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A Note on Intuitionistic Models of ZF

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An analysis of the forcing construction of models of *ZF* by means of intuitionistic logic and Kripke semantics (or sheaf semantics) has been made by many authors (see for example [1], [2], [5]). We feel however that some basic information is still available from this approach, the motto of which could be "think of a sheaf of structures as *a* structure, using intuitionistic logic". In this note, we are not really interested in obtaining independence results, but want to sketch a set-theoretic forcing along lines which closely agree with that motto. In usual forcing constructions, there are two sheaf structures:

$$\langle M, - \epsilon_p - \rangle_{p \epsilon C}$$

and

$$\langle M, p \Vdash - \epsilon - \rangle_{p \epsilon C},$$

and passing from the first to the second is nothing else than a sheaf-version of the extensionalization of a binary relation. This is discussed in Section 2 for Kripke-structures but the proofs are designed so as to immediately extend to sheaf structures as is indicated in Section 4. Application to the construction of models of *ZF* is given in Section 3. To this we add definitions and notions which bring the construction closer to the classical construction of inner models of *ZF* and, as a "test-case", indicate how to adapt the well-known proof of relative consistency of $V \neq L$.

Roughly speaking, we may distinguish between two approaches to forcing. In the first, the model, be it Boolean-valued or intuitionistic, is constructed from the base model M as a union of a sequence of M_α indexed by ordinals of M ([1], [2], [4], [5]). In the second, no hierarchy is present, at least to begin with, and the hierarchy effect remains concentrated in the various forms of induction [7]. Our approach is of the second kind and may be viewed as an intuitionistic version of [7], whereas [2] for example is an intuitionistic construction of the first kind. In fact, this distinction is not so clear-cut and the

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reader will notice that our presentation sometimes parallels [2]. For example, no new ordinals are added in the construction (Proposition 7, Section 3, compared with [2], Ch. 9, Section 6).

In contrast with these approaches, the sheaf and topos-theoretic methods have inspired our Section 4. We do not develop the subject very much, but enough, we think, to suggest comparisons in that direction.

1 ***Preliminaries*** All languages to be considered here are first-order languages which have only the binary predicate symbols to be denoted by 'ϵ', 'E', 'E_1', . . . according to the intended interpretation. They do not contain symbols for equality and '$\neg$', '$\wedge$', '$\rightarrow$', '$\exists$' are their only logical symbols. The use of '$\forall x$' is a mere abbreviation for '$\neg\exists x\neg$' so that notations like '$\forall z \in y\phi$' abbreviate '$\neg\exists z\neg(x \in y \rightarrow \phi)$'. '$\exists z \in y\phi$', '$\phi \vee \psi$', '$\phi \longleftrightarrow \psi$' are also considered as abbreviations. It is important, however, when we come to semantics to consider '$\rightarrow$' as *not* defined from '$\neg$' and '$\wedge$'. Languages will be denoted by $L(\epsilon)$, $L(E)$, $L(E, E_1, \ldots)$, . . . according to their list of typical symbols.

The following definitions are given for a language $L(E)$. The generalization to other languages is immediate.

A theory T (in $L(E)$) is a set of formulas of $L(E)$. We denote by '$T \vdash_i \phi$' (respectively $T \vdash_c \phi$) the fact that ϕ is an intuitionistic (respectively classical) consequence of T. Recall that since we do *not* have universal quantifiers in $L(E)$, the double negation theorem holds: if $T \vdash_c \phi$, then $T \vdash_i \neg\neg\phi$. In the absence of a symbol for equality, the axioms of equality and extensionality, written in $L(E)$, reduce to

$$\exists zEy\, z \underset{E}{\simeq} x \rightarrow xEy,$$

where $x \underset{E}{\simeq} y$ abbreviates $\forall t(tEx \longleftrightarrow tEy)$; we refer to the above formula as (E-Ext). The other axioms of ZF will be taken from [4], pp. 55 ff under the following form and with the usual restrictions on the variables of ϕ:

(E-Found)	$\forall y'Ex\, \exists yEx\, \forall zEy\, \neg zEx$
(E-Pair)	$\exists z(xEz \wedge yEz)$
(E-Union)	$\exists y\, \forall uEx\, \forall vEu\, vEy$
(E-Subset ϕ)	$\exists y\, \forall uEx(\exists v\, \phi \rightarrow \exists vEy\, \phi)$
(E-Inf)	$\exists y(\exists uEy \wedge \forall vEy\, \exists wEy\, vEw)$
(E-Comp ϕ)	$\exists y(\forall uEy(uEx \wedge \phi) \wedge \forall uEx(\phi \rightarrow uEy))$
(E-Subset)	$\exists y\, \forall u(\forall vEu\, vEx \rightarrow uEy)$.

Let $\langle C, \leqslant\rangle$ be a partially ordered set; the elements of C are denoted by 'p', 'q', 'r', . . . A structure for $L(E)$ over $\langle C, \leqslant\rangle$ is an ordered pair $\mathcal{M} = \langle D^{\mathcal{M}}, E^{\mathcal{M}}\rangle$ where $D^{\mathcal{M}} = (D_p^{\mathcal{M}})_{p\in C}$, $E^{\mathcal{M}} = (E_p^{\mathcal{M}})_{p\in C}$ and for every $p, q \in C$, $E_p^{\mathcal{M}} \subseteq (D_p^{\mathcal{M}})^2$ and

$$q \leqslant p \rightarrow D_p^{\mathcal{M}} \subseteq D_q^{\mathcal{M}} \text{ and } E_p^{\mathcal{M}} \subseteq E_q^{\mathcal{M}}.$$

Formulas of $L(E)$ are interpreted in $\mathcal{M}$ by "forcing"; i.e., we define '$\mathcal{M} \Vdash_p \phi(\bar{x})\, [\bar{a}]$' (for $p \in C$, $\bar{a}$ is a sequence of elements of $D_p^{\mathcal{M}}$ whose length matches that of $\bar{x}$) by induction on the form of ϕ:

