Notre Dame Journal of Formal Logic
Volume 24, Number 2, April 1983

Compactness via Prime Semilattices

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1 Introduction Compactness is certainly one of the most fruitful concepts of general topology. Topologically inspired notions of compactness have also proven useful in logic (see [3], [9]) and measure theory (see [8]). In this paper we introduce a definition of compactness for subsets of a prime semilattice. Prime semilattices were introduced by Balbes [2] and their algebraic structure seems just right for presenting the ideas which underlie many compactness arguments.

2 Prime semilattices and Wallman's lemma Let $\langle S, \leqslant\rangle$ be a partially ordered set and suppose $T \subseteq S$. The greatest lower bound, or meet, of T, if it exists, will be denoted by $\wedge T$. The least upper bound, or join, if it exists, will be denoted by $\vee T$. If T is finite, $T = \{t_1, \ldots, t_n\}$, we shall write $t_1 \wedge \ldots \wedge t_n$ and $t_1 \vee \ldots \vee t_n$ for the meet and join of T, respectively.

A partially ordered set $\langle S, \leqslant\rangle$ is a (meet) *semilattice* if every finite, nonempty, set has a meet. A semilattice is said to be *prime* if whenever $s \in S$ and $s_1 \vee \ldots \vee s_n \in S$ then $(s \wedge s_1) \vee \ldots \vee (s \wedge s_n) \in S$ and $s \wedge (s_1 \vee \ldots \vee s_n) = (s \wedge s_1) \vee \ldots \vee (s \wedge s_n)$.

Let $\langle S, \leqslant\rangle$ be a semilattice and suppose $I \subseteq S$, $I \neq \phi$. I is an *ideal* of the semilattice $\langle S, \leqslant\rangle$ if $s \in I$ and $t \leqslant s$ implies $t \in I$; if, in addition, $s, t \in I$ and $s \vee t \in S$ implies $s \vee t \in I$, I will be called a *regular* ideal.

Suppose I is an ideal of the semilattice $\langle S, \leqslant\rangle$. A subset $W \subset S$ *avoids* I if $\wedge W \notin I$; W *finitely avoids* I if $\wedge W_0 \notin I$, for every finite $W_0 \subset W$. The following theorem generalizes a lemma of Wallman [12].

Theorem 1 *Let $\langle S, \leqslant\rangle$ be a prime semilattice and I a regular ideal of $\langle S, \leqslant\rangle$. Suppose $\{b_j\}_{j \in J}$ is a subcollection of S which finitely avoids I and $b_j = a_{j1} \vee \ldots \vee a_{jn_j}$, $j \in J$. Then there is a function f with domain J such that $\{a_{jf(j)}\}_{j \in J}$ finitely avoids I.*

Received August 17, 1981

Proof: Zorn's lemma easily implies there is a maximal subset M of S which includes $\{b_j\}_{j\epsilon J}$ and finitely avoids I.

We show next that $s_1 \vee s_2 \epsilon M$ implies $s_1 \epsilon M$ or $s_2 \epsilon M$. Assume not; that is, assume there exists $s_1, s_2 \epsilon S - M$ with $s_1 \vee s_2 \epsilon M$. Since M is maximal, neither $M \cup \{s_1\}$ nor $M \cup \{s_2\}$ finitely avoids I. Therefore there exists $u_1, \ldots, u_n$, $v_1, \ldots, v_k \epsilon M$ such that $u_1 \wedge \ldots \wedge u_n \wedge s_1 \epsilon I$ and $v_1 \wedge \ldots \wedge v_k \wedge s_2 \epsilon I$. If $w = u_1 \wedge \ldots \wedge u_n \wedge v_1 \wedge \ldots \wedge v_k$, then $w \wedge s_1 \epsilon I$ and $w \wedge s_2 \epsilon I$ since I is an ideal. Therefore $(w \wedge s_1) \vee (w \wedge s_2) \epsilon I$, since I is regular.[1] Thus $w \wedge (s_1 \vee s_2) \epsilon I$. However $w \wedge (s_1 \vee s_2)$ is a meet of elements of M, because $s_1 \vee s_2 \epsilon M$, and therefore cannot belong to I, since M finitely avoids I. Hence $s_1 \vee s_2 \epsilon M$ implies $s_1 \epsilon M$ or $s_2 \epsilon M$ as claimed.

