets. We refer to [5] and [7] for more information. $\Delta(e) = \{x : sym\ x \supseteq fix\ e\}$ is the class of sets supported by e, $supp\ (x)$ is the least support for x; $p\ supp\ (x) = supp\ (px)$. Counterexamples are transferred to ZF-models with similar properties.

3.1 Lemma *In ZF°, if X is $D2$ and $P(X)$ is D-finite, then X has isolated points.*

Proof: The assumptions imply that X is a finite union of discrete D_i, $i \in n$. We apply induction on n. If $n = 1$, X is discrete. If $X = \bigcup_{i \in n+1} D_i$, we choose $x \in D_n$, $O \in \underline{X}$, such that $O \cap D_n = \{x\}$. If $O = \{x\}$, we are finished. Otherwise $O \backslash D_n$ is an open subset (in X) of $\bigcup_{i \in n} D_n$ which has an isolated point by the inductive hypothesis.

3.2 Example $A(L1, D2)$ is provable neither in ZF nor in $ZF^\circ + PW$. Hence PW does not imply $A(D2)$.

Proof: In M_2 a closed interval $X = [a,b]$, $a < b$, of urelements with the order topology is compact and so $L1$. Since $P(X)$ is D-finite but X has not isolated points, X is not $D2$ by 3.1.

3.3 Theorem *In ZF°, $A(D2)$ does not imply MC.*

Proof: We show that $A(D2)$ holds in M_1, where MC is known to fail. Let e be a support of the T_2 space $(X, \underline{X})$. Since

$$X = \cup \{orb\ x : x \in X\}$$

where

$$orb\ x = \{px : p \in fix\ e\} \in \Delta(e),$$

X is D_2, if each $orb\ x$ is discrete. We prove this fact.

We set for $x \in X$ $a(x) = supp(x)\backslash e$. When $a(x) = \phi$, $orb\ x = \{x\}$ is finite. If $a(x) = \{a_i : i \in n\} \neq \phi$, we set $f = \{p((a_i)_{i\in n}, x)) : p \in fix\ e\} \in \Delta(e)$. We observe that f is a function since if $p((a_i)_{i \in n}) = q((a_i)_{i\in n})$, $q^{-1}p \in fix\ (e \cup a(x))$, i.e., $px = qx$. Since $Im\ f = orb\ x$, $dom\ f \subseteq U^{n+1}$, and $P(U^{n+1})$ is D-finite (c.f. [11]), $P(orb\ x)$ also is D-finite.

We now prove by induction on $|a(x)|$, that $orb\ x$ is discrete. We assume that $orb\ x$ is discrete whenever e supports $(X,\underline{X})$ and $|a(x)| \leqslant n$ and we assume ($n = 0$ is trivial) on the contrary, that $orb\ x$ is not discrete for some $x \in X$ such that $|a(x)| = n + 1$.

If $O \in \underline{X}|orb\ x$ is nonempty ($\underline{X}|$. is $\underline{X}$ relativized to.) and $f \supseteq e$ supports O, there is a $y \in O$, such that $supp\ y \cap f \subseteq e$.

Otherwise $(supp\ y\backslash e) \cap f \neq \phi$ for $y \in O$. As $f \supseteq e$, $|supp\ y\backslash f| <$

$|supp\ y\backslash e| = |a(y)|$ and since $y \epsilon\ orb\ x$, $|a(y)| = |a(x)|$. Hence $|supp\ y\backslash f| \leqslant n$ and

$$o(y) = \{py : p \epsilon\ fix\ f\} \epsilon\ \Delta(f)$$

is discrete by the inductive assumption. Since

$$O = \cup \{o(y) : y \epsilon\ O\},$$

O is $D2$. Therefore, by Lemma 3.1, it has isolated points. Hence *orb* x has isolated points and since *fix* e fixes $\underline{X}$, *orb* x is discrete, contradicting our assumption.

We next derive a contradiction to T_2. If O_i, $i \epsilon\ 2$, are nonempty open subsets of *orb* x, $O_0 \cap O_1 \neq \phi$.

Let $f \supseteq e$ support O_i. There are $p_i \epsilon\ fix\ e$ such that $supp\ (p_i x) \cap f \subseteq e$ and $p_i x \epsilon\ O_i$, as follows from the above remark. We define q: $supp(p_0 x) \cup f \rightarrow U_1$ by $q(t) = t$ for $t \epsilon\ f$ and $q(p_0(t)) = p_1(t)$ otherwise. As $p_i \epsilon\ fix\ e$ and $f \cap p_i\ (supp(x)\backslash e) = \phi$, q is injective and so can be extended to a bijective $\bar{q}$ on U_1. Then $\bar{q} \epsilon\ fix\ f$ and $\bar{q}\ p_0(t) = p_1(t)$ for $t \epsilon\ supp\ x$. Therefore, $\bar{q}(p_0 x) = p_1 x$ and $p_1 x \epsilon\ \bar{q}\ O_0 \cap O_1 = O_1 \cap O_1$.

