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An Axiomatization of Predicate Functor Logic

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1 Introduction Predicate Functor Logic is a formal system devised by Quine to provide a natural variable-free equivalent of elementary logic. It is described, in its most recent form, in [9]; but the ideas go back to [8] and [7]. In [9] Quine discusses the problem of exhibiting a simple and complete "proof procedure" for predicate functor logic, i.e., a procedure for recursively enumerating the formulas of predicate functor logic whose elementary logic counterparts are valid. This paper describes one such procedure.[1] We provide an interpretation for predicate functor logic which is consonant with Quine's remarks. The class of formulas valid with respect to this interpretation is axiomatized by a recursive set of axioms and rules. The proof that this axiomatization is complete is an adaptation of the Henkin completeness proof for elementary logic. It does not require translations between predicate functor and elementary logic. The axiomatization is not as simple as might be desired: nonprimitive symbols are needed to display the axioms conveniently. But a closely related system is shown to have a simple and perspicuous axiomatization.

2 Predicate functor logic The language of predicate functor logic contains symbols of two varieties. First, for each $n \geqslant 0$ there is a countable collection of *n-ary atomic predicates*. For convenience we take these to be just the n-ary predicates of elementary logic. Second, there are the *predicate functors*, $\urcorner$, $\cap$, **p**, **P**, [,], and I.

For $n \geqslant 0$ the set of *n-ary predicates* satisfies the following conditions:

1. All n-ary atomic predicates are n-ary predicates.
2. If P^n and Q^m are n-ary and m-ary predicates, respectively, then $(P^n \cap Q^m)$ is a $max(m,n)$-ary predicate.

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3. If P^n is an n-ary predicate, then $\neg P^n$ and $\mathsf{P}P^n$ are n-ary predicates, $[P^n$ is an n+1-ary predicate, $]P^n$ is an n−1-ary predicate (unless $n = 0$, in which case it is an n-ary predicate), and $\mathsf{p}P^n$ is an n-ary predicate (unless $n \leqslant 1$, in which case it is a 2-ary predicate).
4. I is a 2-ary predicate.

A *predicate* is a string of symbols which, for some n, can be shown to be an n-ary predicate on the basis of 1-4. (We make the usual assumptions that the initial collection of symbols is pairwise disjoint, and that juxtaposition in the metalanguage represents concatenation in the object language.) $\mathcal{L}_{PF}$ is the set of all predicates. A *sentence of* $\mathcal{L}_{PF}$ is a 0-ary predicate. Henceforth we use P, Q, R and P^n, Q^n, R^n, as metamathematical variables ranging over predicates in $\mathcal{L}_{PF}$ and n-ary predicates in $\mathcal{L}_{PF}$, respectively. ϕ and ψ are used similarly as variables over formulas of elementary logic.

A *model* is a pair $M = \langle \mathcal{D}, \mathcal{I} \rangle$ where $\mathcal{D}$ is a nonempty set (the *domain* of M) and $\mathcal{I}$ is a function from n-ary atomic predicates of $\mathcal{L}_{PF}$ to subsets of $\mathcal{D}^n$. The members of $\mathcal{D}^\omega$ are called *arrays of individuals*, or simply *arrays*. Suppose $M = \langle \mathcal{D}, \mathcal{I} \rangle$ is a model and $\boldsymbol{a} = \langle d_1, d_2, \ldots \rangle$ is an individual array on M. Then *P is true of $\boldsymbol{a}$ in M* (written '$M \overset{\boldsymbol{a}}{\models} P$' or simply '$\boldsymbol{a} \models P$' when confusion is unlikely) if one of the following holds:

1. P is an atomic n-ary predicate and $\langle d_1, \ldots, d_n \rangle \in \mathcal{I}(P)$
2. $P = \neg Q$ and not $\boldsymbol{a} \models Q$
3. $P = Q \cap R$ and $\boldsymbol{a} \models Q$ and $\boldsymbol{a} \models R$
4. $P = \mathsf{p}Q^n$ and $\langle d_2, d_1, d_3, d_4, \ldots \rangle \models Q^n$
5. $P = \mathsf{P}Q^n$ and $\langle d_n, d_1, \ldots, d_{n-1}, d_{n+1}, \ldots \rangle \models Q^n$
6. $P = [Q$ and $\langle d_2, d_3, \ldots \rangle \models Q$
7. $P =]Q$ and $\langle d_0, d_1, \ldots \rangle \models Q$ for some $d_0 \in \mathcal{D}$
8. $P = I$ and $d_1 = d_2$.