(0) $\mathcal{M} \Vdash_p xEy\, [a, b]$ iff $\langle a, b\rangle \in E_p^{\mathcal{M}}$

(1) $\mathcal{M} \Vdash_p (\neg\psi)(\bar{x})\, [\bar{a}]$ iff $\forall q \leqslant p\ \mathcal{M} \not\Vdash_q \psi(\bar{x})\, [\bar{a}]$,

(2) $\mathcal{M} \Vdash_p (\psi \wedge \chi)(\bar{x})\ [\bar{a}]$ iff $\mathcal{M} \Vdash_p \psi(\bar{x})\ [\bar{a}]$ and $\mathcal{M} \Vdash_p \chi(\bar{x})\ [\bar{a}]$
(3) $\mathcal{M} \Vdash_p (\psi \rightarrow \chi)(\bar{x})\ [\bar{a}]$ iff $\forall q \leqslant p\ (\mathcal{M} \Vdash_q \psi(\bar{x})\ [\bar{a}]$ implies $\mathcal{M} \Vdash_q \chi(\bar{x})\ [\bar{a}])$
(4) $\mathcal{M} \Vdash_p (\exists x \psi)(\bar{x})\ [\bar{a}]$ iff $\exists a \in D_p^{\mathcal{M}}\ \mathcal{M} \Vdash_p \psi(x, \bar{x})\ [a, \bar{a}]$.

From this we define for formulas $\phi(\bar{x})$:

$$\mathcal{M} \Vdash_p \phi(\bar{x}) \text{ iff } \forall \bar{a} \in D_p^{\mathcal{M}}\ \mathcal{M} \Vdash_p \phi(\bar{x})\ [\bar{a}]$$

and $\mathcal{M} \Vdash \phi(\bar{x})$ iff $\forall p\ \mathcal{M} \Vdash_p \phi(\bar{x})$. If T is a theory in $L(E)$, "$\mathcal{M}$ is a model of T" is defined by:

$$\mathcal{M} \Vdash T \text{ iff for every } \phi(\bar{x}) \text{ in } T,\ \mathcal{M} \Vdash \phi(\bar{x}).$$

When there is no risk of confusion, we use simplified notations like:

$$\mathcal{M} \Vdash_p \phi[\bar{a}].$$

The following lemmas are easily established and will often be used without explicit reference:

Extension lemma *If $q \leqslant p$ and $\mathcal{M} \Vdash_p \phi[\bar{a}]$, then $\mathcal{M} \Vdash_q \phi[\bar{a}]$.*

Validity lemma *If $T \vdash_i \phi$ and $\mathcal{M} \Vdash T$, then $\mathcal{M} \Vdash \phi$.*

*2 **Extensionalization of structures*** In this section we fix a partially ordered set $\langle C, \leqslant\rangle$. All structures to be considered are structures for different languages over $\langle C, \leqslant\rangle$; notations like $(\mathcal{M}, E)$, $(\mathcal{M}, E, E_1, \ldots)$ will put in evidence the typical symbols of the underlying languages. All our considerations are carried out "naively" within a background ZF-set theory; adaptation to the construction of models of ZF will be considered in Section 3.

Definition 1 $(\mathcal{M}, E)$ is *ranked* if for every $p \in C$, $\langle a, b\rangle \in E_p^{\mathcal{M}} \rightarrow \text{rank}(a) < \text{rank}(b)$.

Definition 2 Let $\phi(x, \bar{z})$ be a formula in $L(E, E_1, \ldots)$. The *axiom of $E, E_1, \ldots$-induction* for ϕ is the formula ($E, E_1, \ldots$-Ind ϕ): $\forall x(\forall y Ex\ \phi(y, \bar{z}) \rightarrow \phi(x, \bar{z})) \rightarrow \forall x\ \phi(x, \bar{z})$. The *schema of $E, E_1, \ldots$-induction* ($E, E_1, \ldots$-Ind) is the set of all such formulas for ϕ in $L(E, E_1, \ldots)$.

Another intuitionistically equivalent form of ($E, E_1, \ldots$-Ind) is the schema:

$$\exists x\ \psi(x, \bar{z}) \rightarrow \neg\neg\exists x(\psi(x, \bar{z}) \wedge \neg\exists y Ex\ \psi(y, \bar{z})).$$

This form is well adapted to forcing computations as in the following proposition:

Proposition 1 *If $(\mathcal{M}, E)$ is ranked, then every expansion $(\mathcal{M}, E, E_1, \ldots)$ of $(\mathcal{M}, E)$ is a model of ($E, E_1, \ldots$-Ind).*

Proof: The above remark and usual forcing computations lead us to choose in a nonempty set $B = \{b \mid \exists r \leqslant q\ (b \in D_r^{\mathcal{M}} \wedge \mathcal{M} \Vdash_r \psi(b, \bar{c}))\}$ an element b_0 of minimal rank which will be used to prove the thesis.