It follows, by induction, that if $s_1 \vee \ldots \vee s_n \epsilon M$, then $s_i \epsilon M$ for some i, $1 \leqslant i \leqslant n$. Since $b_j \epsilon M$, $a_{ji_j} \epsilon M$ for some i_j, $1 \leqslant i_j \leqslant n_j$. Let $f(j) = i_j$ and then $\{a_{jf(j)}\}_{j\epsilon J}$ finitely avoids I because it is a subcollection of M.

A collection of sets, W, has the *finite intersection property*, or *fip*, if every finite subcollection has a nonempty intersection, that is, W finitely avoids $\{\phi\}$, the ideal consisting of the empty set. Thus we obtain,

Corollary 1 (Wallman) *Suppose $\{B_j\}_{j\epsilon J}$ is a collection of sets with fip and $B_j = A_{j1} \cup \ldots \cup A_{jn_j}$, $j \epsilon J$. Then there is a function f, with domain J, such that $\{A_{jf(j)}\}_{j\epsilon J}$ has fip.*

Next, we give an analog of Theorem 1 for propositional logic. An indexed collection of propositional sentences, $\{S_i\}_{i\epsilon I}$ will be called *finitely satisfiable* if every finite subcollection is satisfiable.

Corollary 2 *Suppose $\{S_i\}_{i\epsilon I}$ is finitely satisfiable where $S_i \equiv T_{i1} \vee \ldots \vee T_{ik_i}$, $i \epsilon I$. Then there is a function f with domain I such that $\{T_{if(i)}\}_{i\epsilon I}$ is finitely satisfiable.*

Proof: This follows from Theorem 1 on passing to the Lindenbaum algebra associated with the propositional logic (see [9]) and letting I be the regular ideal consisting of the equivalence classes of unsatisfiable formulas.

We now derive from Corollary 2 the Compactness Theorem for propositional logic.

Corollary 3 *Let $\{S_i\}_{i\epsilon I}$ be a finitely satisfiable set of propositional sentences. Then $\{S_i\}_{i\epsilon I}$ is satisfiable.*

Proof: Let $\{p_j\}_{j\epsilon J}$ be the set of all propositional letters and let $p_{j1} = p_j$, $p_{j2} = \sim p_j$, $j \epsilon J$. Then $\{S_i\}_{i\epsilon I} \cup \{p_{j1} \vee p_{j2}\}_{j\epsilon J}$ is finitely satisfiable since $\{S_i\}_{i\epsilon I}$ was assumed to be so and $p_{j1} \vee p_{j2}$ is tautologous. By Corollary 2, there is a function f such that $\{S_i\}_{i\epsilon I} \cup \{p_{jf(j)}\}_{j\epsilon J}$ is finitely satisfiable. Since $\{p_{jf(j)}\}_{j\epsilon J}$ contains exactly one of every pair $\{p_j, \sim p_j\}$, $j \epsilon J$, f determines a unique interpretation $\mathcal{I}$ satisfying all $p_{jf(j)}$, $j \epsilon J$. We claim $\mathcal{I}$ also satisfies each S_i, $i \epsilon I$; for if $\{p_j\}_{j\epsilon J_i}$ is the finite set of propositional letters of S_i and $\mathcal{I}'$ satisfies $\{S_i\} \cup \{p_{jf(j)}\}_{j\epsilon J_i}$, $\mathcal{I}$ and $\mathcal{I}'$ must agree on $\{p_j\}_{j\epsilon J_i}$. Hence, $\mathcal{I}$ satisfies S_i, $i \epsilon I$.

We show next that the Ultrafilter Theorem for Boolean algebras is also a corollary of Theorem 1. Let $\langle B, \vee, \wedge, \sim, 0, 1\rangle$ be a Boolean algebra. A subset

$B_0 \subset B$ is *full* if $b \in B_0$ or $\sim b \in B_0$ for every $b \in B$. B_0 has the *finite meet property*, or *fmp*, if B_0 finitely avoids $\{0\}$ in the semilattice $\langle B, \leqslant \rangle$, where $b_1 \leqslant b_2$ if and only if $b_1 \wedge b_2 = b_1$. The following characterization of ultrafilters is sometimes useful.