P is true in M (written '$M \models P$') if $M \overset{\boldsymbol{a}}{\models} P$ for all individual arrays $\boldsymbol{a}$ on M. *P is valid* ('$\models P$') if it is true in all models. If $\Gamma \subseteq \mathcal{L}_{PF}$ then *Γ is true of $\boldsymbol{a}$ in M* ('$M \overset{\boldsymbol{a}}{\models} \Gamma$') if, for all $P \in \Gamma$, $M \overset{\boldsymbol{a}}{\models} P$.

*3 **Abbreviations*** Superscripts on functors or on bracketed groups of functors are used to indicate iterations. For example, '$]^3P$' means $]]]P$, '$(\mathsf{Pp})^2Q$' means $\mathsf{PpPp}Q$, and 'P^0Q' means Q. The following abbreviations will be useful ($n,k \geqslant 0$):

1. $(Q \supset R) = \neg(Q \cap \neg R)$
2. $(Q \# R) = ((Q \supset R) \cap (R \supset Q))$
3. $\perp = (I \cap \neg I)$
4. $\sigma_k Q^n = \begin{cases} \mathsf{P}^{(n+1)-k}(\mathsf{Pp})^{k-1}Q^n \text{ if } k \leqslant n \\ \sigma_k(Q^n \cap [^{k-2} \neg \perp) \text{ otherwise} \end{cases}$
5. $\sigma_k^{-1} Q = \sigma_k^{k-1} Q$
6. $\tau_k Q =]\sigma_{k+1}(I \cap \sigma_{k+1}^{-1} Q)$
7. $\tau\langle k_1, \ldots, k_n \rangle Q = \tau_{k_n} \tau_{k_{n-1}+1} \cdots \tau_{k_1+n-1} Q$
8. *If* $\Gamma \subseteq \mathcal{L}_{PF}$, *then* $\Gamma^* = \{\tau\langle 2, 4, \ldots, 2n \rangle P^n : P^n \in \Gamma\}$.

Notice that if Q is an n-ary predicate $\sigma_k Q$ is a $max(n,k)$-ary predicate, $\tau_k Q$ is a $max(n-1,k)$-ary predicate and $\tau\langle k_1, \ldots, k_n \rangle Q$ is an m-ary predicate where

$m = max(k_1 + n - 1, k_2 + n - 2, \ldots, k_n)$. Outermost parentheses in the name of a predicate are often dropped.

Lemma 1 *For all models* $M = \langle \mathscr{D}, \mathscr{I} \rangle$ *and all* $\boldsymbol{a} = \langle d_1, d_2, \ldots \rangle$ *in* $\mathscr{D}^\omega$,

a. $\boldsymbol{a} \models Q \supset R$ *iff* $\boldsymbol{a} \not\models Q$ *or* $\boldsymbol{a} \models R$
b. $\boldsymbol{a} \models Q \# R$ *iff either* $\boldsymbol{a} \models Q$ *and* $\boldsymbol{a} \models R$ *or* $\boldsymbol{a} \not\models Q$ *and* $\boldsymbol{a} \not\models R$
c. $\boldsymbol{a} \not\models \perp$
d. $\boldsymbol{a} \models \sigma_k Q$ *iff* $\langle d_k, d_1, \ldots, d_{k-1}, d_{k+1}, \ldots \rangle \models Q$
e. $\boldsymbol{a} \models \sigma_k^{-1} Q$ *iff* $\langle d_2, \ldots, d_k, d_1, d_{k+1}, \ldots \rangle \models Q$
f. $\boldsymbol{a} \models \tau_k Q$ *iff* $\langle d_k, d_1, \ldots \rangle \models Q$
g. $\boldsymbol{a} \models \tau_{\langle k_1, \ldots, k_n \rangle} Q$ *iff* $\langle d_{k_1}, \ldots, d_{k_n}, d_1, d_2, \ldots \rangle \models Q$.

4 Elementary logic The models and "arrays" defined in Section 2 are easily recognized as ordinary models and assignments of elementary logic. If $v_1, v_2, \ldots$ is an enumeration of the variables of elementary logic then we can regard a formula of elementary logic as *true in M under* $\boldsymbol{a}$ if it is true in M when the i'th coordinate of $\boldsymbol{a}$ is assigned to v_i. So it makes sense to ask whether there is a predicate in $\mathscr{L}_{PF}$ which is *true under the same conditions* as a given formula of elementary logic. For example, the sentence $\neg\,]\neg(P^1 \cap \neg Q^1)$ of $\mathscr{L}_{PF}$ is true in a model M if and only if the sentence $\forall x(P^1x \wedge \sim Q^1x)$ of elementary logic is true in M. To find equivalents of other sentences requires some ingenuity. Predicate functor logic contains no variables, so the effect of permuting, repeating, and deleting variables must be obtained by appropriate use of functors. But, as Quine remarked, it can be done: whatever is expressible in elementary logic is expressible in $\mathscr{L}_{PF}$ and vice versa.