More generally, every expansion of a ranked structure is a model of $E, E_1, \ldots$-induction in n variables. This results from the following lemma:

Lemma 2 *For every formula $\phi(x_1, \ldots, x_n, \bar{z})$ of $L(E, E_1, \ldots)$,*

$$(E, E_1, \ldots\text{-Ind}) \vdash_i \psi,$$

where ψ is the following formula:

$$\forall x_1 \ldots \forall x_n (\forall y_1 E x_1 \ldots \forall y_n E x_n\, \phi(y_1, \ldots, y_n, \bar{z}) \to \phi(x_1, \ldots, x_n, \bar{z})) \\ \to \forall x_1 \ldots \forall x_n\, \phi(x_1, \ldots, x_n, \bar{z}).$$

Proof: The following method of proof will be repeatedly used in the sequel:

$$(E, E_1, \ldots\text{-Ind}) \vdash_c \psi;$$

therefore, by the double negation theorem:

$$(E, E_1, \ldots\text{-Ind}) \vdash_i \neg\neg\psi;$$

but

$$\vdash_i \neg\neg\psi \to \psi$$

(recall that '$\forall x$' stands for '$\neg\exists x\neg$'); therefore

$$(E, E_1, \ldots\text{-Ind}) \vdash_i \psi.$$

Definition 3 $(\mathcal{M}, E)$ is *extensional* if $(\mathcal{M}, E)$ is a model of (E-Ext). $(\mathcal{M}, R)$ is an *extension* of $(\mathcal{M}, E)$ if

$$(\mathcal{M}, E, R) \Vdash xEy \to xRy,$$

and an *end extension* of $(\mathcal{M}, E)$ if moreover

$$(\mathcal{M}, E, R) \Vdash xRy \to \exists z E y\, z \underset{R}{\sim} x.$$

Proposition 3 *If $(\mathcal{M}, E)$ is ranked, there is at most one extensional end extension of $(\mathcal{M}, E)$.*

Proof: Let $(\mathcal{M}, R)$ and $(\mathcal{M}, S)$ be two such extensions. We consider in $L(E, R, S)$ the theory T formed by the schema (E, R, S-Ind) and formulas (1)-(6):

(1) $xEy \to xRy$,
(2) $xRy \to \exists zEy\, z \underset{R}{\sim} x$,
(3) $xRy \longleftrightarrow \exists zRy\, z \underset{R}{\sim} x$,
(4) $xEy \to xSy$,
(5) $xSy \to \exists zEy\, z \underset{S}{\sim} x$,
(6) $xSy \longleftrightarrow \exists zSy\, z \underset{S}{\sim} x$.

A classical proof by "E-induction" will show that

$$T \vdash_c x \underset{R}{\sim} y \longleftrightarrow x \underset{S}{\sim} y,$$

from which, by the double negation technique used in Lemma 2:

$$T \vdash_i x \underset{R}{\sim} y \longleftrightarrow x \underset{S}{\sim} y.$$

It is then easy to prove $T \vdash_i xRy \to xSy$; taking xRy as a hypothesis, one obtains:

$\exists zEy\, z \underset{R}{\sim} x$	(by (2))
$\exists zEy\, z \underset{S}{\sim} y$	(by the preceding result)
$\exists zSy\, z \underset{S}{\sim} y$	(by (4))
xSy	(by (6)).

Similarly, $T \vdash_i xSy \to xRy$. Finally,

$$T \vdash_i xRy \longleftrightarrow xSy,$$

and the proposition follows from the validity lemma.

Definition 4 The *axiom of E, R extensionalization* is the formula

$$(E, R\text{-Ext}) : xRy \longleftrightarrow \exists zEy\ z \underset{E}{\overset{R}{\sim}} x,$$

where $z \underset{E}{\overset{R}{\sim}} x$ abbreviates $\forall tEz\ tRx \wedge \forall tEx\ tRz$.

Proposition 4 *If* $(\mathcal{M}, E)$ *is ranked, then every extensional end extension* $(\mathcal{M}, R)$ *of* $(\mathcal{M}, E)$ *is a model of* $(E, R\text{-Ext})$.

Proof: Similar to the proof of Proposition 3, using for T the theory constituted by the schema $(E, R\text{-Ind})$ and formulas (1)–(3) of Proposition 3. The main steps are:

$$T \vdash_c x \underset{E}{\overset{R}{\sim}} y \longleftrightarrow x \underset{R}{\sim} y,\ T \vdash_i x \underset{E}{\overset{R}{\sim}} y \longleftrightarrow x \underset{R}{\sim} y,\ T \vdash_i (E, R\text{-Ext})$$

and the thesis follows from the validity lemma.

Proposition 5 *If* $(\mathcal{M}, E)$ *is ranked, there exists an extensional end extension* $(\mathcal{M}, E^*)$ *of* $(\mathcal{M}, E)$.

Proof: By Proposition 4, such an extension is necessarily a model of $(E, E^*\text{-Ext})$. This means that for every $p \in C$, $a, b \in D_p^{\mathcal{M}}$,

$$(\mathcal{M}, E, E^*) \Vdash_p aE^*b$$

iff

$$\begin{aligned}&\forall q \leqslant p\ \forall d \in D_q^{\mathcal{M}}\ (\langle d, a\rangle \in E_q^{\mathcal{M}} \to \exists r \leqslant q\ \mathcal{M} \Vdash_r dE^*b)\\ &\wedge \forall q \leqslant p\ \forall d \in D_q^{\mathcal{M}}\ (\langle d, b\rangle \in E_q^{\mathcal{M}} \to \exists r \leqslant q\ \mathcal{M} \Vdash_r dE^*a).\end{aligned}$$

The usual rank-argument shows that there exists a formula $\psi(x, y, z)$ of the background language of ZF such that for every $p \in C$,

$$\psi(a, b, p) \longleftrightarrow a \in D_p^{\mathcal{M}} \wedge b \in D_p^{\mathcal{M}} \wedge (\mathcal{M}, E, E^*) \Vdash_p aE^*b;$$

thus, defining

$$E_p^{*\mathcal{M}} = \{\langle a, b\rangle |\ \psi(a, b, p)\}$$

will give the required structure $(\mathcal{M}, E^*)$. To see that this is indeed an extensional end extension of $(\mathcal{M}, E)$, we form the theory T constituted by the schema $(E, E^*\text{-Ind})$ and $(E, E^*\text{-Ext})$ and prove successively:

$$T \vdash_i x \underset{E}{\overset{E^*}{\sim}} x$$

$$T \vdash_i x \underset{E}{\overset{E^*}{\sim}} y \to y \underset{E}{\overset{E^*}{\sim}} x$$

$$T \vdash_i x \underset{E}{\overset{E^*}{\sim}} y \wedge y \underset{E}{\overset{E^*}{\sim}} z \to x \underset{E}{\overset{E^*}{\sim}} z$$

$$T \vdash_i xEy \to xE^*y$$

$$T \vdash_i x \underset{E}{\overset{E^*}{\sim}} y \longleftrightarrow x \underset{E^*}{\sim} y$$

$T \vdash_i xE^*y \to \exists zEy\ z \underset{E^*}{\simeq} x$

$T \vdash_i (E^*\text{-Ext})$.