We note without a proof that if X is a D-finite T_2 space in M_1, X is covered by a countable family of discrete sets.

4 The cardinality of Ai-spaces In M_1, U_1 is amorphous: Infinite subsets are cofinite. U_1 with the discrete topology is not P for $P \epsilon\ \{B1, L1, L2, S, C, W1, W2\}$. Also U_1^+ is amorphous and since U_1 is not MW (if $P(X)$ is D-finite and X is infinite, X is not MW), U_1^+ is not P, for $P \epsilon\ \{B2, B3, D1\}$ (c.f. the proof of 2.8). More interesting are $A1$ and $A2$. We shall prove that amorphous spaces are $A1$ and if $P(X)$ is D-finite, X is $A2$. We also give an application of this result, demonstrating the use of $A2$. The cardinality behavior of Ai-spaces depends on the model. In M_2 there is a space with a D-finite power set, which is not $A2$. When G is the family of all two-element subsets of U_1, which is known to have no choice function, the construction of 2.5 yields a space with a D-finite power set which is not $A1$. The construction of 2.9 applied to the family of infinite subsets of U_1 gives a D-finite space which is not $A2$. So our results on the cardinality of Ai-spaces in M_1 cannot be improved.

4.1 Theorem *Let X be a space in M_1. If X is amorphous, X is $A1$ and if $P(X)$ is D-finite, X is $A2$.*

Proof: Let e be a support of X, $\underline{X}$, $\underline{O}$, where $\underline{O}$ is open cover of X. If X is amorphous, there is a mapping $m : U_1\backslash e \rightarrow X$, such that $X\backslash Im(m)$ is finite [3]. Hence we may assume $X = U_1$. Because $P(U_1)$ is D-finite, the mapping $n(a) = min\ \{|supp\ (V)| : a \epsilon\ V \epsilon\ \underline{O}\}$, $a \epsilon\ U_1$, which is in the model, has an upper bound N. $f \supseteq e$ is a subset of U_1 with $|e| + N + 1$ elements. Consider the mapping $G \epsilon\ \Delta(f)$, defined through $G(a) = \{V \epsilon\ \underline{O} : a \epsilon\ V \epsilon\ \Delta(f \cup \{a\})\}$.

We show: $G(a) \neq \phi$. For, let $a \epsilon\ V \epsilon\ \underline{O}$ and $|supp\ V| = n(a)$. As

$$|e \cup supp(V)| \leqslant |f|,$$

there is a permutation $p \epsilon\ fix\ e \cup \{a\}$, such that $p\ supp(V) \subseteq f \cup \{a\}$, whence

$a = pa \in V \in \underline{O} = p\underline{O}$ and

$$pV \in \Delta(f \cup \{a\}).$$

Thus $pV \in G(a)$. We now apply a lemma from [6], p. 116, and get:

There is a function $g \in \Delta(f)$ such that $g(a) \in G(a)$. Since this means $a \in g(a) \in \underline{O}$, g is an $A1$-mapping.

If $P(X)$ is D-finite, we define $n(x)$ as above and there is a $N \in \omega$, such that always $|supp\ x| + |n(x)| < N$. When $f \supseteq e$ is a subset of U with $N + |e|$ elements, as above $G(x) = \{V \in \underline{O} : x \in V \in \Delta\ (f \cup supp\ x)\}$ is nonempty. Since $P(X)$ is D-finite and $G(x)$ is w.o., $G(x)$ is finite and $f(x) = \cap G(x)$ is open. f is the desired $A2$-function.

We note that the following improved result holds: If Y is a subspace of X, $P(Y)$ is D-finite, and $\underline{O} \subseteq \underline{X}$ covers Y ($\underline{X}$ the X-open sets), there is a mapping $f : Y \to \underline{X}$, such that $x \in f(x)$ and $f'Y$ refines $\underline{O}$. The proof uses [6], which provides us with an $O(R_x) \in G(x)$ for each ordering R_x of $supp\ x$ and $f(x) = \bigcap_{R_x} O(R_x)$. A similar improvement is valid, if Y is amorphous.