To show this it is convenient to use a fact noted by Tarski: every formula of elementary logic with identity is equivalent to one in which each k-ary predicate (except '=') is followed by $v_1, \ldots, v_k$. Let $\mathscr{L}_T$ be the set of formulas of elementary logic which have this property and let $\mathscr{L}$ be the set of all formulas of elementary logic. We define a translation t_1 from $\mathscr{L}_T$ into $\mathscr{L}_{PF}$ by induction:

$$t_1(Pv_1 \ldots v_k) = P$$
$$t_1(v_j = v_k) = \begin{cases} \sigma_j \sigma_k I \text{ if } j \leqslant k \\ \sigma_k \sigma_j I \text{ if } j > k \end{cases}$$
$$t_1(\neg\phi) = \neg t_1(\phi)$$
$$t_1(\phi \wedge \psi) = t_1(\phi) \cap t_1(\psi)$$
$$t_1((\exists v_j)\phi) = \,]\,\sigma_{j+1}\,[\,\sigma_j^{-1} t_1(\phi).$$

Similarly, we define a translation t_2 from $\mathscr{L}_{PF}$ into $\mathscr{L}$:

$$t_2(I) = v_1 = v_2$$
$$t_2(P^n) = Pv_1 \ldots v_n$$
$$t_2(\neg Q) = \sim t_2(Q)$$
$$t_2(Q \cap R) = t_2(Q) \wedge t_2(R)$$
$$t_2(\mathbf{p}Q) = t_2(Q)[v_1, v_2/v_2, v_1]$$
$$t_2(\mathbf{P}Q^n) = t_2(Q)[v_1, \ldots, v_n/v_n, v_1, \ldots, v_{n-1}]$$
$$t_2(\,[\,Q^n) = t_2(Q)[v_1, \ldots, v_n/v_2, \ldots, v_{n+1}]$$
$$t_2(\,]\,Q^n) = \exists v_{n+1}(t_2(Q)[v_1, \ldots, v_{n+1}/v_{n+1}, v_1, \ldots, v_n]).$$

Lemma 2 (a) *If* $M \overset{a}{\models} \phi$, *then* $M \overset{a}{\models} t_1(\phi)$. (b) *If* $M \overset{a}{\models} P$, *then* $M \overset{a}{\models} t_2(P)$.

Corollary *If* ϕ *and* P *are sentences of* $\mathcal{L}_T$ *and* $\mathcal{L}_{PF}$, *then* $\models \phi$ *implies* $\models t_1(\phi)$, $\models P$ *implies* $\models t_2(P)$ *and* $\models \phi \leftrightarrow t_2(t_1(\phi))$. *In other words, predicate functor logic and elementary logic are equivalent in the sense of Kotas and Pieczkowski [1].*

5 Axiomatization To state the axioms and rules properly we need to know conditions under which the k'th coordinate in the array $\boldsymbol{a}$ is relevant to P's being true of $\boldsymbol{a}$.[2]

Definition *P depends on coordinate k* if $k \geqslant 1$, $P \neq \bot$ and one of the following hold:

a. P is an n-ary atomic predicate and $k \leqslant n$
b. $P = \neg Q$ and Q depends on coordinate k
c. $P = (Q \cap R)$ and either Q or R depends on coordinate k
d. $P = \mathbf{P}Q^n$ and either $k = 1$ and Q^n depends on coordinate n or $2 \leqslant k \leqslant n$ and Q^n depends on coordinate $n - 1$
e. $P = \mathbf{p}Q^n$ and either $k = 1$ and Q^n depends on coordinate 2 or $k = 2$ and Q^n depends on coordinate 1 or $3 \leqslant k \leqslant n$ and Q depends on coordinate k
f. $P = [Q$ and Q depends on coordinate $k - 1$
g. $P =]Q$ and Q depends on coordinate $k + 1$.

Lemma 3 (a) *If P does not depend on coordinate k, then* $\langle d_1, \ldots, d_k, \ldots \rangle \models P$ *iff* $\langle d_1, \ldots, d_k', \ldots \rangle \models P$. (b) *If* $\tau_{\langle k_1, \ldots, k_{n+p} \rangle} P^n$ *depends on coordinate j, then* $j \in \{k_1, \ldots, k_n\}$.

Let PF be the smallest subset of $\mathcal{L}_{PF}$ which contains all instances of the following axiom schemes and is closed under the following rules.

