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A New Foundation for the Theory of Relations

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Relation algebras are characterized using certain multivalued algebraic systems called polygroupoids. The connection between these concepts provides a basis for an alternative to the usual approach to the study of relations. Examples of polygroupoids are given as well as an application to the theory of relations.

1 Introduction The purpose of this paper is to outline a new approach to the calculus of relations. Relation algebras were introduced by Tarski in [11] as an abstract algebraic system defined by a natural set of axioms. The principal models for these axioms are obtained from collections of binary relations on a set using the set-theoretic operations of union, intersection, relation composition, and converse. Such algebras are known as representable relation algebras. Not all models of Tarski's axioms are representable (cf. [9], [10]). The present study developed as an outgrowth of an investigation into ways of characterizing "nonrepresentable" relation algebras (cf. [3], [4]). The characterization given in Section 4 is an extension of the relationship (cf. [8], Section 5) between certain relation algebras and systems called Brandt groupoids. Nowadays, these systems are just called "groupoids" in category theory (cf. [6]). Basically, the idea in the treatment below is to replace the use of groupoids by multivalued groupoids in Tarski's complex algebra construction and thereby extend the relationship in [8] to all relation algebras.

The results in this paper can be developed using the language of category

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theory; however, the categories involved are "multicategories" which have not been studied to the author's knowledge. Instead we employ standard terminology from universal algebra.

Tarski's theory of relation algebras is reviewed in Section 2. The notion of a polygroupoid is introduced in Section 3 with examples. The basic characterization is presented in Section 4. Ideas similar to those in this section have been independently worked out by Brian McEvoy. In a final section we consider a problem about relation algebras raised by G. Birkhoff at the Jónsson Symposium. We settle Birkhoff's question using the polygroupoid approach to relation algebras that is the main thesis of this paper.

2 ***Relation algebras*** In [8] a *relation algebra* (*RA*) is defined as an algebra of the type $\langle A, +, \cdot, 0, 1, ;, 1', ^{\cup}\rangle$ where 0, 1, and $1'$ are elements of A, $+$, $\cdot$, and ; are binary operations on A, $^{\cup}$ is a unary operation on A, and the following axioms hold:

R_0 $\langle A, +, \cdot, 0, 1\rangle$ *is a Boolean algebra.*

R_1 $(x;y);z = x;(y;z)$ *for all* $x, y, z \in A$.

R_2 $1';x = x = x;1'$ *for all* $x \in A$.

R_3 *the formulas* $(x;y)\cdot z = 0$, $(x^{\cup};z)\cdot y = 0$, *and* $(z;y^{\cup})\cdot x = 0$ *are equivalent for all* $x, y, z \in A$.

The following class of examples motivated the definition.

Example 2.1: Consider a system $\langle \mathcal{A}, \cup, \cap, \phi, X^2, |, ^{-1}, I_X\rangle$ where $\mathcal{A}$ is a collection of binary relations on a set X that contains ϕ, X^2, and $I_X = \{(x, x)\colon x \in X\}$ and $\mathcal{A}$ is closed under the operations of union $\cup$, intersection $\cap$, relation composition $|$, and converse $^{-1}$. Such a system is called a *proper relation algebra.* A relation algebra is *representable* if it is a subalgebra of a product of proper relation algebras.

Before introducing another class of examples we recall the notion of a complex algebra ([8], Definition 3.8). Consider a system

(*) $\mathfrak{A} = \langle A, R_0, R_1, \ldots\rangle$

where $R_i \subseteq A^{n_i+1}$ for each i. The *complex algebra* of $\mathfrak{A}$, denoted $\mathfrak{C}[\mathfrak{A}]$, is the system

$$\mathfrak{C}[\mathfrak{A}] = \langle \mathcal{P}(A), \cup, \cap, \phi, A, R_0^*, R_1^*, \ldots\rangle$$

where $\mathcal{P}(A)$ is the collection of all subsets of A and, for each i,

$$R_i^*\colon \mathcal{P}(A)^{n_i} \to \mathcal{P}(A)$$

is defined for $X_0, \ldots, X_{n_i-1} \subseteq A$ by

$$R_i^*(X_0, \ldots, X_{n_i-1}) = \{x \in A\colon \exists x_0 \in X_0, \ldots, \exists x_{n_i-1} \in X_{n_i-1}\cdot (x_0, \ldots, x_{n_i-1}, x) \in R_i\}.$$