It follows that U_1 and U_1^+ are $H1$. The following application of 4.1 is less obvious:

4.2 Example A discrete F-space in M_1 which is not $A1$.

Proof: Let $Y = (U \cup \{U\})^2$ be discrete. As in 4.1 Y is $A1$.

$$X = U \cup \{U\} \cup \{\{a,b\} : a \neq b \text{ in } U\}$$

with the discrete topology is not $A1$, as follows from the proof of 2.5. But X is F, since $f : Y \to X$, where $f(U,U) = U$, $f(a,U) = f(U,a) = f(a,a) = a$, $a \in U$, $f(a,b) = f(b,a) = \{a,b\}$, $a \neq b$ in U, is finite-to-one, onto, continuous, and open.

The following example gives an application of $A2$. As is well-known, in $ZF^\circ + AC$ metacompact countably compact spaces are compact [1]. If X is infinite and $P(X)$ is D-finite, X with the discrete topology is countably compact, paracompact and metrizable, but not compact. Need metacompact spaces with a D-finite power set be paracompact?

4.3 Example In M_1 there is a metacompact space with a D-finite power set, which is not paracompact.

Proof: Let $U_1^+ = U_1 \cup \{U_1\}$ be the one point compactification of the discrete space U_1.

$$X = (U_1^+)^2 \setminus \{(U_1, U_1)\}$$

is T_2 and not T_4, as was shown in [3]. Therefore, in ZF° X is not paracompact [4]. It follows from [11], that $P(X)$ is D-finite.

Let $\underline{O} \in \Delta(e)$ be an open cover of X. As was shown in 4.1, there is an $A2$-mapping G for $\underline{O}$.

We next observe that X is metacompact if X is $A2$ and there is a mapping $g : X \to \underline{X}$, the topology, such that $x \in g(x)$, and

$$\{y \in X : x \in g(y)\} = g'(y)$$

is finite. Let $\underline{O}$ be an open cover and G be an $A2$-mapping. $h(x) = g(x) \cap G(x)$ defines a refinement $H = Im(h)$ of $\underline{O}$ and $\underline{H}$ is point-finite, since $h'(x) \subseteq g'(x)$ is finite.

X is $A2$ by 4.1 and g is defined as follows: $g(a,b) = \{(a,b)\}$, when a,b are in U_1, $g(U_1,b) = U_1^+ \times \{b\}$, $g(a,U_1) = \{a\} \times U_1^+$.

In M_2 the situation is quite different: $A2$-spaces are rare.

4.4 Example In M_2 there is a space X with a D-finite power set which is C, but not $B3$, $A2$, $D2$, S.

Proof: Let $(X,\underline{X})$ be U_2 with the order topology. As was shown in [3], X is hereditarily C. Because $P(X)$ is D-finite and X has no isolated points, X is not D_2 (3.1) or $B3$. Since in permutation models S spaces are well-orderable [3], X is not S. We prove not $A2$.

Consider the open cover $O = \{(a,\rightarrow) : a \in U_2\}$ ("$\rightarrow$" means "∞"). Let f be an $A2$ mapping for O. Since $Im(f)$ refines O, for each a there is a $b \in U_2$, such that $f(a) \subseteq (b,\rightarrow)$, and because $a \in f(a) \in \underline{X}$, there are $c < a$ in $f(a)$. Therefore $g(a) = inf\ f(a)$ exists and it satisfies $g(a) < a$. But then $\{g^n(a) : n \in \underline{w}\}$ is an infinite countable subset of U_2, contradicting D-finiteness.

4.5 Lemma *M_2 satisfies $A(B3, D1)$ and $A(H1, W1)$.*

Proof: Let X be $B3$. As was shown in [2], $X = \cup W$, where W is a w.o. family of sets Y, such that $P(Y)$ is D-finite. But then from [2] and $B3$ each $Y \in W$ is closed and discrete and therefore $A(B3, D1)$.

By PW $A(H1, W1) \Leftrightarrow A(H1, W2)$. Suppose X is not $W2$. Then it follows from [3] that X contains an infinite subset Y which is a copy of an interval $(a,b) \subseteq U_2$, $a < b$, and $\underline{X} \mid (a,b)$ is finer than the order topology ($\underline{X}$: topology of X) which is not $A1$ by 4.4. Hence X is not $H1$.

5 ***Conclusion*** We complete the proof of Figure 2. If $A(P,Q)$ is provable in $ZF°$, we insert a "$\rightarrow$", and if $A(P,Q)$ implies PW, MC, or AC in $ZF°$, we write this form into the matrix. The corresponding results were proved in Section 2. We show by counterexamples that the remaining $A(P,Q)$ also depend on AC with some exceptions.