A1 *Any predicate schema which can be obtained from a tautology by a uniform substitution of predicate variables for sentence letters,* $\neg$ *for* $\sim$ *and* $\cap$ *for* $\wedge$ *(for example* $P \supset (Q \supset P)$, $Q \# Q$)
A2 $\tau_{\langle 1, \ldots, n \rangle} P^n \# P^n$
A3 $\tau_{\langle k_1, \ldots, k_n \rangle} P^n \# \tau_{\langle k_1, \ldots, k_{n+p} \rangle} P^n$
A4 $\tau_{\langle k_1, \ldots, k_n \rangle} \neg P \# \neg \tau_{\langle k_1, \ldots, k_n \rangle} P$
A5 $\tau_{\langle k_1, \ldots, k_n \rangle} (P^i \cap Q^j) \# (\tau_{\langle k_1, \ldots, k_i \rangle} P^i \cap \tau_{\langle k_1, \ldots, k_j \rangle} Q^j)$, *where* $n = max(i,j)$
A6 $\tau_{\langle k,k \rangle} I$
A7 $\tau_{\langle j,k \rangle} I \supset \tau_{\langle k,j \rangle} I$
A8 $(\tau_{\langle i,j \rangle} I \cap \tau_{\langle j,k \rangle} I) \supset \tau_{\langle i,k \rangle} I$
A9 $(\tau_{\langle k_1, \ldots, k_i, \ldots, k_n \rangle} P \cap \tau_{\langle k_i, j_i \rangle} I) \supset \tau_{\langle k_1, \ldots, j_i, \ldots, k_n \rangle} P$
A10 $\tau_{\langle k_1, \ldots, k_n \rangle} \mathbf{P} P^n \# \tau_{\langle k_n, k_1, \ldots, k_{n-1} \rangle} P^n$
A11 $\tau_{\langle k_1, \ldots, k_n \rangle} \mathbf{p} P^n \# \tau_{\langle k_2, k_1, k_3, \ldots, k_n \rangle} P^n$
A12 $\tau_{\langle k_1, \ldots, k_n \rangle} [P^{n-1} \# \tau_{\langle k_2, \ldots, k_n \rangle} P^{n-1}$
A13 $\tau_{\langle k_0, \ldots, k_n \rangle} P^{n+1} \supset \tau_{\langle k_1, \ldots, k_n \rangle}] P^n$.

R1 If P and $P \supset Q$ are in PF, then so is Q.
R2 If $(P \cap \tau_{\langle k_0, \ldots, k_n \rangle} Q) \supset R$ is in PF and neither $P \cap \tau_{\langle k_1, \ldots, k_n \rangle}] Q$ nor R depends on coordinate k_0, then $(P \cap \tau_{\langle k_1, \ldots, k_n \rangle}] Q) \supset R$ is in PF.

It is easy to check that each instance of A1-A13 is valid and that R1 and R2 preserve validity. Hence every member of *PF* is valid. To prove the converse we need a few definitions and lemmas.

Definitions Suppose $\Gamma \subseteq PF$. Γ is *consistent* if it does not contain $G_1, \ldots, G_n$ such that $(G_1 \cap \ldots \cap G_n) \supset \perp$ is in *PF*. Γ is *maximal consistent* if it is consistent and it has no proper extension which is consistent. Γ is *saturated* if it is maximal consistent and in addition it contains $\tau\langle k_0, \ldots, k_n\rangle P$ for some k_0 whenever it contains $\tau\langle k_1, \ldots, k_n\rangle P$.

Lemma 4 *For all* Γ, *if* Γ^* *is consistent it can be extended to a saturated set* $\Gamma^+ \subseteq \mathcal{L}_{PF}$.