Theorem 3.10 of [8] asserts that every normal Boolean algebra with operators is isomorphic to a regular subalgebra of the complex algebra of some system of type (*). We show in Section 4 that, in the case of relation algebras, natural systems of multivalued groupoids can be used. As a prelude we mention another class of examples of relation algebras treated in [8], Section 5.

Example 2.2: A *generalized Brandt groupoid* is a partial algebraic system

$$\mathfrak{A} = \langle A, \cdot, I, {}^{-1}\rangle$$

where $\cdot$ is a partial binary operation on A, $I \subseteq A$, ${}^{-1}$ is an operation on A, and the following axioms hold:

(i) For every $x \in A$, there is a $y \in A$ such that $x \cdot y \in A$.
(ii) For any $x, y, z \in A$, if $x \cdot y$ and $(x \cdot y) \cdot z$ are in A or $y \cdot z$ and $x \cdot (y \cdot z)$ are in A, then all four elements are in A and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
(iii) For any $x \in A$, $x \in I$ iff $x \cdot x = x$.
(iv) The formulas $x \cdot y = z$, $x^{-1} \cdot z = y$, and $z \cdot y^{-1} = x$ are equivalent for all $x, y, z \in A$.

The complex algebra $\mathfrak{C}[\mathfrak{A}]$ of a generalized Brandt groupoid $\mathfrak{A}$ is a complete atomic relation algebra such that $0 \neq 1$ and every atom is a functional element (Theorem 5.5 of [8]). Conversely, every such *RA* is isomorphic to the complex algebra of some generalized Brandt groupoid.

Remark 2.3: The notion of a generalized Brandt groupoid is the same as the categorical notion of groupoid, i.e., a category in which every morphism is invertible (see [6]). Moreover, the notion of a Brandt groupoid introduced in [8] is exactly that of a connected groupoid in the categorical sense.

*3 **Polygroupoids*** The goal of this section is to introduce a multivalued version of a generalized Brandt groupoid. First some terminology.

A *partial multivalued operation* f of rank n on a set A is a function from A^n into $\mathcal{P}(A)$. We call f a *multivalued operation* on A if $dom(f) = A^n$ where $dom(f) = \{(x_1, \ldots, x_n) \in A^n : f(x_1, \ldots, x_n) \neq \phi\}$. A *partial multivalued algebra* is a system $\mathfrak{A} = \langle A, f_0, f_1, \ldots\rangle$ where each f_i is a partial multivalued operation on A. These are just systems of type (*) with a different notation. A system $\mathfrak{A}$ is a *multivalued algebra* if $dom(f_i) = A^{n_i}$ for each f_i with rank n_i.

Several conventions are useful. A partial operation on A extends to an operation on $\mathcal{P}(A)$. Namely, if $f\colon A^n \to \mathcal{P}(A)$ and $X_1, \ldots, X_n \subseteq A$, then

$$f(X_1, \ldots, X_n) = \cup \{f(x_1, \ldots, x_n) : x_1 \in X_1, \ldots, x_n \in X_n\}.$$

In particular, if $X_i = \phi$ for some i, $f(X_1, \ldots, X_n) = \phi$. We also identify $\{x\}$ with x when convenient.

