Equivalent Versions of a Weak Form of the Axiom of Choice

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We make the following definition:

**Definition** Let \((Q, \leq)\) be a quasi-order (i.e., \(\leq\) is reflexive and transitive). Two elements \(x, y\) of \(Q\) are said to be incompatible if there does not exist \(z \in Q\) such that \(z \leq x\) and \(z \leq y\). A subset \(I\) of \(Q\) is said to be an incompatible set if any two elements of \(I\) are incompatible. For each \(x \in Q\), let \(l(x)\) denote the set of lower bounds of \(x\); and let \(c(x)\) denote the set of elements of \(Q\) that are compatible with \(x\). "Countable" is used here to mean "countably infinite".

Let \(U_{K0}^f (U_{K0}^g)\) denote the statement that the union of a countable collection of pairwise disjoint nonempty finite (countable) sets is countable.

Let \(AC^f_{K0} (AC^g_{K0})\) denote the statement that there exists a choice function for any countable family of finite (countable) nonempty sets.

Let \(ACS^f_{K0} (ACS^g_{K0})\) denote the statement that for any countable family of finite (countable) nonempty sets there exists a countable subfamily for which a choice function exists.

It is known that in \(ZF\); \(U_{K0}^f\) is equivalent to \(AC^f_{K0}\). (Let \(C = \{C_n; n \in \mathbb{N}\}\) be a countable collection of pairwise disjoint nonempty finite sets. Let \(|C_n| = m_n\); and, by \(AC^f_{K0}\), choose for each \(n \in \mathbb{N}\) a function \(f_n: C_n \to m_n \times \{n\}\) such that \(f_n\) is 1-1 and onto. Then \(\bigcup_{n \in \mathbb{N}} f_n\) is 1-1; \(\bigcup_{n \in \mathbb{N}} f_n \left[ \bigcup_{n \in \mathbb{N}} C_n \right]\) is an infinite subset of \(\mathbb{N} \times \mathbb{N}\) (which is countable)—and therefore \(\bigcup_{n \in \mathbb{N}} C_n\) is countable. Conversely, if \(C = \{C_n; n \in \mathbb{N}\}\) is a countable collection of finite nonempty sets then \(D = \{C_n \times \{n\}; n \in \mathbb{N}\}\) is a countable collection of pairwise disjoint nonempty finite sets and hence, by \(U_{K0}^f\), \(\bigcup_{n \in \mathbb{N}} (C_n \times \{n\})\) is countable. Therefore there exists a \(g: \bigcup_{n \in \mathbb{N}} (C_n \times \{n\}) \to \mathbb{N}\), such that \(g\) is 1-1 and onto. A choice

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function for $C$ is then given by $f(C_n) = \text{the element } x \text{ of } C_n \text{ such that } g(x, n)$ is the least element of $g[C_n \times \{n\}]$.

It is also known that in $ZF \cup \text{AC}^{\aleph_0}$, $AC^{\aleph_0}$ implies $AC^{\aleph_0}_\kappa$, but it is not known if $AC^{\aleph_0}_\kappa$ implies $U^{\aleph_0}_\kappa$ ([7], pp. 203, 324).

$AC^{\aleph_0}_\kappa$ is also equivalent to König’s Infinity Lemma ([5], p. 298; [1], pp. 202, 203), Ramsey’s Theorem [6], and a limited version of Tychonoff’s Theorem [4]. It is clear that $AC^{f}_\kappa$ implies $ACS^{f}_\kappa$, and it will be shown that $ACS^{f}_\kappa$ implies $ACS^{f}_\kappa$.

Let $Q^{\kappa}_\alpha(P^{\kappa}_\alpha)$ denote the statement that if $(Q, \leq)$ is a countable quasi-order (partial-order) that contains incompatible sets of arbitrarily large finite cardinality then $Q$ contains a countable incompatible set. ($P^{\kappa}_\alpha$ follows from an old result of Erdős and Tarski [2].)