Proof: Let $P_1, P_2, \ldots$ be an enumeration of $\mathcal{L}_{PF}$ such that if $P_i = \tau\langle k_1, \ldots, k_n\rangle]Q$ then $P_{i+1} = \tau\langle k_0, \ldots, k_n\rangle Q$ for some k_0 such that k_0 is odd, k_0 is distinct from $k_1, \ldots, k_n$, and none of the P_i's which appeared previously in the list depend on k_0. (Such a k_0 will always be available because each P_i depends on at most finitely many coordinates.) We define a sequence of subsets of $\mathcal{L}_{PF}$ by induction: $\Gamma_0 = \Gamma^*$. $\Gamma_{i+1} = \Gamma \cup \{P_{i+1}\}$ if this is consistent and $\Gamma_{i+1} = \Gamma \cup \{\neg P_{i+1}\}$ otherwise. Let $\Gamma^+ = \bigcup_{i<\omega} \Gamma_i$. It is easy to check that Γ^+ is a maximal consistent extension of Γ^*. To see that Γ^+ is saturated, suppose $\tau\langle k_1, \ldots, k_n\rangle]Q \in \Gamma^+$. $\tau\langle k_1, \ldots, k_n\rangle]Q$ must appear in the enumeration of $\mathcal{L}_{PF}$, say it is P_j. It is sufficient to show $P_{j+1} \in \Gamma_{j+1}$. But this can fail only if $\Gamma_j \cup \{P_{j+1}\}$ is inconsistent. Since Γ^+ is consistent Γ_j must contain P_j. Hence, if $P_{j+1} \notin \Gamma_{j+1}$ then $(G_1 \cap \ldots \cap G_n \cap \tau\langle k_0, \ldots, k_n\rangle Q) \supset \perp$ is in *PF* for some $G_1, \ldots, G_n$ in Γ_j. But k_0 is odd, so nothing in Γ^* depends on it. Furthermore k_0 was chosen so that $P_1, \ldots, P_j$ and $\tau\langle k_1, \ldots, k_n\rangle]Q$ do not depend on it. By Rule 2, therefore, $(G_1 \cap \ldots \cap G_n \cap \tau\langle k_1, \ldots, k_n\rangle]Q) \supset \perp$ is in *PF* which violates the consistency of Γ_j.

Lemma 5 *If* Γ^* *is consistent there is a model* M *and an array* $\boldsymbol{a}$ *such that* $M \overset{\boldsymbol{a}}{\models} \Gamma^*$.

Proof: Let Γ^+ be a saturated extension of Γ^*. Let $\sim$ be the binary relation on positive integers defined:

$$i \sim j \text{ iff } \tau\langle i,j\rangle I \in \Gamma^+.$$

By Axioms A6, A7, and A8 it follows that $\sim$ is an equivalence relation. Let $[i]$ be the equivalence class of i under $\sim$. Let $\mathcal{D} = \{[i]: i < \omega\}$; let $a = \langle [1], [2], \ldots \rangle$; and for atomic P^n, let $(P^n) = \{\langle [k_1], \ldots, [k_n]\rangle: \tau\langle k_1, \ldots, k_n\rangle P^n \in \Gamma^+\}$. Then $M = \langle \mathcal{D}, \mathcal{I}\rangle$ is a model and a is an array on M. It can be proved by induction on the length of P^n that $\langle [k_1], [k_2], \ldots \rangle \models P^n$ iff $\tau\langle k_1, \ldots, k_n\rangle P^n \in \Gamma^+$. We do three cases and leave the others to the reader.

Case i. $P^n = I$. $\langle [k_1], [k_2], \ldots \rangle \models P^n$ iff $[k_1] = [k_2]$ iff $\tau\langle k_1, k_2\rangle I \in \Gamma^+$.
Case ii. $P^n = \mathbf{P}Q^n$ and $n \geqslant 2$. $\langle [k_1], [k_2], \ldots \rangle \models \mathbf{P}Q^n$ iff $\langle [k_n], [k_1], \ldots, [k_{n-1}], [k_{n+1}], \ldots \rangle \models Q^n$ iff $\tau\langle k_n, k_1, \ldots, k_{n-1}\rangle Q^n \in \Gamma^+$ iff $\tau\langle k_1, \ldots, k_n\rangle \mathbf{P}Q^n \in \Gamma^+$ (by A10).
Case iii. $P^n =]Q^{n+1}$. If $\langle [k_1], [k_2], \ldots \rangle \models]Q^{n+1}$, then $\langle k_0, k_1, \ldots \rangle \models Q^{n+1}$ for some k_0. By induction hypothesis, $\tau\langle k_0, \ldots, k_n\rangle = Q^{n+1} \in \Gamma^+$ for some k_0. By

A13 $\tau\langle k_1,\ldots,k_n\rangle]Q^{n+1} \in \Gamma^+$. Conversely, if $\tau\langle k_1,\ldots,k_n\rangle]Q^{n+1} \in \Gamma^+$, then since Γ^+ is saturated $\tau\langle k_0,\ldots,k_n\rangle Q^{n+1} \in \Gamma^+$ for some k_0. By induction hypothesis $\langle k_0, k_1, \ldots\rangle \models Q^{n+1}$. Hence $\langle k_1, k_2, \ldots\rangle \models]Q^{n+1}$.

To complete the proof of Lemma 5 observe that $M \overset{a}{\models} P^n$ iff $\tau\langle 1,\ldots n\rangle P^n \in \Gamma^+$ iff $P^n \in \Gamma^+$ (by A2). Since $\Gamma^* \subseteq \Gamma^+$, $M \overset{a}{\models} \Gamma^*$.

Lemma 6 *If* Γ *is consistent so is* Γ^*.