Lemma *A subset of a Boolean algebra which is full and has fmp is an ultrafilter.*

Proof: Let U be full with fmp. It suffices to show U is a filter. Let $u_1, u_2 \in U$; since $u_1 \wedge u_2 \wedge \sim(u_1 \wedge u_2) = 0$, $\sim(u_1 \wedge u_2) \notin U$, because U has fmp. Therefore $u_1 \wedge u_2 \in U$, since U is full. Let $u \in U$ and $b \in B$; since $\sim(u \vee b) \wedge u = \sim u \wedge \sim b \wedge u = 0$, $\sim(u \vee b) \notin U$. Hence $u \vee b \in U$, completing the proof.

Corollary 4 (Ultrafilter Theorem) *Any Boolean algebra contains an ultrafilter.*

Proof: Let $\langle B, \vee, \wedge, \sim, 0, 1\rangle$ be a Boolean algebra. Then surely $\langle B, \leqslant \rangle$ is a prime semilattice. Consider the collection $C = \{b \vee \sim b\}_{b \in B}$. Since $b \vee \sim b = 1$, C has fmp, that is, C finitely avoids $\{0\}$. Theorem 1 yields a set U with fmp which is also full. The lemma implies U is an ultrafilter.

*3 **Compactness*** Let $\langle S, \leqslant \rangle$ be a prime semilattice with ideal I and let κ be an infinite cardinal. For any set K, let $\overline{\overline{K}}$ denote the cardinality of K. A subset $C \subset S$ is *κ-compact with respect to I* if for every $K \subset C$ with $\overline{\overline{K}} \leqslant \kappa$, K finitely avoids I implies K avoids I. C is *compact with respect to I* if for every $K \subset C$, K finitely avoids I implies K avoids I. If it is clear what ideal I we are referring to, we shall simply write κ-compact or compact as the case may be.

Marczewski [8] defines a class F of subsets of a set X to be compact if for each sequence $P_n \in F$, $P_1 \cap \ldots \cap P_n \neq \phi$ implies $\bigcap P_n \neq \phi$, that is, F is $\aleph_0$-compact in the prime semilattice $\langle P(X), \leqslant\rangle^2$ with $I = \{\phi\}$. He then proves that if a class F is compact, so are the class of all countable intersections of members of F and the class of all finite unions of members of F. The next two theorems generalize these results.

If $Q \subset S$, we let $\Pi(Q) = \{\wedge T | T \subset Q \text{ and } \wedge T \in S\}$ and $\Sigma(Q) = \{\vee T | T \subset Q$ and $\vee T \in S\}$. If T is only allowed to range over subsets of Q of cardinality $\leqslant \kappa$ we write $\Pi_\kappa(Q)$ and $\Sigma_\kappa(Q)$, respectively; whereas if T only ranges over finite subsets of Q, we write $\Pi_F(Q)$ and $\Sigma_F(Q)$, respectively.

Theorem 2 *Let $\langle S, \leqslant\rangle$ be a prime semilattice with ideal I. If $Q \subset S$ is κ-compact, then $\Pi_\kappa(Q)$ is κ-compact. If Q is compact, $\Pi(Q)$ is compact.*

Proof: Suppose $K \subset \Pi_\kappa(Q)$, $\overline{\overline{K}} \leqslant \kappa$, and K doesn't avoid I, that is, $\wedge K \in I$. If $k \in K$, $k = \wedge\{q_{jk} | q_{jk} \in Q, j \in J_k\}$ where $\overline{\overline{J_k}} \leqslant \kappa$ and $\wedge K = \wedge\{q_{jk} | j \in J_k, k \in K\}$. If $D = \{q_{jk} | j \in J_k, k \in K\}$, $\overline{\overline{D}} \leqslant \kappa \times \kappa = \kappa$. Since $\wedge D = \wedge K \in I$ and Q is κ-compact, there is a finite $W \subset D$ such that $\wedge W \in I$. Since W is finite, there is a finite $K_0 \subset K$ such that $q_{jk} \in W$ implies $k \in K_0$ and then $\wedge K_0 \leqslant \wedge W$. Hence $\wedge K_0 \in I$ for K_0 a finite subset of K.