We now know by Propositions 3 and 5 that, if $(\mathcal{M}, E)$ is ranked, there exists a unique extensional end extension of $(\mathcal{M}, E)$, to be denoted hereafter by $(\mathcal{M}, E^*)$. To complete our study of extensionalization, we should look for formulas preserved when passing from $(\mathcal{M}, E)$ to $(\mathcal{M}, E^*)$: since $(\mathcal{M}, E^*)$ is essentially a "quotient" of $(\mathcal{M}, E)$ (see remark at the end of this paragraph) it is not surprising that "positive" formulas are preserved.

Definition 5 If ϕ is a formula of $L(E, E^*)$, ϕ^* is the formula (of $L(E^*)$) obtained by replacing in ϕ every occurrence of E by E^*.

It will spare notation to define T^* as follows:

Definition 6 T^* is the theory in $L(E, E^*)$ formed by the schema $(E, E^*\text{-Ind})$ and the axiom $(E, E^*\text{-Ext})$.

Definition 7 A formula $\phi(x, \bar{z})$ of $L(E^*)$ is *substitutive in* x if $T^* \vdash_i x \underset{E}{\simeq} y \wedge \psi(x, \bar{z}) \to \psi(y, \bar{z})$.

Proposition 6 *For any formula ϕ of $L(E^*)$ and any variable x, $\neg\phi$ is substitutive in x.*

Proof: Prove $T^* \vdash_i \exists x\,(x \underset{E^*}{\simeq} y \wedge \neg\psi\,(x, \bar{z})) \longleftrightarrow \neg\psi(y, \bar{z}))$ by induction on the form of ψ.

Definition 8 In $L(E, E^*)$, $Pos(E)$ and $Neg(E)$ formulas are simultaneously defined by induction:

(1) formulas of $L(E^*)$ are $Pos(E)$ and $Neg(E)$
(2) xEy is $Pos(E)$
(3) if ϕ is $Pos(E)$, $(\neg\phi)$ is $Neg(E)$
 if ϕ is $Neg(E)$, $(\neg\phi)$ is $Pos(E)$
(4) if ϕ and ψ are $Pos(E)$, $(\phi \wedge \psi)$ is $Pos(E)$
 if ϕ and ψ are $Neg(E)$, $(\phi \wedge \psi)$ is $Neg(E)$
(5) if ϕ is $Neg(E)$ and ψ is $Pos(E)$, $(\phi \to \psi)$ is $Pos(E)$
 if ϕ is $Pos(E)$ and ψ is $Neg(E)$, $(\phi \to \psi)$ is $Neg(E)$
(6) if ϕ is $Pos(E)$, $(\exists x\phi)$ is $Pos(E)$
 if ϕ is $Neg(E)$, $(\exists x\phi)$ is $Neg(E)$
(7) if ϕ is $Neg(E)$ and ϕ^* substitutive in x, $(\exists xEy\phi)$ is $Neg(E)$.

Note that if ϕ is $Pos(E)$, $(\forall xEy\ \phi)$ is, up to intuitionistic equivalence, a $Pos(E)$ formula (apply Proposition 6).

Proposition 7 *If ϕ is $Pos(E)$ and ψ is $Neg(E)$, then $T^* \vdash_i \phi \to \phi^*$ and $T^* \vdash_i \psi^* \to \psi$.*

Proof: A simultaneous induction on the form of $Pos(E)$ and $Neg(E)$ formulas.

In terms of models, this proposition means that $Pos(E)$ formulas of $L(E, E^*)$ can be transferred from $(\mathcal{M}, E, E^*)$ to $(\mathcal{M}, E^*)$ by "starring". In particular,

Proposition 8 *For any formula* ψ *of* $L(E^*)$, $T^* \vdash_i (E^*, E^*\text{-Ind } \psi)$; $T^* \vdash_i (E^*\text{-Found})$.

Proof: $(E^*, E^*\text{-Ind } \psi)$ is obtained by starring $(E, E^*\text{-Ind } \psi)$, a $Pos(E)$ formula, consequence of T^*. The formula (E^*-Found) (with z as a free variable) is intuitionistically equivalent to $(E^*, E^*\text{-Ind } (\neg x E^* z))$.

We summarize the results of this section as follows:

Theorem 1

(1) *If* $(\mathcal{M}, E)$ *is a ranked structure, there exists a unique extensional end extension* $(\mathcal{M}, E^*)$ *of* $(\mathcal{M}, E)$.
(2) *Such an extension is a model of the theory* T^* *formed by the schema* $(E, E^*\text{-Ind})$ *and the axiom* $(E, E^*\text{-Ext})$.
(3) $T^* \vdash_i \phi \to \phi^*$ *for any* $Pos(E)$ *formula* ϕ *of* $L(E, E^*)$.
(4) $T^* \vdash_i (E^*\text{-Ext})$.
(5) $T^* \vdash_i (E^*, E^*\text{-Ind})$.
(6) $T^* \vdash_i (E^*\text{-Found})$.