Proof: If not, then $(G_1 \cap \ldots \cap G_n) \supset \perp$ is in PF for G of the form $\tau\langle 2,\ldots,2k_i\rangle P_i$ where $P_i \in \Gamma$. By A3 and A5, it follows that $\tau\langle 2,4,\ldots,2,m\rangle(P_1 \cap \ldots \cap P_n) \supset \perp$ is in PF where $m = max\{k_i\colon 1 \leqslant i \leqslant n\}$. By m applications of R2, $]^m(P_1 \cap \ldots \cap P_n) \supset \perp$ is in PF. But by m applications of A13, $\tau\langle 1,\ldots,m\rangle(P_1 \cap \ldots \cap P_n) \supset\ ]^m(P_1 \cap \ldots \cap P_n)$ is in PF. Hence $(P_1 \cap \ldots \cap P_n) \supset \perp$ is in PF which violates the consistency of Γ.

Theorem 1 *If* Γ *is consistent there is a model M and an array a such that* $M \overset{a}{\models} \Gamma$.

Proof: If Γ is consistent, then by Lemma 6, so is Γ^*. So by Lemma 5 there is a model M and an array $a = \langle d_1, d_2, \ldots\rangle$ such that $M \overset{a}{\models} \Gamma^*$. But if $\boldsymbol{a}' = \langle d_2, d_4, \ldots\rangle$ then $M \overset{a'}{\models} \Gamma$.

Corollary $\models P$ *iff* $P \in PF$.

*6 **Logic without identity*** The version of predicate functor logic described here is that of [9]. In [8] the 0-ary functor I is not present. In its stead is a unary functor S called "reflection" with the truth condition:

$$(a_1, \ldots, a_n) \models SA \text{ iff } (a_1, a_1, a_2, \ldots, a_n) \models A.$$

The resulting system is equivalent to predicate logic without identity.

Our axiomatization and completeness proof can easily be adapted to this system. First, for all numbers m and n, we can define a complex functor $i_{m,n}$ such that $(a_1, \ldots, a_m, \ldots, a_p) \models i_{m,n}A$ iff $(a_1, \ldots, a_{m-1}, a_n, a_{m+1}, \ldots, a_p) \models A$. A6-A9 are replaced by:

B6 $i_{m,n}i_{n,m}P \# i_{m,n}P$
B7 $i_{m,n}i_{p,m}P \# i_{m,n}i_{p,n}P$
B8 $\tau\langle k_1,\ldots,k_m,\ldots,k_p\rangle i_{m,n}P \# \tau\langle k_1,\ldots,k_{m-1},k_n,k_{m+1},\ldots,k_p\rangle P$.

The remaining axioms are kept intact. The completeness proof is unchanged except that the definition of the relation $\sim$ of Lemma 5 becomes:

$$m \sim n \text{ iff } A \# i_{m,n}A \text{ is a member of } \Gamma^* \text{ for all } A.$$

*7 **Restoring free variables*** In the first section of [8] Quine points out that the introduction of variable binding operators in mathematics coincides with the failure of some attractive rules of substitution. Since all variable binding can be reduced to variable binding by quantifiers, the replacement of quantifiers by functors can be regarded as the reduction of a convenient, but conceptually complicated, idea to a simpler (though less convenient) one. It should provide, as Quine says, "... an analysis of the idea of the bound variable: an explanation with all the clarity of the discrete and blocklike terms and simple substitutions characteristic of algebra" ([9], p. 215). But predicate functor logic is a more radical departure from elementary logic than is needed for this reduction. The language of predicate functor logic dispenses not only with variable binding operators, but with the variables themselves—both bound and free. In this section we consider a variant of $\mathcal{L}_{PF}$ in which the free variables are restored. We call the new language $\mathcal{L}_{PFV}$.

A *basic formula* of $\mathcal{L}_{PFV}$ is an n-ary predicate (not necessarily atomic) followed by n individual variables of $\mathcal{L}$. A *formula* of $\mathcal{L}_{PFV}$ is something built up from the basic formulas in the usual way using the connectives $\wedge$ and $\neg$. A *sentence* of $\mathcal{L}_{PFV}$ is a formula containing no individual variables. For example $P^0 \wedge]\mathbf{P}\ ](Q^2 \cap R^1)$ is a sentence of $\mathcal{L}_{PFV}$. $\mathcal{L}_{PFV}$ is identified with the set of its formulas. α, β, γ are used as metamathematical variables ranging over $\mathcal{L}_{PFV}$. *Truth* for formulas of $\mathcal{L}_{PFV}$ can be defined succinctly by using the notion of truth for predicates of $\mathcal{L}_{PF}$ defined previously: If M is a model and $\boldsymbol{a}$ is an array on M then

1. $\boldsymbol{a} \models P^n v_{k_1} \ldots v_{k_n}$ iff $\langle d_{k_1}, \ldots, d_{k_n}, d_1, \ldots \rangle \models P^n$
2. $\boldsymbol{a} \models \neg\alpha$ iff not $\boldsymbol{a} \models \alpha$
3. $\boldsymbol{a} \models \alpha \wedge \beta$ iff $\boldsymbol{a} \models \alpha$ and $\boldsymbol{a} \models \beta$.