The proof of the second assertion is virtually the same.

Theorem 3 *Let $\langle S, \leqslant\rangle$ be a prime semilattice and let I be a regular ideal of*

$\langle S, \leqslant\rangle$. If $Q \subset S$ is compact (κ-compact) then $\Sigma_F(Q)$ is compact (κ-compact).

Proof: We give the proof for κ-compactness. Suppose $K \subset \Sigma_F(Q)$, $\bar{\bar{K}} \leqslant \kappa$ and K finitely avoids I. Let $k = \{k_j\}_{j\epsilon J}$, $\bar{\bar{J}} \leqslant \kappa$, with $k_j = q_{j1} \vee \ldots \vee q_{jn_j}$, $q_{ji} \in Q$, $1 \leqslant i \leqslant n_j$. Theorem 1 gives a function f, with domain J, such that $D = \{q_{jf(j)}\}_{j\epsilon J}$ finitely avoids I and $\bar{\bar{D}} \leqslant \bar{\bar{J}} \leqslant \kappa$. Therefore D avoids I since Q is κ-compact. Hence K avoids I since $\wedge D \leqslant \wedge K$ and I is an ideal.

Suppose $\langle S, \leqslant\rangle$ is a prime semilattice and $C \subset S$. Then B is a *base* for C if $C = \Pi(B)$; A is a *subbase* for C if $C = \Pi(\Sigma_F(A))$. If we replace Π by Π_κ in the above definitions, we define a *κ-base* and *κ-subbase*, respectively. Theorems 2 and 3 now yield the following generalization of the Subbase Theorem of Alexander [1].

Theorem 4 *Let $\langle S, \leqslant\rangle$ be a prime semilattice and let I be a regular ideal. If a subbase (κ-subbase) for C is compact (κ-compact), then C is compact (κ-compact).*

Let X be a topological space; then X is compact if and only if the closed sets are compact with respect to $\{\phi\}$ in the prime semilattice $\langle P(X), \subseteq\rangle$. The following result was used by Alexander [1] to prove the Tychonoff Theorem.

Corollary (Subbase Theorem) *Let X be a topological space and let A be a subbase for the closed sets. Then X is compact if no subset of A with fip has an empty intersection.*

Proof: The hypothesis says that the subbase A is compact. The conclusion now follows from Theorem 4.

We show next how the methods of this section can be used to prove some well known infinite combinatorial results. We begin by proving a general theorem which we stated in [5]. A partially ordered set $\langle W, \leqslant\rangle$ is *directed* if $w_1, \ldots, w_n \in W$ implies there is a $w \in W$ with $w_i \leqslant w$, $1 \leqslant i \leqslant n$.

Theorem 5 *Let $\langle W, \leqslant\rangle$ be a directed partially ordered set and for each $w \in W$, let F_w be a finite, nonempty, set of functions with domain D_w. Suppose that $w_1 \leqslant w_2$ and $f \in F_{w_2}$ implies $f \upharpoonright D_{w_1} \in F_{w_1}$. Then there is a function f such that $f \upharpoonright D_w \in F_w$ for all $w \in W$.*