Remark: If $(\mathcal{M}, E)$ is not ranked, the existence of an extensional end extension $(\mathcal{M}, E^*)$ of $(\mathcal{M}, E)$ can still be proved by adapting for example the construction of [3]; a sequence of $(\mathcal{M}, E^\alpha)$ (α ordinal) is defined, letting for $p \in C$,

$$\langle a, b\rangle \in E_p^0 \longleftrightarrow \langle a, b\rangle \in E_p$$
$$\langle a, b\rangle \in E_p^{\alpha+1} \longleftrightarrow \exists c(\langle c, b\rangle \in E_p^\alpha \wedge \forall q \leqslant p\ \forall d \in D_q^{\mathcal{M}}(\langle d, c\rangle \in E_p^\alpha \longleftrightarrow \langle d, a\rangle \in E_p^\alpha)$$
$$\langle a, b\rangle \in E_p^\lambda \longleftrightarrow \exists \alpha < \lambda\ \langle a, b\rangle \in E_p^\alpha \ (\lambda \text{ limit});$$

since $\alpha < \beta$ implies $E_p^\alpha \subseteq E_p^\beta$, a cardinality argument shows that for some δ, $(\mathcal{M}, E^\delta)$ is identical with $(\mathcal{M}, E^{\delta+1})$; $(\mathcal{M}, E^\delta)$ is an extensional end extension of $(\mathcal{M}, E)$. Such an extension is not necessarily unique (see [4] for classical structures), and in general uniqueness clearly depends on some "well-founded" character of $(\mathcal{M}, E)$: we proved it for ranked structures. It can also be proved for "well-founded" structures: $(\mathcal{M}, E)$ is *well-founded* if there is no family $\langle a_n, p_n\rangle_{n<\omega}$ such that for every $n < \omega$, $p_{n+1} \leqslant p_n$, $a_n \in D_{p_n}^{\mathcal{M}}$ and $\langle a_{n+1}, a_n\rangle \in E_{p_{n+1}}^{\mathcal{M}}$; well-founded structures in fact satisfy Proposition 1. Note also that "ranked" implies "well-founded". We did not take these lines of proof because they do not seem to carry over to "class-structures" as is required for the construction of models of *ZF*.

3 Application to the construction of models of ZF To construct models of *ZF* by "forcing", one usually fixes a (classical) transitive model M of *ZF* and a partially ordered set $\langle C, \leqslant\rangle$ in M. A (Kripke) structure $(\mathcal{M}, E)$ arises when defining for $p \in C$, $a, b \in M$

$$D_p^{\mathcal{M}} = M$$
$$\langle a, b\rangle \in E_p^{\mathcal{M}} \longleftrightarrow \exists q \geqslant p\ \langle a, q\rangle \in b$$

(we refer to this as Forcing (A); see for example [7]). In Heyting-valued versions, one fixes a complete Heyting algebra H in M; if H happens to be the Heyting algebra of all open subsets of $\langle C, \leqslant\rangle$ for the order-topology, a (Kripke) structure also arises with the following definitions:

$$D_p^{\mathfrak{M}} = \text{the Heyting-valued universe } M^H$$
$$\langle a,b\rangle \in E_p^{\mathfrak{M}} \longleftrightarrow a \in dom(b) \wedge p \in b(a)$$

(Forcing (B); see for example [5]). In this section we apply results of Section 2 and ideas underlying it to give sufficient conditions on a general $(\mathfrak{M}, E)$ to obtain models $(\mathfrak{M}, E^*)$ of $ZF(E^*)$. We also develop a few definitions which bring the whole construction closer to the (classical) construction of inner models of ZF. Our definitions and propositions are designed not only to cover Forcing (A) and Forcing (B) but to suggest generalizations to sheaves which will be briefly discussed in Section 4. We will omit generics since these are dispensable from the present point of view (see [2]).

We fix a (classical) transitive model M of ZF and a partially ordered set $\langle C, \leqslant\rangle$ of M. To apply Section 2, the construction of $(\mathfrak{M}, E^*)$ from $(\mathfrak{M}, E)$ should be done "inside" M. We therefore restrict our attention to M-definable structures:

Definition 1 $(\mathfrak{M}, E)$ is *M-definable* if there exist formulas $\delta(x, z, t)$ and $\varepsilon(x, y, z, t)$ of the language of ZF such that for any $p \in C$,

$$D_p^{\mathfrak{M}} = \{a | a \in M \text{ and } M \models \delta(a, p, C)\}$$
$$E_p^{\mathfrak{M}} = \{\langle a, b\rangle | a, b \in M \text{ and } M \models \varepsilon(a, b, p, C)\}.$$

$(\mathfrak{M}, E, E_1, \ldots)$ is *M-definable* if $(\mathfrak{M}, E)$, $(\mathfrak{M}, E_1)$, . . . are M-definable.

Using this definition, all results of Section 2 hold when "structure" is replaced by "M-definable structure". Thus, for example, Proposition 1 is proved by the same forcing computations but uses the fact that M is a (classical) transitive model of ZF; more precisely, although B (in the proof of Section 2, Proposition 1) is no longer an element of M, usual ϵ-induction applied in M will furnish the desired b_0 of minimal rank. The only other point which really needs adaptation is in the proof of Proposition 5; one needs the fact that M allows definitions by transfinite induction and from this the well-known "definability of forcing" will follow:

Definability lemma *If $(\mathfrak{M}, E)$ is M-definable, then $(\mathfrak{M}, E^*)$ is M-definable and more generally, to every formula ϕ of $L(E, E^*)$ one can associate a formula $\phi^{\#}$ of the language of ZF in such a way that for all $p \in C$, $\bar{a} \in D_p^{\mathfrak{M}}$,*

$$(\mathfrak{M}, E, E^*) \Vdash_{\overline{p}} \phi[\bar{a}] \textit{ iff } M \models \phi^{\#}(\bar{a}, p, C).$$

(In fact, the whole construction could be carried out "inside M", interpreting the validity lemma, the double negation theorem, etc. . . ., as instructions to compute forcing).