$M \models \alpha$ iff for all $\boldsymbol{a}$, $(M,\boldsymbol{a}) \models \alpha$.

PFV is the smallest subset of $\mathcal{L}_{PFV}$ containing all instances of the following schemas and closed under the following rules.

B1 All tautologous formulas
B2 $(P^m \cap Q^n)x_1 \ldots x_{max(m,n)} \leftrightarrow (P^m x_1 \ldots x_m \wedge P^n x_1 \ldots x_n)$
B3 $\neg Px_1 \ldots x_n \leftrightarrow {\sim}Px_1 \ldots x_n$
B4 Ixx
B5 $Ixy \rightarrow Iyx$
B6 $(Ixy \wedge Iyz) \rightarrow Ixz$
B7 $(Px_1 \ldots x_n \wedge Ix_1y_1) \rightarrow Py_1x_2 \ldots x_n$
B8 $\mathbf{P}Px_1 \ldots x_n \leftrightarrow Px_nx_1 \ldots x_{n-1}$
B9 $\mathbf{p}Px_1 \ldots x_n \leftrightarrow Px_2x_1 \ldots x_n$
B10 $[Px_1 \ldots x_n \leftrightarrow Px_2 \ldots x_n$
B11 $Px_0 \ldots x_n \rightarrow]Px_1 \ldots x_n$.

S1 If α and $\alpha \rightarrow \beta$ are in *PFV* then so is β.
S2 If $(\alpha \wedge Px_0 \ldots x_n) \rightarrow \beta$ is in *PFV* and x_0 does not occur in β or $\alpha \wedge]Px_1 \ldots x_n$, then $\alpha \wedge]Px_1 \ldots x_n \rightarrow \beta$ is in *PFV*.

Lemma 7 *PFV contains all instances of the following schemas*:

(a) $\sigma_k Q^n x_1 \ldots x_n \leftrightarrow Q^n x_k x_1 \ldots x_{k-1}x_{k+1} \ldots x_n$, *if* $k < n$
(a′) $\sigma_k Q^n x_1 \ldots x_k \leftrightarrow Q^n x_k x_1 \ldots x_{n-1}$, *if* $k \geqslant n$
(b) $\tau_k Q^n x_1 \ldots x_m \leftrightarrow Q^n x_k x_1 \ldots x_{n-1}$, *where* $m = max(k,n)$
(c) $\tau_{\langle k_1, \ldots, k_{n+p}\rangle} Q^n x_1 \ldots x_m \leftrightarrow Q^n x_{k_1} \ldots x_{k_n}$, *where* $m = max(k_1, \ldots, k_{n+p}, n + p)$.

Proof: Lemmas 7a and 7a′ can be proved easily using B5 and B6. 7c follows from 7b. We prove 7b for the case $k < n$. First, notice that each conditional in the following list is in *PFV*:

$$Q^n x_k x_1 \ldots x_{n-1} \to \overset{-1}{\sigma}_{k+1} Q^n x_k x_k x_1 \ldots x_{k-1} \ldots x_{n-1} \quad \text{(by 7a,a')}$$
$$Q^n x_k x_1 \ldots x_{n-1} \to I x_k x_k \wedge \overset{-1}{\sigma}_{k+1} Q^n x_k x_k x_1 \ldots x_{k-1} x_{k+1} \ldots x_{n-1} \quad \text{(by B4)}$$
$$Q^n x_k x_1 \ldots x_{n-1} \to (I \cap \overset{-1}{\sigma}_{k+1} Q^n) x_k x_k x_1 \ldots x_{k-1} x_{k+1} \ldots x_{n-1} \quad \text{(by B2)}$$
$$Q^n x_k x_1 \ldots x_{n-1} \to \sigma_{k+1}(I \cap \overset{-1}{\sigma}_{k+1} Q^n) x_k x_1 \ldots x_{n-1} \quad \text{(by 7a,a')}$$
$$Q^n x_k x_1 \ldots x_{n-1} \to \,]\,\sigma_{k+1}(I \cap \overset{-1}{\sigma}_{k+1} Q^n) x_1 \ldots x_{n-1} \quad \text{(by B11)}$$
$$Q^n x_k x_1 \ldots x_{n-1} \to \tau_k Q^n x_1 \ldots x_{n-1}.$$

