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The Transitivity of Implication in Tree Logic

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A *If* $\vdash S_1$ *and* $\vdash S_1 \supset S_2$ *then* $\vdash S_2$.
B *If* $S_1 \vdash S_2$ *and* $S_2 \vdash S_3$ *then* $S_1 \vdash S_3$.

These are, obviously, desirable metatheorems. The first states that the set of theorems is closed under modus ponens and one would like to appeal to A in doing a relative soundness proof, in showing, for example, that *this* system (whatever it is) is at least as strong as some other system, given by axioms and MP. (Whence, if *this* system is provably sound, so is that other.) Proposition B states that [syntactical] implication is transitive—whence the title of this paper.

These are syntactical metatheorems. (They are interderivable for tree logic without too much trouble.) What is surprising is the difficulty of getting *purely syntactical* proofs of them for tree logic. Standard procedure is to prove completeness and soundness of tree logic and then, from the corresponding semantical theorems, get quick proofs of A and B.[1]

But suppose we did the same thing for the Deduction Theorem in traditional logics, i.e., first proved completeness and soundness, without the Deduction Theorem and then got the Deduction Theorem from the corresponding semantical theorem *via* completeness and soundness. The feeling would be that we would have missed out on important insights which a purely syntactical proof of the Deduction Theorem supplies. In the same way, it seems to me, we miss out on some insights into tree logic if we do not have purely syntactical proofs of Propositions A and B. This paper aims at supplying such proofs as Metatheorems 12 and 13.

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1 Basic tree logic Tree logic has its origins in the method of Semantic Tableaux (see Beth [1]). The terms "semantical tableaux" and "truth trees" (Jeffrey [2]) both seem to me to be inappropriate, since they suggest that the methods are somehow semantical. On the contrary, they have to be regarded as syntactical, for the tree rules are stated purely in terms of the syntactical character of the sentences in question. There is no appeal to truth or any other semantical notion.

1.1 Grammar We will use 'S', 'S_1', 'S_2', etc., as metalinguistic variables ranging over the sentences of our object language L, and 's', 's_1', 's_2' as variables ranging over just the elementary (or, atomic) sentences of L and negations of elementary sentences (the "basic" sentences). Initially we shall use Quine's corners, subsequently leaving them to be supplied by the reader.

Nonlogical signs of L: 'A', 'B', 'C', etc., with or without subscripts (these are the *elementary sentences* of L).

Logical signs of L: '$\sim$', '&', 'v', and '$\supset$'.

We shall assume the usual definition of "sentence of L".

1.2 Syntax: The Tree Rules ("Rules of Arboriculture" after Jeffrey [2]) Each rule specifies for sentences of a certain kind what sentence or sentences must be written in each path beneath the given sentence, and whether the offspring sentences will be written as stacked or branched. It is assumed without proof (what is in fact true) that, given a sequence of initial sentences ("the trunk"), it does not matter in what order the sentences are "handled" (have the relevant rule applied to them).

	Stacking Rule	Branching Rule
Negation	Given: $\ulcorner\sim\sim S\urcorner$ Offspring: S	
Conjunction	Given: $\ulcorner S_1 \,\&\, S_2\urcorner$ Offspring: S_1 S_2	Given: $\ulcorner\sim(S_1 \,\&\, S_2)\urcorner$ Offspring: $\ulcorner\sim S_1\urcorner$ $\ulcorner\sim S_2\urcorner$
Disjunction	Given: $\ulcorner\sim(S_1 \vee S_2)\urcorner$ Offspring: $\ulcorner\sim S_1\urcorner$ $\ulcorner\sim S_2\urcorner$	Given: $\ulcorner S_1 \vee S_2\urcorner$ Offspring: S_1 S_2
Conditional	Given: $\ulcorner\sim(S_1 \supset S_2)\urcorner$ Offspring: S_1 $\ulcorner\sim S_2\urcorner$	Given: $\ulcorner S_1 \supset S_2\urcorner$ Offspring: $\ulcorner\sim S_1\urcorner$ S_2

1.3 Definitions

A *basic sentence* is a sentence which is either elementary or the negation of an elementary sentence.