Summarizing, we may state:

Theorem 2 *If $(\mathfrak{M}, E)$ is a ranked M-definable structure, there exists a unique extensional M-definable end extension $(\mathfrak{M}, E^*)$ of $(\mathfrak{M}, E)$.*

In the case of Forcing (A) for example, $(\mathfrak{M}, E^*)$ is in more usual notations the family of structures $\langle M, p \Vdash - \epsilon -\rangle_{p \in C}$; Theorem 2 thus characterizes the so-called "forcing for atomic formulas".

From now on, we fix a ranked M-definable $(\mathfrak{M}, E)$. By Theorem 1 (4) and (6), $(\mathcal{M}, E^*)$ is a model of (E^*-Ext) and (E^*-Found). We now investigate when $(\mathcal{M}, E^*)$ is a model of the other axioms of $ZF\,(E^*)$.

Definition 2 $ZF\,(E, E^*)$ is the theory T^* together with the axioms (E-Pair), (E-Union), (E-Inf), (E-Subst ϕ), (E-Compr ϕ) for any formula ϕ of $L(E^*)$ and (Subsets): $\exists y \forall u\, (\forall v E u\, v E x \to \exists u_1 E y\, \forall w (w E u_1 \longleftrightarrow w E u))$.

Proposition 1 $ZF(E, E^*) \vdash_i ZF(E^*)$.

Proof: Apply Theorem 1, (4) and (6), to get (E^*-Ext) and (E^*-Found). Except for (E^*-Subset), every other axiom of $ZF\,(E, E^*)$ follows from the corresponding axiom of $ZF\,(E, E^*)$ by Theorem 1, (3). To prove (E^*-Subset) from $ZF\,(E, E^*)$, write (Subsets) as $\exists y \psi(y)$ and prove

$$ZF\,(E, E^*) \vdash_i \psi(y) \to \forall u (\forall v E u\, v E^* x \to u E^* y);$$

the usual double negation argument shows that a classical proof (using a comprehension axiom) suffices.

As a consequence of Proposition 1, a sufficient condition for $(\mathcal{M}, E^*)$ to be a model of $ZF\,(E^*)$ is that $(\mathcal{M}, E, E^*)$ be one of $ZF\,(E, E^*)$. This in turn will hold as soon as $(\mathcal{M}, E)$ is "large enough". It is also convenient (but not necessary) to assume from now on that $\langle C, \leqslant\rangle$ has a greatest element, to be denoted by 'o'. This allows us to talk of "global sections" of $(\mathcal{M}, E)$. (In case C has no greatest element, replace global sections by coherent families $(a_p)_{p \in C}$).

Definition 3 An *M-class* d of $(\mathcal{M}, E)$ is a family $(d_p)_{p \in C}$ such that

(1) $d \in M$
(2) $\forall p,q\; q \leqslant p \to d_p \subseteq d_q$
(3) $\forall p\; d_p \subseteq D_p^{\mathcal{M}}$.

A *global section* of $(\mathcal{M}, E)$ is an element of $D_0^{\mathcal{M}}$. $(\mathcal{M}, E)$ is *M-universal* if there exists a term $\sigma(x)$ (in an extension by definitions) of ZF such that for every M-class $d = (d_p)_{p \in C}$ of $(\mathcal{M}, E)$,

(1) $\sigma(d)$ is a global section of $(\mathcal{M}, E)$
(2) $\forall p (\langle a, \sigma(d)\rangle \in E_p^{\mathcal{M}} \longleftrightarrow a \in d_p)$.

Briefly stated, M-universality means that there are enough global sections to represent small classes. Note that structures $(\mathcal{M}, E)$ arising in the cases of Forcing (A) and (B) are clearly M-universal for the following definitions, given for an M-class $d = (d_p)_{p \in C}$.

Forcing (A): $\sigma(d) = \{\langle a, p\rangle | a \in d_p,\, p \in C\}$

Forcing (B): $\sigma(d)$ is characterized by

$$dom(\sigma(d)) = \cup\,\{d_p | p \in C\}$$

and for $a \in dom(\sigma(d))$;

$$\sigma(d)\,(a) = \{p | a \in d_p\}.$$

Proposition 2 *If $(\mathcal{M}, E)$ is M-universal, then $(\mathcal{M}, E, E^*)$ is a model of $ZF\,(E, E^*)$.*

Proof: To prove that $(\mathcal{M}, E, E^*)$ is a model of (E-Pair), let $p \in C$, $a, b \in D_p^{\mathcal{M}}$ and consider the M-class $d = (d_q)_{q \in C}$ defined by $d_q = \{a, b\}$ for $q \leqslant p$ and $d_q = 0$ otherwise; by M-universality, $\langle a, \sigma(d)\rangle \in E_p^{\mathcal{M}}$ and $\langle b, \sigma(d)\rangle \in E_p^{\mathcal{M}}$, which shows that $\sigma(d)$ is the required set z in $(\mathcal{M}, E, E^*) \Vdash_p \exists z(aEz \wedge bEz)$. A similar proof applies to (E-Union), (E-Inf), (E-Subst ϕ) and (E-Compr ϕ). For (Subsets), let $p \in C$, $a \in D_p^{\mathcal{M}}$ and consider the M-class b defined by

$$b_q = \{\sigma(d) | d \ M\text{-class of } (\mathcal{M}, E), \forall p' \not\leqslant p \ d_{p'} = 0,$$
$$\mathcal{M} \Vdash_p \forall w E\sigma(d) \, wEa\}$$

for $q \leqslant p$ and $b_q = 0$ otherwise; $\sigma(b)$ is the required set y in

$$\mathcal{M} \Vdash_p \exists y \forall u(\forall vEu \; vEa \rightarrow \exists u_1 Ey \; \forall w(wEu_1 \longleftrightarrow wEu)).$$

It remains to discuss the traditional embedding of (M, ϵ) in $(\mathcal{M}, E^*)$. From now on we assume that $(\mathcal{M}, E)$ is M-universal. We associate to every $a \in M$ the global section $\hat{a}$ defined by induction on the rank of a:

Definition 5 $\quad \hat{a} = \sigma((\{\hat{b} | b \in a\})_{p \in C})$.