Let $Q^{U^{f}_\kappa}(Q^{U^{f}_\kappa}_\kappa)$ denote the statement that if $(Q, \leq)$ is a quasi-order that contains incompatible sets of arbitrarily large finite cardinality, and if $Q$ can be written as a countable union of finite (countable) sets, then $Q$ contains a countable incompatible set. Let $P^{U^{f}_\kappa}, P^{U^{f}_\kappa}_\kappa$ denote the analogous statements for partial orders.

Let $Q^{M^{f}_\kappa}(Q^{M^{f}_\kappa}_\kappa)$ denote the statement that if $(Q, \leq)$ is a quasi-order that contains incompatible sets of arbitrarily large finite cardinality, and if $Q$ can be written as a countable union of finite (countable) sets, then $Q$ contains a maximal countable incompatible set. Let $P^{M^{f}_\kappa}, P^{M^{f}_\kappa}_\kappa$ denote the analogous statements for partial orders.

It will be shown that in $ZF, Q^{U^{f}_\kappa}, P^{U^{f}_\kappa}, Q^{M^{f}_\kappa}, P^{M^{f}_\kappa}_\kappa$ are equivalent to each other and to $AC^{f}_\kappa$.

**Theorem 1** $AC^{f}_\kappa$ is a theorem of $ZF \cup \{ACS^{f}_\kappa\}$.

**Proof:** It will be shown that $U^{f}_\kappa \kappa$ is a theorem of $ZF \cup \{ACS^{f}_\kappa\}$.

Let $C$ be a countable collection of pairwise disjoint nonempty finite sets; then $C$ can be written as $C = \{C_n : n \in \mathbb{N}\}$. Each $C_n$ is finite, and the union of a finite collection of finite sets is finite, so for each $n \in \mathbb{N}$ there exists a natural number $k_n$ such that $\bigcup_{i=0}^{n} C_i = k_n$. Therefore for each $n \in \mathbb{N}$ there exists a 1-1 function from $\bigcup_{i=0}^{n} C_i$ onto $k_n \times \{n\}$.

For each $n, n \geq 1$, let $g_n$ denote the restriction of $f_{i_n}$ to $\bigcup_{i=0}^{t_n} C_i$. Then $g_n$ is a 1-1 map of $\bigcup_{i=0}^{t_n} C_i$ into $k_{i_n} \times \{i_n\}$, and therefore $g = \bigcup_{n \in \mathbb{N}} g_n$ is a 1-1 map of $\bigcup_{n \in \mathbb{N}} C_n$ into $\mathbb{N} \times \mathbb{N}$. Since the $g_n$'s have pairwise disjoint ranges, $g\left(\bigcup_{n \in \mathbb{N}} C_n\right)$ is a countable subset of $\mathbb{N} \times \mathbb{N}$.

Therefore (since $g$ is 1-1) $\bigcup_{n \in \mathbb{N}} C_n$ is countable, and $U^{f}_\kappa \kappa$ is a theorem of
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ZF \cup \{AC_{\aleph_0}^f\}.  Thus \(AC_{\aleph_0}^f\) is a theorem of \(ZF \cup \{AC_{\aleph_0}^f\}\). (Theorem 1 can also be proved by showing that König’s Infinity Lemma is a theorem of \(ZF \cup \{AC_{\aleph_0}^f\}\), but the proof of this is more involved than the proof given above.)

Theorem 2  \(Q_{\aleph_0}\) is a theorem of \(ZF \cup \{AC_{\aleph_0}^f\}\).

Proof: Assume that \((Q, \leq)\) is a countable quasi-order that contains incompatible sets of arbitrarily large finite cardinality. Let \(Q = \{q_n : n \in \mathbb{N}\}\).

Let \(C\) be the collection of incompatible subsets, \(I\), of \(Q\) such that \(|I| \geq 2\) and such that there exists \(y \in I\) such that \(I(y)\) contains incompatible sets of arbitrarily large finite cardinality.