Two basic sentences are said to be *opposites* of each other if one is elementary and the other its negation.

A path through a tree is *finished* iff every nonbasic sentence in that path has been handled (has had the appropriate tree rule applied to it).

A tree is *finished* iff every path in the tree is finished.

A path is *closed* iff it contains an elementary sentence s and also its opposite $\sim s$ (in which case we put a cross 'X' under it).

On occasion, it will be convenient to put an '0' under a finished open path. In practice, of course, we terminate a path with 'X' as soon as s and $\sim s$ appear (or, given MThl, S and $\sim S$). But in some of the proofs to follow, where one finished tree is tacked onto the end of an open path of another finished tree, it is necessary to assume that a closing path is finished in the sense defined, before termination.

A tree is *closed* iff every path through that tree is closed.

A set K of sentences is *syntactically inconsistent* iff the tree under K is closed. (Strictly speaking, that should be: *a* tree under K. It does not matter, since every properly constructed tree will close if one such tree does.)

A sentence S is *refutable* iff S is syntactically inconsistent (i.e., iff the tree under S closes).

A sentence S is *provable* (i.e., $\vdash S$) iff $\sim S$ is refutable.

$S_1 \ldots S_n$ *syntactically imply* S (i.e., $S_1, \ldots S_n \vdash S$) iff the set $\{S_1, \ldots S_n, \ulcorner \sim S \urcorner\}$ is syntactically inconsistent.

2 Syntactical metatheorems

MThl (The General Closing Law) *For any sentence S, if both S and $\sim S$ occur in a finished path $\mathcal{P}$, then that path closes (i.e., there is an elementary sentence s, such that both s and $\sim s$ occur in $\mathcal{P}$).*

Prove by induction on the length of S.

Normally, the closing conditions are given as what is stated in MThl. Yet I believe that in the interests of articulateness, it is preferable to give narrow closing conditions, leaving it to the theorem to establish the wider closing conditions. Moreover, in many of the proofs to follow, we have to assume that where closing occurs, it is due to the clash of two opposite *basic* sentences.

With MThl we may now offer an

Example: We show that: $S_1 \supset S_2 \vdash (S_2 \supset S_3) \supset (S_1 \supset S_3)$
Premise (1) $S_1 \supset S_2$
Neg. of Concl. (2) $\sim[(S_2 \supset S_3) \supset (S_1 \supset S_3)]$
Stack (2) (3) $S_2 \supset S_3$
(4) $\sim(S_1 \supset S_3)$
Stack (4) S_1
$\sim S_3$

All paths close in virtue of MThl. So, the relevant tree closes and the implication holds.

MTh2 (The Deduction Theorem) *If $S_1, \ldots S_n \vdash S$ then and only then $S_1, \ldots S_{n-1} \vdash S_n \supset S$.*

Proof: The two trees:

$$\text{[a]}\quad\begin{array}{c} S_1 \\ \vdots \\ S_n \\ \sim S \\ \text{etc.} \end{array} \qquad\qquad \text{[b]}\quad\begin{array}{c} S_1 \\ \vdots \\ S_{n-1} \\ \sim[S_n \supset S] \\ S_n \\ \sim S \\ \text{etc.} \end{array}$$

close or stay open together.[2]

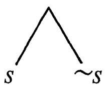
MTh3 $\vdash(S_1 \supset S_2) \supset [(S_2 \supset S_3) \supset (S_1 \supset S_3)]$.

Proof: From the Example and the Deduction Theorem.

From MTh3 and Proposition A, B is quite easily derivable, using the Deduction Theorem. (A is easily derivable from B using the Deduction Theorem and the lemma: $\vdash \sim S_2 \supset S_2$ iff $\vdash S_2$, which in turn is certified by trees.)

All the subsequent work aims at getting the General Forking Law (MTh11, below) which is interesting in its own right and is the essential lemma for getting Propositions A and B.