Next, let y be any variable distinct from $x_1, \ldots, x_{n-1}$. Then the following are also in PFV:

$$\sigma_{k+1}(I \cap \overset{-1}{\sigma}_{k+1} Q^n) y x_1 \ldots x_{n-1} \to (I \cap \overset{-1}{\sigma}_{k+1} Q^n) x_k y x_1 \ldots x_{k-1} x_{k+1} \ldots x_{n-1} \quad \text{(by 7a)}$$
$$\sigma_{k+1}(I \cap \overset{-1}{\sigma}_{k+1} Q^n) y x_1 \ldots x_{n-1} \to (I x_k y \wedge \overset{-1}{\sigma}_{k+1} Q^n x_k y x_1 \ldots x_{k-1} x_{k+1} \ldots x_{n-1}) \quad \text{(by B2)}$$
$$\sigma_{k+1}(I \cap \overset{-1}{\sigma}_{k+1} Q^n) y x_1 \ldots x_{n-1} \to (I y x_k \wedge Q^n y x_1 \ldots x_{n-1}) \quad \text{(by B5 and 7a)}$$
$$\sigma_{k+1}(I \cap \overset{-1}{\sigma}_{k+1} Q^n) y x_1 \ldots x_{n-1} \to Q^n x_k x_1 \ldots x_{n-1} \quad \text{(by B7)}$$
$$]\,\sigma_{k+1}(I \cap \overset{-1}{\sigma}_{k+1} Q^n) x_1 \ldots x_{n-1} \to Q^n x_k x_1 \ldots x_{n-1} \quad \text{(by S2)}$$
$$\tau_k Q^n x_1 \ldots x_{n-1} \to Q^n x_k x_1 \ldots x_{n-1}.$$

Hence, $\tau_k Q^n x_1 \ldots x_{n-1} \leftrightarrow Q^n x_k x_1 \ldots x_{n-1}$ is in PFV.

We now define translations between $\mathcal{L}_{PF}$ and $\mathcal{L}_{PFV}$:

Let $s_1(P^n) v_1 \ldots v_n$.
Let $s_2(P^n v_{k_1} \ldots v_{k_n}) = \tau_{\langle k_1, \ldots, k_n \rangle} P^n$
$s_2(\alpha \wedge \beta) = s_2(\alpha) \cap s_2(\beta)$
$s_2(\sim\alpha) = \neg s_2(\alpha)$.

Lemma 8

(a) $M \overset{a}{\models} P$ *iff* $M \overset{a}{\models} s_1(P)$
(b) $M \overset{a}{\models} \alpha$ *iff* $M \overset{a}{\models} s_2(\alpha)$
(c) *if* $P \in PF$, $s_1(P) \in PFV$
(d) $s_1 s_2(\alpha) \leftrightarrow \alpha \in PFV$.

Proof: Lemma 8a follows from the definitions. 8b can be proved by a routine formula induction. To prove 8c, notice first that by Lemma 7c, $s_1(\tau_{\langle k_1, \ldots, k_n \rangle} P) \leftrightarrow P v_{k_1} \ldots v_{k_n}$ is in PFV. Using this fact it is easy to check that the translations of A1, . . ., A12 are in PFV and that PFV is closed under the translations of R1 and R2. 8d can also be easily proved by using Lemma 7c.

Theorem 2 $\models \alpha$ *iff* $\alpha \in PF$.

Soundness is routine. To prove the other direction, suppose $\models \alpha$. By Lemma 8b, $\models s_2(\alpha)$. By the corollary to Theorem 1, $s_2(\alpha) \in PF$. By Lemma 8c, $s_1(s_2(\alpha)) \in PFV$, and by Lemma 8d, $\alpha \in PFV$.

NOTES

1. Axiomatizations of similar systems have been given by Bernays [1], Nolin [6], Howard [3], and Craig [2]. Craig's axioms are particularly simple. Only Nolin's system, however,

is "autonomous" in the sense Quine requests, and his completeness proof rests on translations with a previously axiomatized version of predicate logic.

2. The definition and lemma which follow are needed only to establish strong completeness. Every valid formula of $\mathcal{L}_{PF}$ can be derived in the axiom system obtained by replacing R2, which follows axiom schema A13, by R2w: If $(P^m \cap \tau\langle k_0,\ldots,k_n\rangle Q^p) \supset R^s$ is in PF and $k_0 > max(m,n,p,s,k_1,\ldots,k_n)$ then $(P^m \cap \tau\langle k_1,\ldots,k_n\rangle] Q^p) \supset R^s$ is in PF.

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