Definition 3.1 A *polygroupoid* is a partial multivalued algebra

$$\mathfrak{A} = \langle A, \cdot, I, {}^{-1}\rangle$$

where $\cdot$ is a partial multivalued binary operation on A, $I \subseteq A$, and ${}^{-1}$ is an operation on A that satisfies the following axioms:

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in A$
(ii) $x \cdot I = x = I \cdot x$ for all $x \in A$
(iii) the formulas $x \in y \cdot z$, $y \in x \cdot z^{-1}$, and $z \in y^{-1} \cdot x$ are equivalent for all $x, y, z \in A$.

Notice that both sides of (i) could be ϕ. Interpret (i) as saying that if either side is nonempty, then both sides are and the sets are equal.

The notion of a polygroup studied in [3] and [4] is a special case of 3.1. A polygroupoid $\mathfrak{A} = \langle A, \cdot, e, {}^{-1}\rangle$ is a *polygroup* if $e \in A$ and $dom(\cdot) = A^2$. That is, a polygroup is a polygroupoid with a single identity element and total operations.

We conclude this section with some examples.

Example 3.2: An ordinary group is a polygroup. More generally, suppose H is a subgroup of a group G. Define a system

$$G/\!/H = \langle \{HgH: g \in G\}, *, H, {}^{-1}\rangle$$

where $(HgH)^{-1} = Hg^{-1}H$ and $(Hg_1H) * (Hg_2H) = \{Hg_1hg_2H: h \in H\}$. The algebra of double cosets $G/\!/H$ is a polygroup introduced in [5].

The next example is an extension of Example 3.2 that produces polygroupoids which are not, in general, polygroups.

Example 3.3: Assume G is a finite group and $H_1, \ldots, H_m$ are subgroups. Let $A = \{H_igH_j: i, j = 1, \ldots, m \text{ and } g \in G\}$ and $I = \{G_i: i = 1, \ldots, m\}$. Define ${}^{-1}$ and $*$ on A by $(H_igH_j)^{-1} = H_jg^{-1}H_i$ and $(H_ixH_j) * (H_ryH_s) = \{H_ixgyH_s: g \in H_j \text{ and } j = r\}$. The double coset system $\langle A, *, I, {}^{-1}\rangle$ is a polygroupoid. Its complex algebra is a representable RA in the sense of Example 2.1, the representation being constructed on the disjoint union of the right coset spaces.

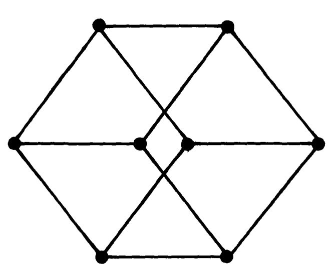
The next example has a different flavor.

Example 3.4: Suppose Γ is a distance regular graph with vertex set V (see [1]). Let $d(u, v)$ denote the shortest distance between vertices u and v in Γ. Define $*$ and ${}^{-1}$ on the set $A = \{1, \ldots, d\}$ where d is the diameter of Γ as follows:

$$i^{-1} = i$$

$$k \in i * j \text{ iff for every } u, v \in V \text{ with } d(u, v) = k \text{ there exist } w \in V \text{ such that } d(u, w) = i \text{ and } d(w, v) = j.$$

Notice that the definition above implies that 0 is the identity element. The system $\langle A, *, 0, {}^{-1}\rangle$ is a polygroup. As a concrete example consider the cube Q_3



which has diameter 3. The polygroup derived from Q_3 on 0, 1, 2, 3 has a multiplication table for $*$ given below.

$*$	0	1	2	3
0	0	1	2	3
1	1	02	13	2
2	2	13	02	1
3	3	2	1	0

The example above has many extensions. For example, in [4] it is shown that a polygroup can be derived from an association scheme (cf. [2]) and from coherent configurations (cf. [7]).

*4 **The characterization*** In this section we extend the relationship mentioned in Example 2.2 to all relation algebras. The following lemma summarizes some useful elementary properties of polygroupoids.