MTh4 (The Special Forking Law)

(a) *If s is any elementary sentence, then, if the fork*  *is added to the end of any finished open path $\mathcal{P}$, then at least one of the augmented paths is open.* For, if s is incompatible with some basic sentence of $\mathcal{P}$, then $\sim s$ is compatible with every basic sentence of $\mathcal{P}$, otherwise there would be incompatible basic sentences within $\mathcal{P}$, contrary to the assumption that $\mathcal{P}$ is open.

(b) *Further, if neither s nor ~s occurs in $\mathcal{P}$, then both the augmented paths are open.*

Definition Given a set K of sentences, a *MaxConBasic set* for K is a set C of sentences such that

(a) for every elementary sentence s in a member of K, either s or $\sim s$ is a member of C (maximal)
(b) but not both (consistent)
(c) no other sentences are members of C.

Note: Each MaxConBasic set for K is incompatible with every other distinct MaxConBasic set for K.

Definition A *fully forked finished tree* (a FFF tree) is a tree each of whose open paths is successively forked with those elementary sentences of the initial sentences that occur neither in themselves nor in their negations in the initial open path.

Note: It follows that the basic sentences of an open path of a FFF tree under a sentence S constitute a MaxConBasic set for S.

MTh5 (The Upward and Downward Syntactical Implication Theorem) *If C is a MaxConBasic set for S_1 then $C \vdash S_1$ if and only if $C \vdash S$ for every S in some twig that is the result of applying the appropriate rule to S_1.*

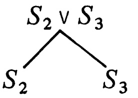
Proof: By cases.

Case 1. $S_1 = \sim\sim S_2$ for some S_2. Then, according to the tree rule we have $\left\{\begin{array}{l}\sim\sim S_2 \\ \quad S_2\end{array}\right.$ with S_2 as the twig. Consider the trees:

$$\begin{array}{rcrc} & C & & C \\ [a] & \sim[\sim\sim S_2] & [b] & \sim[S_2] \\ & \sim S_2 & & \end{array}$$

These trees have the same closing conditions. So, $C \vdash \sim\sim S_2$ (i.e., S_1) if and only if $C \vdash S_2$.

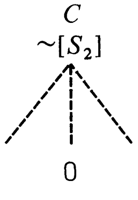
Case 2. $S_1 = S_2 \vee S_3$ for some S_2, S_3. Then by the tree rule we will have S_1 branching to yield two twigs:



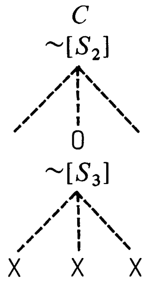
Then (if we suppose that $C \vdash S_1$) the following tree closes:

$$\begin{array}{rc} & C \\ [a] & \sim[S_2 \vee S_3] \\ & \sim S_2 \\ & \sim S_3 \\ & \vdots \\ & \mathrm{X} \end{array}$$

Now suppose: Not ($C \vdash S_2$). That is, the following FFF tree remains open:



Place $\sim[S_3]$ and its FFF tree at the end of any open path. From [a] we know that every path closes.



Suppose that in the case of one closing path, the incompatibility among the basic sentences arises between the $\sim[S_3]$ part of the path and the $\sim[S_2]$ part of the path. But then an inconsistency would have arisen between the $\sim[S_3]$ part of the path and C. (For, if s in the $\sim[S_3]$ part clashes with its opposite, s_1, in the $\sim[S_2]$ part, then it would also clash with s_1 in C, for the path being open prior to the $[S_3]$ part, C must contain s_1 rather than s.) Thus the tree under $\left\{\begin{matrix} C \\ \sim[S_3] \end{matrix}\right.$ closes and $C \vdash S_3$. So $C \vdash S$ for every S in some twig yielded by S_1.

Conversely, if $C \vdash S_3$ then the tree under $\left\{\begin{matrix} C \\ \sim[S_3] \end{matrix}\right.$ closes. But then so does the following tree:

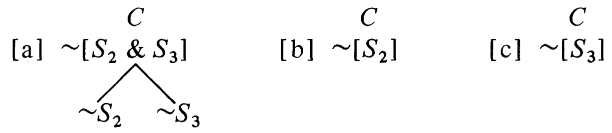
$$\begin{matrix} C \\ \sim[S_2 \vee S_3] \\ \sim S_2 \\ \sim S_3 \end{matrix}$$

and $C \vdash S_2 \vee S_3$. Similarly, if we suppose that $C \vdash S_2$. The cases for the other branching rules are essentially the same.