X_1 is U_1 with the discrete topology in M_1. It was studied in Section 4.

X_2 is U_1^+, the one-point-extension of X_1 which was already constructed in Section 4.

X_3 is the space of 4.2.

X_4 is the space of 4.4.

X_5 is 3.2.

The following counterexamples are constructed in M_3, the Cohen-Halpern-Levy model, where an infinite set A of Cohen-generic reals are adjoined to a ground model of $ZF + AC$. The arguments of [7] show that we may assume that A is an infinite, D-finite subset of $\mathbf{R}$, such that $inf\ A = \leftarrow$ and A has no isolated points. It is well-known that M_3 satisfies AC_{fin} (cf. [5]). If $B \subseteq A$ is infinite, B is not a w.o. union of finite sets (MW), otherwise B is w.o. by AC_{fin} (or use the ordering of $\mathbf{R}$). There is no mapping $f : B \rightarrow B$ such that $f(b) < b$, $b \in B$, since this contradicts D-finiteness (cf. 4.4).

5.1 Example X_6 is A with the subspace topology of $\mathbf{R}$. X_6 is *B1* but not *D2* or *A1*.

Proof: Since $\mathbf{R}$ is second countable, X_6 is *B1*. We conclude that discrete subsets are C (2.1) and therefore finite (D-finiteness). A is not *MW*, whence X_6 is not *D2*. Since AC_{fin} holds in M_3, not F follows from not *A1*. Let $O = \{(a,\rightarrow): a \in A\}$. Because $\mathit{inf}\ A = \leftarrow$, O is an open cover of X_6. If f is an *A1*-mapping, $f(x) = (g(x),\rightarrow)$ defines a function g, such that $g(x) < x$. This was already shown to be impossible.

5.2 Example X_7 is $\mathbf{R}^2$, the topology is an extension of the Euclidean by the complements of end-point-free straight-line segments. X_7 is S and *B3*, but not *D2* or *A2*.

Proof: A basic neighborhood at p is an open ball around p from which finitely many lines through p have been removed. T_2, S, and *B3* are obvious. Since the subspace topology on the unit circle coincides with the usual, X_6 is a subspace of X_7 which therefore is not *D2*.

X_7 is not *A2*: Let $L_{p,a}$ be the straight-line containing $(p,0)$ and $(a,1)$, $p \in \mathbf{R}, a \in A$. $E_p(V) = \{a \in A : L_{p,a} \cap V = \{(p,0)\}\}$ (i.e., $L_{p,a}$ has been removed). If V is open, $E_p(V)$ is finite. It follows from the definition of the topology $\underline{X}$ ($\mathbf{R} \times \{O\}$ is discrete), that $\underline{O}_1 = \{V \in \underline{X}:(p,O) \in O \Rightarrow |E_p(V)| > p\}$ is an open covering of $\mathbf{R} \times \{O\}$, and $\underline{O} = \underline{O}_1 \cup \{\mathbf{R}^2 \backslash (\mathbf{R} \times \{O\})\} \subseteq \underline{X}$ covers X_7. Let f be an *A2*-mapping. Since $(n,O) \in f(n,O) \subseteq V \in \underline{O}$ yields $|E_n(f(n,O))| \geqslant |E_n(V)| > n$ because $V \in O_1$, it follows that $B = \bigcup_{n \in \underline{w}} E_n(f(n,O))$ is an infinite *MW* subset of A, a contradiction, proving not *A2*.

5.3 Example X_8 is $\mathbf{R} \times \mathbf{Q}$, where a basic neighborhood at p is an open ball around p and countably many straight-lines not containing their endpoints are removed. X_8 is *D1* but not *A2*.

Proof: *D1* is obvious since $\mathbf{R} \times \{q\}$ for $q \in \mathbf{Q}$ is closed and discrete. Not *A2* is proved as in 5.2 because A is D-finite and $E_p(V)$ is finite when V is open.

Other remarkable spaces are $\mathbf{R}$ which is *B1* + S, but not *A1* (improving [7]), $\mathbf{R}^A$ which is S [10] and not *B2*, and the Moore-Niemytzky plane, which is S and *B2* but not *L2*. X_7 and X_8 are B_{3a}, but 2^A is not B_{3a}, since it is compact (*BPI* holds in M_3) and compact B_{3a} spaces are *B2*.