Lemma 4.1 *If $\langle A, \cdot, I, {}^{-1}\rangle$ is a polygroupoid, then the following hold for all $x, y \in A$:*

(1) *If $y \in I$ and $x\cdot y \neq \phi$, then $x\cdot y = x$. Similarly, $x\cdot y = y$ if $x \in I$ and $x\cdot y \neq \phi$.*
(2) *$x \in I$ implies $x\cdot x = x$.*
(3) *$x\cdot x^{-1} \cap I \neq \phi$ and $x^{-1}\cdot x \cap I \neq \phi$.*
(4) *$x\cdot y \cap I \neq \phi$ implies $y = x^{-1}$.*
(5) *$(x^{-1})^{-1} = x$.*
(6) *$x \in I$ implies $x^{-1} = x$.*
(7) *$|x\cdot x^{-1} \cap I| = 1$ and $|x^{-1}\cdot x \cap I| = 1$.*
(8) *$(x\cdot y)^{-1} = y^{-1}\cdot x^{-1}$.*

Proof: The proofs are routine. To give one illustration, we establish (4). Suppose $e \in x\cdot y \cap I$. Then by 3.1(iii) and 3.1(ii), $y \in x^{-1}\cdot e \subseteq x^{-1}\cdot I = \{x^{-1}\}$. Therefore, $y = x^{-1}$.

When the general complex algebra construction (Section 2) is applied to a polygroupoid $\mathcal{M} = \langle M, \cdot, I, {}^{-1}\rangle$ we obtain the system

$$\mathfrak{C}[\mathcal{M}] = \langle \mathcal{P}(M), \cup, \cap, \phi, M, \cdot^*, I^*, {}^{-1^*}\rangle$$

where $I^* = I$ and, for $X, Y \subseteq M$,

$$X^{-1^*} = \{x^{-1}: x \in X\},$$

and

$$X\cdot^* Y = \{z \in M: z \in x\cdot y \text{ for some } x \in X \text{ and } y \in Y\}.$$

The $*$ will be dropped when the meaning is clear from the context. The following result characterizes complete atomic RA's.

Theorem 4.2 *The complex algebra $\mathfrak{C}[\mathcal{M}]$ of a polygroupoid $\mathcal{M}$ is a complete atomic relation algebra with $0 \neq 1$. Conversely, if*

$$\mathfrak{A} = \langle A, +, \cdot, 0, 1, ;, 1', {}^{\cup}\rangle$$

is a complete atomic RA with $0 \neq 1$, M is the set of all atoms of $\mathfrak{A}$, and $I = \{x \in M: x \leqslant 1'\}$, then $\mathcal{M} = \langle M, ;, I, {}^{\cup}\rangle$ is a polygroupoid and $\mathfrak{A} \cong \mathfrak{C}[\mathcal{M}]$.

Proof: It is easy to check that $\mathfrak{C}[\mathcal{M}]$ is a complete atomic RA: axiom R_0 is obvious and axioms R_1, R_2, and R_3 are easy extensions of 3.1(i), (ii), and (iii) to subsets. Conversely, it is also straightforward to verify that the system $\mathfrak{M}$ is a polygroupoid. Now define a map $F: A \to \mathcal{P}(M)$ by

$$F(a) = \{u \in M: u \leqslant a\}$$

for all $a \in A$. It is again routine to check that F is a one-one homomorphism since $\mathfrak{A}$ is atomic. Ontoness follows from the fact that $\mathfrak{A}$ is complete.

The system $\mathfrak{M}$ constructed in Theorem 4.2 is called the *atomic structure* of the relation algebra $\mathfrak{A}$. The following corollary is immediate from Theorem 4.2 and Theorem 4.21 of [8].

Corollary 4.3 *Every relation algebra is embeddable in the complex algebra of a polygroupoid.*

Both 4.2 and 4.3 can be specialized in various ways. Classes of RA's give rise to classes of polygroupoids and vice versa. For example, in [3] and [4] it was shown that integral RA's correspond to polygroups. As another variation on this theme we consider the important class of simple RA's.