Case 3. $S_1 = S_2 \,\&\, S_3$ for some S_2, S_3. By the tree rule we have the stack:

$$\begin{matrix} S_2 \,\&\, S_3 \\ S_2 \\ S_3 \end{matrix}$$

Now consider the trees:



Obviously, [a] closes if and only if both [b] and [c] close. So, $C \vdash S_2 \,\&\, S_3$ if and only if $C \vdash S_2$ and $C \vdash S_3$. The case for the other stacking rules is essentially the same.

MTh6[3] *If C is a MaxConBasic set for K and $C \vdash S$ for every S in K, then there is an open path $\mathcal{P}$ through the FFF tree under K, such that $C \vdash S$ for every S in $\mathcal{P}$.*

Proof: By strong induction on n, the number of nonbasic sentences in the FFF tree under K.

Basis. There are no nonbasic sentences: the members of K are all basic, $K = C$, and the theorem holds trivially.

Step. We assume as induction hypothesis that the theorem holds for every FFF tree with fewer than n nonbasic sentences. Consider the tree under K and the fork or stack immediately under K—obtained by applying a tree rule to one of the members of K, S_1, say. By the antecedent of the theorem, $C \vdash S_1$, and so, by MTh5, $C \vdash S$ for every S in some branch (or the stack). Now delete S_1 and the other fork (if any) and its branches. We are left with one new initial sentence (if S_1 called for branching) or two new initial sentences (if S_1 called for stacking) in a tree with fewer than n nonbasic sentences. So, by the induction hypothesis, $C \vdash S$ for every S in some open path $\mathcal{P}$ in the diminished tree. Now reintroduce S_1. Then $\mathcal{P}$, extended to include S_1, is a path in the tree under K, such that $C \vdash S$ for every S in that path. Hence the theorem.

MTh7 *If C is a MaxConBasic set for S and* $C \vdash S$, *then there is an open path through the FFF tree under S whose basic sentences are precisely those of C.*

For, from MTh6 and the antecedent of the theorem, there is an open path $\mathcal{P}$ through the FFF tree under S such that $C \vdash S$ for every sentence of $\mathcal{P}$, and a fortiori, for every basic sentence s of $\mathcal{P}$. From the definition of a FFF tree, the set K of basic sentences of $\mathcal{P}$ is a MaxConBasic set for S.

Let s be a member of K. Then $C \vdash s$, which is to say, the tree

$$[\mathrm{a}]\quad \begin{array}{c} C \\ {\sim}[s] \end{array}$$

closes. Since C is MaxConBasic, [a] closes only if s is a member of C.

Let s be a member of C. Suppose its opposite, s_1, is a member of K. Then, from above, $C \vdash s_1$. But in that case s_1 must be a member of C, contrary to the fact that C is consistent. So, $s_1 \notin K$. But since K is MaxConBasic for S, K contains either s or s_1; so, lacking s_1, it must contain s.

Hence, $s \in K$ if and only if $s \in C$; and $K = C$.

MTh8 *If C is a MaxConBasic set for S, and there is no open path through the FFF tree under S whose basic sentences are precisely those of C, then* $C \vdash {\sim}S$.

For, consider the FFF tree under:

$$[\mathrm{a}]\quad \begin{array}{c} C \\ {\sim}[{\sim}S] \\ S \end{array}$$

C will clash with some basic sentence in every otherwise open path in the FFF tree under S in isolation. So, every path in [a] will close. Whence $C \vdash {\sim}S$.

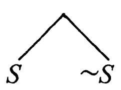
MTh9 *If C is a MaxConBasic set for S and there is no path through the FFF tree under S whose basic sentences are precisely those of C, then C is identical with the set of basic sentences in some open path in the FFF tree under* ${\sim}S$.

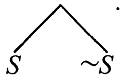
Directly from MTh8 and MTh7.

