There are Denumerably Many Ternary
Intuitionistic Sheffer Functions

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In [1] Došen asks what is the number of mutually nonequivalent ternary indigenous Sheffer functions for \{\to, \land, \lor, \neg\} in the intuitionistic propositional calculus (IPC). The answer is: denumerably many.

Following [2] we shall say that a set of functions \(F\) is an indigenous Sheffer set for a set of functions \(G\) iff every member of \(G\) can be defined by a finite number of compositions from the members of \(F\) and vice versa. A function \(f\) is an indigenous Sheffer function for \(G\) iff \([f]\) is an indigenous Sheffer set for \(G\). The \(n\)-ary propositional functions \(f_1\) and \(f_2\) are mutually equivalent iff for some permutation \(P\) of the sequence \(A_1, \ldots, A_n\) in the propositional calculus we can prove \(f_1(A_1, \ldots, A_n) \leftrightarrow f_2(P)\). We work all the time in IPC. Expressions of the form \(\vdash A\) (or \(\not\vdash A\)) mean that \(A\) is provable (or unprovable) in IPC.

Kuznetsov [3] and Hendry [2] have shown that there is no binary indigenous Sheffer function for \{\to, \land, \lor, \neg\} in IPC. The first example of a ternary indigenous Sheffer function was given in [3]. Here we use one of the three examples given in [1].

The Rieger–Nishimura Lattice of one variable \(X\), \(\text{RNL}(X)\) is recursively defined as follows:

\[
P_0(X) = X \land \neg X,\quad P_1(X) = X,\quad P_2(X) = \neg X,\quad P_\omega(X) = X \rightarrow X,\quad P_{2n+3}(X) = P_{2n+1}(X) \lor P_{2n+2}(X),\quad P_{2n+4}(X) = P_{2n+3}(X) \rightarrow P_{2n+1}(X),
\]

for \(n \geq 0\). For every \(i > j\), \(\not\vdash P_i(X) \rightarrow P_j(X)\) (see [5] or [4]).

First, we have one simple lemma:

**Lemma** For every \(i \geq 5\):

1. \(\vdash \neg X \land P_i(X) \leftrightarrow \neg X\)
2. \(\vdash P_i(\bot)\).

**Proof:** (1) For every \(i \geq 5\), we have \(\neg X \vdash P_i(X)\) directly from \(\text{RNL}(X)\). We obtain (2) by using \(\vdash \neg \bot\) and (1).

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Next, we give the following definition:

**Definition** \( f_i(A,B,C) = t(A,B,C) \land P_i(B \land C) \), where \( t(A,B,C) = ((A \lor B) \leftrightarrow (C \land \neg B)) \lor (A \leftrightarrow (C \land \neg B)) \) (see [1]), and \( P_i(B \land C) \in RNL(B \land C) \).

**Theorem** For every \( i \geq 5 \), \( f_i \) is an indigenous Sheffer function for \( \{ \rightarrow, \land, \lor, \neg \} \) and for every \( i \geq 5 \) and every \( j > i \), \( \vdash f_i \leftrightarrow f_j \).

**Proof:** For every \( i \geq 5 \), \( f_i \) is an indigenous Sheffer function for \( \{ \rightarrow, \land, \lor, \neg \} \) because:

\[
\begin{align*}
f_i(A,A,A) & \leftrightarrow t(A,A,A) \land P_i(A) \\
& \leftrightarrow \neg A \land P_i(A) \\
& \leftrightarrow \neg A \quad \text{(by using Lemma (1));}
\end{align*}
\]

\[
\begin{align*}
f_i(A,B,\neg B) & \leftrightarrow t(A,B,\neg B) \land P_i(B \land \neg B) \\
& \leftrightarrow (A \lor B) \land P_i(\bot) \\
& \leftrightarrow A \lor B \quad \text{(by using Lemma (2));}
\end{align*}
\]

\[
\begin{align*}
f_i(A,\neg (A \lor \neg A),B) & \leftrightarrow t(A,\bot,B) \land P_i(\bot \land B) \\
& \leftrightarrow (A \leftrightarrow B) \land P_i(\bot) \\
& \leftrightarrow (A \leftrightarrow B) \quad \text{(by using Lemma (2));}
\end{align*}
\]

and we know that \( \{ \rightarrow, \land, \lor, \neg \} \) is an indigenous Sheffer set for \( \{ \rightarrow, \land, \lor, \neg \} \) (we have: \( \vdash (A \rightarrow B) \leftrightarrow ((A \lor B) \leftrightarrow B), \vdash (A \land B) \leftrightarrow ((A \lor B) \leftrightarrow (A \rightarrow B)) \)).

If for some \( i \geq 5 \) and some \( j > i \), \( \vdash f_i \leftrightarrow f_j \), we have \( \vdash f_i(\bot, B, B) \leftrightarrow f_j(\bot, B, B) \) which implies \( \vdash P_i(B) \leftrightarrow P_j(B) \), and that implies \( \vdash P_i(B) \leftrightarrow P_j(B) \), which is a contradiction.

Note that this theorem is also valid for \( i = 3 \).

For every \( i \geq 5 \) and \( j > i \), \( f_i \) and \( f_j \) are mutually nonequivalent because \( f_i(A,B,C) \) is classically equivalent only with \( f_j(C,B,A) \), but not in IPC (if it is, we have: \( \vdash f_i(\bot, T, C) \leftrightarrow f_j(C, T, \bot) \); then \( \vdash t(\bot, T, C) \land P_i(C) \leftrightarrow T \), and then \( \vdash P_i(C) \), which is a contradiction). Since we have at most denumerably many nonequivalent ternary indigenous Sheffer functions (consider them as words in the alphabet \( \{ A,B,C,\rightarrow, \land, \lor, \neg \} \)), we may conclude that there are exactly denumerably many of them.

For every \( n > 3 \), there exist denumerably many \( n \)-ary Sheffer functions for \( \{ \rightarrow, \land, \lor, \neg \} \) (we substitute \( A_1 \land \ldots \land A_{n-2} \) for \( A \) in \( f_i \)).

We conclude this note with two questions:

(1) Is it true that for every ternary Sheffer function in the classical propositional calculus there exists a classically equivalent function which is a Sheffer function in IPC?

(2) What structure is produced by all ternary Sheffer functions in IPC?

**REFERENCES**


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