A polygroupoid $\mathfrak{M} = \langle M, \cdot, I, ^{-1}\rangle$ is called *connected* if for all $x, y \in I$ there exist $z \in M$ such that $x \cdot z = z$ and $z \cdot y = z$. Thinking of $\mathfrak{M}$ as a "multicategory" the condition just means that the underlying graph is connected.

The following lemma and its proof are analgous to Theorem 5.4 in [8].

Lemma 4.4 *For every polygroupoid $\mathfrak{M}$ the following are equivalent:*

(1) *$\mathfrak{M}$ is connected*
(2) *for all x, y there exist z such that $x \cdot z \neq \phi$ and $z \cdot y \neq \phi$*
(3) *for all x, y there exist u, v such that $x \in u \cdot y \cdot v$.*

Theorem 4.5 *The complex algebra of a connected polygroupoid is a simple RA. Conversely, if $\mathfrak{A}$ is a simple RA, its atomic structure $\mathfrak{M}$ is connected.*

Proof: By 4.4, if $\mathfrak{M}$ is connected and $\phi \neq X \subseteq M$, then $M \cdot X \cdot M = M$. It follows that $\mathfrak{C}[\mathfrak{M}]$ is simple using Theorem 4.10 in [8]. Conversely, if $\mathfrak{A}$ is simple, then for all $x, y \in M$, $x \leqslant 1; y; 1$ which implies that $x \in u; y; v$ for some $u, v \in M$. Thus $\mathfrak{M}$ is connected by 4.4.

Since simplicity is preserved under complete extensions (cf., 4.21 of [8]) we obtain

Corollary 4.6 *Every simple relation algebra is embeddable in the complex algebra of a connected polygroupoid.*

5 ***An application*** The results in Section 4 suggest that relation algebras (and hence properties of relations) can be studied via polygroupoids. In [4] it was shown how polygroups introduce a wide range of combinatorial configurations into the study of integral relation algebras. In this section we illustrate how polygroupoids are useful in the verification of properties of relation algebras. The particular implication verified in 5.1 below gives an affirmative answer to a question raised by G. Birkhoff at the Jónsson Symposium.

Proposition 5.1 *The implication $R; S = S; R = 1' \Rightarrow S = R^{-1}$ holds in every relation algebra.*

Proof: Since the statement is universal it suffices to verify it in all complete atomic RA's. By the representation 4.2 we may regard R and S as subsets of a polygroupoid $\langle M, \cdot, I, ^{-1}\rangle$ and assume $R\cdot S = S\cdot R = I$ (hence both are nonempty). In order to show $R^{-1} \subseteq S$ assume $r \in R$. Now, using 3.1(ii), $I = R\cdot S$, and 3.1(i)

$$r^{-1} = r^{-1}\cdot I = r^{-1}\cdot(R\cdot S) = (r^{-1}\cdot R)\cdot S,$$

so $r^{-1} \in u\cdot s$ for some $s \in S$ and $u \in r^{-1}\cdot R$. By 3.1(iii) and the fact that $S\cdot R = I$ implies $R^{-1}\cdot S^{-1} = I^{-1} = I$ (use 4.1(8)), it follows that

$$u \in r^{-1}\cdot s^{-1} \subseteq R^{-1}\cdot S^{-1} = I.$$

Therefore, $u\cdot s \neq \phi$ and $u \in I$ which yields $u\cdot s = s$ by 4.1(1). Hence $r^{-1} \in u\cdot s = s$ which gives $r^{-1} = s \in S$. Since r was arbitrary, $R^{-1} \subseteq S$. Similarily, $S^{-1} \subseteq R$ which implies $S \subseteq R^{-1}$. Thus $R^{-1} = S$ as desired.

Although a direct proof of the above from the axioms for RA's no doubt exists, the line of reasoning above illustrates that properties of relation algebras and their proofs are "almost like" group (or groupoid) theory when treated from the viewpoint of polygroupoids.

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