MTh10 *If there are n elementary sentences occurring in S, then there are* 2^n *MaxConBasic sets for S and each one is identical with the set of basic sentences in some open path in the FFF tree under* $S \vee {\sim}S$.

Proof: From MTh9 and the tree rule for disjunctions.

MTh11 (The General Forking Law) *If the end of a finished open path* $\mathcal{P}$ *of any tree is forked with* *for any S and then finished, then there is at least one open path beneath the fork.*



Proof: We shall suppose that the tree in question is FFF before the fork is added to an open path. We also suppose that what is added is the FFF tree under 

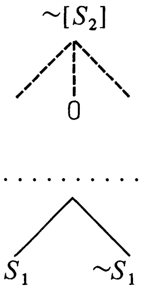
. Let S contain n elementary sentences. The general idea is that since the set of basic sentences in the path above the fork (i.e., $\mathcal{P}$) is self-compatible it must be compatible with at least one of the 2^n MaxConBasic sets for S represented in the path segments beneath the fork. Specifically, let us consider a path through the whole tree, a path which passes through a self-consistent sub-path below the fork. Suppose this path, considered as a path through the whole tree, is closed. Since the path was open down to the fork, there is no contradiction among the basic sentences above the fork. So, the closing of the whole path must be due to the clash of basic sentences $s_{1_1}, \ldots s_{1_i}$ above the fork with their opposites $s_{2_1}, \ldots s_{2_i}$ below the fork. But by MTh10, there is another path containing $s_{1_1}, \ldots s_{1_i}, s_{2_{i+1}}, \ldots s_{2_n}$ below the fork. This whole path will be free of contradiction, hence open.

MTh12 *If* $\vdash S_1 \supset S_2$ *and* $\vdash S_1$, *then* $\vdash S_2$.

From the antecedent, the following trees close:

$$[a] \quad \begin{matrix} \sim[S_1 \supset S_2] \\ S_1 \\ \sim S_2 \end{matrix} \qquad\qquad [b] \quad \sim[S_1]$$

Consider the tree under $\sim[S_2]$. Suppose this tree has an open path, $\mathcal{P}$. Then fork it with S_1 and $\sim S_1$. By the General Forking Law the resulting tree must be open. But from [a] we see that the left branch closes and from [b] we see that the right branch closes. So the tree under $\sim[S_2]$ by itself must be closed. Hence $\vdash S_2$.

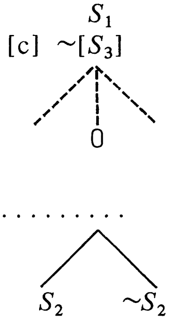


MTh13 *If* $S_1 \vdash S_2$ *and* $S_2 \vdash S_3$, *then* $S_1 \vdash S_3$.

From the antecedent, the following trees both close:

$$[a] \quad \begin{matrix} S_1 \\ \sim[S_2] \end{matrix} \qquad\qquad [b] \quad \begin{matrix} S_2 \\ \sim[S_3] \end{matrix}$$

Consider the tree [c] and suppose it to be open. Then fork an open path with S_2 and $\sim S_2$. By the General Forking Law there must be at least one open path through the finished tree. But by [a] we see that the right-hand branch closes and by [b] that the left branch closes. So the tree $\left\{\begin{matrix} S_1 \\ \sim[S_3] \end{matrix}\right.$ could not have been open. Hence $S_1 \vdash S_3$.



These last two theorems are the sought-after Propositions A and B. The fact that it should be so difficult to prove them for tree logic says something about tree logic.

NOTES

1. See, e.g., van Fraassen, [3], p. 195, question 3.2, and the solution, p. 209; also, Jeffrey, [2], p. 99, question 5.7.
2. The ease of proving the Deduction Theorem in tree logic, together with the difficulty of proving A and B, illustrate the principle: what you make on the peanuts you lose on the popcorn.
3. This proof is adapted from Jeffrey's semantic proof of his 5.5 in [2], p. 90.

REFERENCES

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[2] Jeffrey, R. C., *Formal Logic: Its Scope and Limits,* McGraw-Hill, New York, 1967.

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