We conclude with some words on $A(B1, W2)$, which was the starting point of this paper. As follows from 2.11, in ZF° $A(B1, W2) \Leftrightarrow PW$. Hence in ZF $A(C, W2) \Leftrightarrow AC$, thus answering in part a question from [3]. $A(her\ L2, W2)$ is–in ZF°–weaker than $A(L2, W2) \Leftrightarrow AC$. For as was shown in [3], $D_2 \Rightarrow A(her\ L2, W2)$. D_2 is the axiom that a set is well-orderable if each infinite subset has a Dedekind-finite power set. In ZF°, $D_2 \Rightarrow PW$ but $D_2 \not\Rightarrow AC$, $PW \not\Rightarrow D_2$. In contrast, $A(her\ L1, W2)$ does not imply AC in ZF. D_1 is the axiom that a set is well-orderable if each infinite subset is Dedekind-infinite. D_1 holds in the Cohen model and therefore it does not imply AC in ZF (cf. [3]).

5.4 Remark In ZF°, $D_1 \Rightarrow A(her\ L1, W2)$.

Proof: "*her L1*" is "hereditarily *L1*." Let X be *her L1*. If X is not w.o. there is an infinite D-finite $Y \subseteq X$. We show that Y is compact. Then Y is hereditarily compact and therefore finite, a contradiction. If Y is not compact, there is a w.o. cover ($L1$) without a finite subcover. Hence there is a sequence $(O_n)_{n\epsilon\omega}$ of open sets such that $O_n \subseteq O_{n+1}$ and $O_n \neq O_{n+1}$. We set $Y' = \bigcup_{n\epsilon\omega} O_n$, $\underline{V}_n = \{O_{n+1}\backslash\{p\} : p \in O_{n+1}\backslash O_n\}$ and $\underline{V} = \bigcup_{n\epsilon\omega} \underline{V}_n$. $\underline{V}$ is an open cover of Y' and so it has a w.o. subcover $(P_\alpha)_{\alpha\epsilon\beta}$, β an ordinal number. We set $n(\alpha) = min\ \{n \in \omega : P_\alpha \in \underline{V}_n\}$ and $\{a_\alpha\} = O_{n(\alpha)+1}\backslash P_\alpha$. Then $a_\alpha \in O_{n(\alpha)+1}\backslash \bigcup_{n\leqslant n(\alpha)} O_n$ and $P_\alpha \subseteq O_{n(\alpha)+1}$. Because Y is D-finite, $A = \{a_\alpha : \alpha \in \beta\} \subseteq Y$ is finite, and therefore there is a $n \in \omega$ such that $\bigcup_{\alpha\epsilon\beta} P_\alpha \subseteq O_n \neq Y'$, a contradiction.

5.4 is the reason that questions concerning $A(H1, Q)$ are difficult.

NOTES

1. Added in proof: $A(L2, A1) \Leftrightarrow AC^{wo}$. "$\Leftarrow$" is obvious and for "$\Rightarrow$" let $(F_\alpha)_{\alpha\epsilon\beta}$ be a w.o. family of nonempty sets such that $F_\alpha \cap F_{\alpha'} = F_\alpha \cap \beta = \phi, \alpha \neq \alpha'$ in β. We set $F = \bigcup_{\alpha\epsilon\beta} F_\alpha$ and $X = F \cup \beta \cup \{p\}$ for some $p \notin F \cup \beta$. Points $x \in F \cup \beta$ are isolated, while neighborhoods of p are of the form $\{p\} \cup F\backslash E$, E finite. This defines a $T_2 + D_2$ topology on X. It is $L2$. Let $\underline{O}$ be an open cover. There is a $V \in \underline{O}$ such that $p \in V$. Since $F\backslash V$ is finite, $\{V\} \cup \{\{x\} : x \in \beta \cup F\backslash V\}$ is a w.o. open refinement of $\underline{O}$. By $A(L2, A1)$ the following cover $\underline{O}$ has an $A1$-mapping $a : X \to \underline{O}; \underline{O} = \{\{p\} \cup F\} \cup \{\{\alpha\} \cup F_\alpha\backslash\{x\} : \alpha \in \beta, x \in F_\alpha\}$. $\{f(\alpha)\} = F_\alpha\backslash a(\alpha)$ defines the required choice function (since $\alpha \in a(\alpha)$).

2. $X_q = \mathbf{Q}$ in M_3. Let $A \subseteq \mathbf{R}$ be D-finite and $inf\, A = \leftarrow$. $\underline{O} = \{\mathbf{Q} \cap (a, \rightarrow) : a \in A\}$ is an open cover. If $f : \mathbf{Q} \to \underline{O}$ is A_1, then $a(x) = inf\, f(x) \in A$; $B = a'\mathbf{Q} \subseteq A$ is countable and unbounded. Therefore X_9 is not A_1 and not $F(AC_{fin})$.

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