Proof: Let $D = \cup D_w (w \in W)$. For each $d \in D$, let $A_d = \{f(d) | f \in F_w, d \in D_w\}$ and let $F = \bigtimes_{d\epsilon D} A_d$. We consider the prime semilattice $\langle P(F), \subseteq\rangle$ and let $I = \{\phi\}$. For each $w \in W$, let $\tilde{F}_w = \{f \in F | f \upharpoonright D_w \in F_w\}$. For each $d \in D$ and $a \in A_d$, let $\langle d, a\rangle = \{f | f \in F$ and $f(d) = a\}$ and let $C = \{\langle d, a\rangle | d \in D, a \in A_d\}$. Then C is compact; for if $\bigcap_{i\epsilon I} \langle d_i, a_i\rangle = \phi$, it must be that $d_{i_1} = d_{i_2}$ and $a_{i_1} \neq a_{i_2}$, and then $\langle d_{i_1}, a_{i_1}\rangle \cap \langle d_{i_2}, a_{i_2}\rangle = \phi$, By Theorems 2 and 3 $\Sigma_F(\Pi(C))$ is compact. But $\tilde{F}_w \in \Sigma_F(\Pi(C))$; in fact if $F_w = \{f_1, \ldots, f_n\}$, $F_w = \bigcup_{j=1}^{n} \bigcap_{d\epsilon D_w} \langle d, f_j(d)\rangle$. Also, given any $w_1, \ldots, w_n \in W$, $\tilde{F}_{w_1} \cap \ldots \cap \tilde{F}_{w_n} \neq \phi$, since if $w \in W$ and $w_i \leqslant w$, $1 \leqslant i \leqslant n$, and $f \in F_w$, then $f \upharpoonright D_{w_i} \in \tilde{F}_{w_i}$, $1 \leqslant i \leqslant n$, and so $\tilde{F}_w \subseteq \tilde{F}_{w_1} \cap \ldots \cap \tilde{F}_{w_n}$. Therefore $\cap \tilde{F}_w \neq \phi$ and if $f \in \cap \tilde{F}_w$, $f \upharpoonright D_w \in F_w$, for all $w \in W$.

As our first application of Theorem 5, we prove the theorem of De Bruijn and Erdös [6] that a graph is n-colorable if all its finite subgraphs are n-colorable. Let G be a graph with vertex set V and let n be a positive integer. An n-coloring of G is a function $f\colon V \to \{0, \ldots, n-1\}$ such that $f(v_1) \neq f(v_2)$ if v_1 and v_2 are connected by an edge of G.[3]

Theorem 6 *A graph is n-colorable if every finite subgraph is n-colorable.*

Proof: If H_1, H_2 are finite subgraphs of G, we let $H_1 \leqslant H_2$ if H_1 is a subgraph of H_2. If H is a subgraph of G, let F_H be the set of n-colorings of H. Surely if $H_1 \leqslant H_2$ and $f \in F_{H_2}$ then $f \upharpoonright V(H_1) \in F_{H_1}$ where $V(H_1)$ is the set of vertices of H_1. Theorem 5 yields a function f such that $f \upharpoonright V(H) \in F_H$ for all H and so f n-colors G.

A special case of Theorem 6 deserves mention: let $\langle P, \leqslant \rangle$ be a partially ordered set and let G be the graph whose set of vertices is P and whose edges connect those pairs $\{p_1, p_2\}$ which are incomparable, that is, $p_1 \nleqslant p_2$ and $p_2 \nleqslant p_1$. Then an n-coloring of G is equivalent to a partition of P into at most n chains (C is a chain if $p_1, p_2 \in C$ implies $p_1 \leqslant p_2$ or $p_2 \leqslant p_1$). Hence Theorem 6 implies,

Theorem 7 *If $\langle P, \leqslant \rangle$ is a partially ordered set and n, a positive integer and any finite $P_0 \subset P$ can be partitioned into at most n chains, then P itself can be so partitioned.*

A subset S of a partially ordered set P is an antichain if every two distinct elements of S are incomparable: Dilworth [7] proved the following decomposition theorem by first considering the case where P is finite and then using the finite case along with a nontrivial transfinite argument to generalize to infinite P.

Theorem 8 *If a partially ordered set P contains no antichain of cardinality $n + 1$, then P is the union of at most n chains.*

Proof: The general case follows from the finite case[4] and Theorem 7.

Theorem 5 has other applications as well. It can be considered a variant of the Rado selection lemma (see [4] and [10] for other variants) and has the same applications as Rado's lemma. In fact many of the compactness results we have considered can be used interchangeably; we have given primacy to the semilattice presentation to demonstrate the algebraic assumptions which are implicit in many compactness arguments.

NOTES

1. $(w \wedge s_1) \vee (w \wedge s_2) \in S$, since $\langle S, \leqslant \rangle$ is prime and $s_1 \vee s_2 \in S$.
2. $P(X)$ is the set of all subsets of X.
3. See Wilson [13] for more on graphs.
4. See Tverberg [11] for an elegant proof of the finite case.

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