If \(C = \emptyset\) then let \(D\) be the collection of all incompatible subsets of \(Q\). If \(D\) is not countable then \(Q\) contains a countable incompatible set. Assume that \(D\) is countable; then \(D\) can be written as \(D = \{D_n : n \in \mathbb{N}\}\). Let \(g_1\) be the least natural number such that \(D_1\) is a proper subset of \(D_{g_1}\) \((g_1\) exists since \(D_1\) is finite and for each \(y \in D_1\) there exists a maximal finite incompatible set \(I(y)\) \((I(y)\) contains incompatible sets of arbitrarily large finite cardinality). Therefore the largest incompatible subset of \(\bigcup_{y \in D_1} c(y)\) has cardinality \(\sum_{y \in D_1} |I(y)|\), and hence \(Q \neq \bigcup_{y \in D_1} c(y)\). Let \(m\) be the least natural number such that \(q_m \in Q - \bigcup_{y \in D_1} c(y)\). Then \(D_1 \cup \{q_m\} \in \emptyset\). For each \(n > 1\) define, by induction, \(g_{n+1}\) to be the least natural number such that \(D_{g_{n+1}}\) is a proper subset of \(D_{g_n}\). Then \(\bigcup_{n \in \mathbb{N}} D_n \subseteq Q\) is a countable incompatible set, therefore \(Q\) contains a countable incompatible set.

Assume \(C \neq \emptyset\). If \(C\) is not finite and not countable then \(Q\) contains a countable incompatible set. Assume that \(C\) is countable (the proof for the case that \(C\) is finite is similar); then \(C\) can be written as \(C = \{I_n : n \in \mathbb{N}\}\). For each \(n \in \mathbb{N}\), choose \((AC_{\aleph_0}^f)\) \(p_n \in I_n\) such that \(l(p_n)\) contains incompatible sets of arbitrarily large finite cardinality; and for each \(n \in \mathbb{N}\), let \(C_n = \{I \in C : I \subseteq l(p_n)\}\). If \(C_n = \emptyset\) for any \(n\), then \((by an argument similar to that given above for \(C = \emptyset\))\) \(Q\) contains a countable incompatible set. Therefore assume that \(C_n \neq \emptyset\) for all \(n \in \mathbb{N}\). Choose \(r_1 \in I_1 - \{p_1\}\) and let \(\lambda(1)\) be the least natural number such that \(I_{\lambda(1)} \in C_1\). Choose \(r_2 \in I_{\lambda(1)} - \{p_{\lambda(1)}\}\) and let \(\lambda(2)\) be the least natural number such that \(I_{\lambda(2)} \in C_{\lambda(1)}\). Assume that \(r_n\) is defined \((n \geq 2)\), \(r_n \in I_{\lambda(n-1)} - \{p_{\lambda(n-1)}\}\). Let \(\lambda(n)\) be the least natural number such that \(I_{\lambda(n)} \in C_{\lambda(n-1)}\). Then choose \((by AC_{\aleph_0}^f)\) \(r_{n+1} \in I_{\lambda(n)} - \{p_{\lambda(n)}\}\). By construction, \(p_{\lambda(1)} \leq p_{\lambda(2)} \leq p_{\lambda(3)} \leq \ldots\) Thus it follows that for each \(m < n \in \mathbb{N}\), \(r_m\) and \(r_n\) are incompatible. Then \(\{r_n : n \in \mathbb{N}\}\) is a countable incompatible subset of \(Q\).

Therefore \(Q_{\aleph_0}\) is a theorem of \(ZF \cup \{AC_{\aleph_0}^f\}\).

It follows from Theorem 2 that \(QU_{\aleph_0}^f\) is a theorem of \(ZF \cup \{AC_{\aleph_0}^f\}\) (since \(AC_{\aleph_0}^f\) implies \(U_{\aleph_0}^f\)).

Theorem 3  \(AC_{\aleph_0}^f\) is a theorem of \(ZF \cup \{QU_{\aleph_0}^f\}\).

Proof: To prove \(AC_{\aleph_0}^f\) it suffices (by Theorem 1) to prove \(ACS_{\aleph_0}^f\).