That $\hat{a}$ is indeed a global section of $(\mathcal{M}, E)$ is proved by induction on the rank of a. As usual,

Proposition 3 *For every $a, b \in M$, $p \in C$, $c \in D_p^{\mathcal{M}}$,*

(1) $(\mathcal{M} \Vdash_p \hat{a}E^*\hat{b}) \longleftrightarrow a \in b$
(2) $(\mathcal{M} \Vdash_p cE^*\hat{b}) \rightarrow \exists q \leqslant p \, \exists a \in M \ \mathcal{M} \Vdash_q c \sim \hat{a}$.

(In (2) and in the sequel, $\sim$ is $\underset{E^*}{\approx}$ or $\overset{E^*}{\underset{E}{\approx}}$: these are equivalent in $\mathcal{M}$). Proposition 3 (1) means that the formula $(x \in y)$ is "absolute" and Proposition 3 (2) means that it is "conservative" in the following sense:

Definition 6 To every formula ϕ of $L(\epsilon)$ associate the formula ϕ_* of $L(E^*)$ obtained by replacing every occurrence of ϵ by E^*.

(1) The formula $\phi(x_1, \ldots, x_n)$ of $L(\epsilon)$ is *absolute* if for every $a_1, \ldots, a_n \in M$ and $p \in C$,

$$(\mathcal{M} \Vdash_p \phi_*(\hat{a}_1, \ldots, \hat{a}_n)) \longleftrightarrow M \models \phi(a_1, \ldots, a_n).$$

(2) The formula $\phi(y_1, \ldots, y_n, x_1, \ldots, x_m)$ of $L(\epsilon)$ is *conservative* in $y_1, \ldots, y_n$ if for every $a_1, \ldots, a_m \in M$, $p \in C$ and $b_1, \ldots, b_n \in D_p^{\mathcal{M}}$,

$$(\mathcal{M} \Vdash_p \phi_*(b_1, \ldots, b_n, \hat{a}_1, \ldots, \hat{a}_m))$$
$$\rightarrow \exists q \leqslant p \, \exists b'_1, \ldots, b'_n \in M \ \mathcal{M} \Vdash_q b_1 \sim \hat{b}'_1 \wedge \ldots \wedge b_n \sim \hat{b}'_n.$$

(Note that we could as well enrich our language $L(E, E^*)$ with constants $\underline{a}$ $(a \in M)$ interpreting $\underline{a}$ in $\mathcal{M}$ by $(\hat{a})_{p \in C}$ and consider (M, ϵ) as a constant structure over $\langle C, \leqslant\rangle$ for the language $L(E, E^*, \underline{a})_{a \in M}$, interpreting E and E^* by ϵ and $\underline{a}$ by a. This consideration of constants would require a slight extension of the notions of Section 2 but would have the advantage of giving a better formulation of Definition 6.)

Lemma 4

(1) *If ϕ and ψ are absolute, so are $\neg\phi$, $\phi \wedge \psi$, $\phi \rightarrow \psi$.*
(2) *If ϕ is absolute and conservative in $y, y_1, \ldots, y_n$, then $\exists y\phi$ is absolute and conservative in $y_1, \ldots, y_n$.*

Proof: Immediate from the definitions.

Using the classical definition of Δ_1^{ZF}-formulas (but always reading '$\forall x$' as '$\neg\exists x\neg$'), the following holds:

Proposition 5 *If ϕ is Δ_1^{ZF}, then $\neg\neg\phi$ is absolute.*

Proof: Lemma 4 and Proposition 3 give the result for formulas with bounded quantifiers. This is easily extended to Δ_1^{ZF}-formulas.

Proposition 5 allows us to transfer automatically classical results:

Proposition 6

(1) *The formula $\neg\neg Ord(x)$ where $Ord(x)$ expresses (classically without universal quantifiers) that x is an ordinal is absolute.*
(2) *The formula $\neg\neg L(y, x)$ where $L(y, x)$ expresses (classically without universal quantifiers) that y is the x-th set of the constructible hierarchy is absolute.*

Proposition 7 *The formula $Ord(x)$ is conservative in x.*

Proof: Let $p \in C$, $c \in D_p^{\mathcal{M}}$ and $\mathcal{M} \Vdash_p^- Ord(c)$. Define $\alpha = \{\beta | \beta$ ordinal of M, $\exists q \leqslant p\ \mathcal{M} \Vdash_q^- \hat{\beta}E^*c\}$; prove first that α is an ordinal of M; then compare $\hat{\alpha}$ and c in $\mathcal{M}$ by the (double negation of the) law of trichotomy; the fact that $\exists q \leqslant p\ \mathcal{M} \Vdash_q^- c \sim \hat{\alpha}$ will follow.

To show that $L(y, x)$ is conservative in y and x, we use the following notion:

Definition 7 The formula $\phi(y, x)$ is *strongly conservative* in x if for every $p \in M$, $b, a \in D_p^{\mathcal{M}}$

$$(\mathcal{M} \Vdash_p^- \phi_*(b, a)) \rightarrow \exists q \leqslant p\ \exists a' \in M\ \mathcal{M} \Vdash a \sim \hat{a}'.$$

Lemma 8

(1) *If $\phi(y, x)$ is absolute and strongly conservative in x and if $ZF \vdash_c^- \exists! y\phi$, then ϕ is conservative in y, x.*
(2) *If $ZF \vdash_c^- \phi(y, x) \rightarrow \psi(x)$ and $\psi(x)$ is conservative in x, then ϕ is strongly conservative in x.*

Applied to $L(y, x)$ and $Ord(x)$, Lemma 8 gives the desired result:

Proposition 9 *The formula $L(y, x)$ is conservative in y and x.*

Finally, denoting by $\Lambda(x)$ the formula $\exists y(x \in y \wedge \exists z \neg\neg L(y, z))$ which expresses that x is constructible and using the preceding observations, we obtain:

Proposition 10 *$\Lambda(x)$ is absolute and conservative in x.*

This fact is what is essentially required to prove the relative consistency of $V \neq L$: define $\langle C, \leqslant\rangle$ and $(\mathcal{M}, E)$ as usual, say with forcing (A) and $p \in C \longleftrightarrow \exists$ finite $x \subseteq \omega$, $p: x \to \{0, 1\}$

$$p \leqslant q \longleftrightarrow p \supseteq q;$$

consider $c = \{\langle \hat{m}, p\rangle | p \in C, \langle m, 1\rangle \in p\}$; using the conservative character of $\Lambda(x)$, it is easy to turn $\mathcal{M} \not\Vdash \neg\Lambda_*(c)$ into a contradiction.

4 Generalization to sheaves Instead of interpreting languages in structures over partially ordered sets, it is possible to interpret them in sheaves over a site (and in particular over a topological space). We recall here this interpretation due to Joyal (see [6]). A *site* $\mathcal{S}$ is an ordered pair $\langle \mathcal{C}, Cov\rangle$ where $\mathcal{C}$ is a small category and Cov a set of *covering families* $(Cov(p))_{p \in C_o}$, C_o denoting the set of objects of $\mathcal{C}$, satisfying the usual conditions. Any partially ordered set is a site for the definition $Cov(p) = \{p\}$. The case of general Heyting algebras (not necessarily associated with a partially ordered set) is also covered by the definition $Cov(p) = \{(p_i)_{i \in I} | \bigvee_{i \in I} p_i = p\}$. A *sheaf over* $\mathcal{S}$ is a functor D: $\mathcal{C}^{op} \to$ Set satisfying the glueing condition. A *sheaf of structures for a language $L(E)$ over* $\mathcal{S}$ is an ordered pair $\langle D^{\mathcal{M}}, E^{\mathcal{M}}\rangle$ where $D^{\mathcal{M}}$ is a sheaf over $\mathcal{S}$ and $E^{\mathcal{M}}$ is a subsheaf of $(D^{\mathcal{M}})^2$. Formulas of $L(E)$ are interpreted in $\mathcal{M}$ by an inductive definition of $\mathcal{M} \Vdash_{\bar{p}} \phi(\bar{x})\ [\bar{a}]$ for objects p of $\mathcal{C}$ and sequences $\bar{a}$ of elements of $D_p^{\mathcal{M}}$, whose length matches that of $\bar{x}$. This definition is that of Section 1 for atomic formulas and conjunctions: the other cases are given below:

(1) $\mathcal{M} \Vdash_{\bar{p}} (\neg\psi)(\bar{x})\ [\bar{a}]$ iff $\forall q \xrightarrow{f} p$ in $\mathcal{C}$ $\mathcal{M} \not\Vdash_{\bar{q}} \psi(\bar{x})\ [D_f^{\mathcal{M}}\bar{a}]$.
(2) $\mathcal{M} \Vdash_{\bar{p}} (\psi \to \chi)(\bar{x})\ [\bar{a}]$ iff $\forall q \xrightarrow{f} p$ in $\mathcal{C}$, $\mathcal{M} \Vdash_{\bar{q}} \psi(\bar{x})\ [D_f^{\mathcal{M}}\bar{a}]$ implies $\mathcal{M} \Vdash_{\bar{q}} \chi(\bar{x})\ [D_f^{\mathcal{M}}\bar{a}]$.
(3) $\mathcal{M} \Vdash_{\bar{p}} (\exists x\psi)(\bar{x})\ [\bar{a}]$ iff there exists a family $(p_i \xrightarrow{f_i} p)_{i \in I} \in Cov(p)$ and a family $(a_i)_{i \in I} \in \prod_{i \in I} D_{p_i}^{\mathcal{M}}$ such that for every $i \in I$, $\mathcal{M} \Vdash_{\bar{p_i}} \psi(x, \bar{x})\ [a_i, D_{f_i}^{\mathcal{M}}\bar{a}]$.

Minor changes in Section 1 will again give an Extension Lemma and a Validity Lemma, to which the local character of this forcing may be added:

Local Character Lemma *If $(p_i \xrightarrow{f_i} p)_{i \in I} \in Cov(p)$, $\bar{a} \in D_p^{\mathcal{M}}$ and for every $i \in I$, $\mathcal{M} \Vdash_{\bar{p_i}} \phi(\bar{x})\ [D_{f_i}^{\mathcal{M}}\bar{a}]$, then $\mathcal{M} \Vdash_{\bar{p}} \phi(\bar{x})\ [\bar{a}]$.*

Since Section 2 is mainly based on syntactic considerations and the validity lemma holds, definitions and proofs will carry over without difficulty and Theorem 1 will still hold for sheaves of structures over a site $\mathcal{C}$.

Sheaves can also be used to give models of *ZF* but details of the adaptation of Section 3 are more delicate than for Section 2. Essentially, $\mathcal{S}$ (and in particular Cov) should be an element of M and "things" to be considered are not sheaves in the external sense but sheaves in the sense of M; these are ordered pairs $\langle D^{\mathcal{M}}, E^{\mathcal{M}}\rangle$ given by formulas of the language of *ZF* corresponding not only to $D_p^{\mathcal{M}}$ and $E_p^{\mathcal{M}}$ but also to $D_f^{\mathcal{M}}$ and $E_f^{\mathcal{M}}$ (f is an arrow of $\mathcal{C}$), and such that, using these formulas, $M \models$ "$\langle D^{\mathcal{M}}, E^{\mathcal{M}}\rangle$ is a sheaf over $\mathcal{S}$". In this connection, the remark following the Definability Lemma of Section 3 may be useful.

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