Let \(C\) be a countable family of finite sets. It can be assumed (without loss of generality) that the sets of \(C\) are pairwise disjoint. Define \(\leq\) on \(Q = \bigcup_{A \in C} A\)
by \( x \leq y \) iff there exists \( A \in C \) such that \( x \in A \) and \( y \in A \). Then \( \leq \) is a quasi-order on \( Q \); and two elements \( u, v \) are incompatible iff there exist \( A, B \in C, A \neq B \), with \( u \in A \) and \( v \in B \).

Let \( n \in \mathbb{N} \), and let \( A_1, \ldots, A_n \) be elements of \( C \). Choose \( x_1 \in A_1, x_2 \in A_2, \ldots, x_n \in A_n \) (this can be done in \( ZF \)). Then \( \{x_1, \ldots, x_n\} \) is an incompatible subset of \( Q \). Therefore \( (Q, \leq) \) contains incompatible sets of arbitrarily large finite cardinality, and hence (by \( QUk_0 \)) \( (Q, \leq) \) contains a countable incompatible set, \( I \).

Let \( D = \{A \in C: A \cap I \neq \emptyset\} \), and define \( f \) on \( D \) by \( f(A) = A \cap I \). Then \( D \) is a countable subfamily of \( C \) for which a choice function exists.

Therefore \( ACS_{k_0} \) is a theorem of \( ZF \cup \{QU_{k_0}\} \) — and hence \( AC_{k_0} \) is a theorem of \( ZF \cup \{QU_{k_0}\} \).

Therefore \( QU_{k_0} \) is equivalent to \( AC_{k_0} \).

It is clear that \( Q_{k_0} \) implies \( P_{k_0} \). The converse is given in the following:

**Claim 1** \( Q_{k_0} \) is a theorem of \( ZF \cup \{P_{k_0}\} \).

**Proof:** Let \( (Q, \leq) \) be a countable quasi-order that has incompatible subsets of arbitrarily large finite cardinality. Then \( Q \) can be written as \( Q = \{q_n: n \in \mathbb{N}\} \).

For each \( n \in \mathbb{N} \), let \( A_n = \{q_i \in Q: q_i \leq q_n \text{ and } q_n \leq q_i\} \), and let \( P = \{A_n: n \in \mathbb{N}\} \). Define \( \leq \) on \( P \) by \( A_i \preceq A_j \) iff \( q_i \leq q_j \). Then \( \leq \) is a partial-order on \( P \).

Since \( (Q, \leq) \) has incompatible subsets of arbitrarily large finite cardinality so does \( (P, \leq) \), and hence (by \( P_{k_0} \)), \( P \) contains a countable incompatible set, \( I \).

Let \( J \) be the set of natural numbers, \( n \), such that \( I = \{A_n: n \in J\} \) and such that for any \( m \in \mathbb{N} \), if \( m < n \) then \( A_m \neq A_n \).

Then \( \{q_i: i \in J\} \) is a countable incompatible subset of \( Q \); and thus \( Q_{k_0} \) is a theorem of \( ZF \cup \{P_{k_0}\} \).

Note that the statement: “If \( (R, \leq) \) is a quasi-order that contains incompatible sets of arbitrarily large finite cardinality, then \( R \) contains a countable incompatible set” is not a theorem of \( ZF \cup \{AC_{k_0} \} \) — since from this statement it follows that any infinite set contains a countable subset (simply define \( \leq \) on an infinite set \( R \) by \( x \preceq y \) iff \( x = y \)) — but this result requires a stronger form of the Axiom of Choice than \( AC_{k_0} \) ([7], pp. 322, 323).

It follows from Claim 1 that \( PU_{k_0} \) is equivalent to \( QU_{k_0} \).

**Theorem 4** In \( ZF \), \( QM_{k_0} \) is equivalent to \( AC_{k_0} \).

**Proof:** \( QM_{k_0} \) clearly implies \( QU_{k_0} \), thus \( AC_{k_0} \) is a theorem of \( ZF \cup \{QM_{k_0}\} \).

Assume that \( (Q, \leq) \) is a quasi-order that contains incompatible sets of arbitrarily large finite cardinality, and that \( Q \) can be written as a countable union of finite sets. Then (assuming \( AC_{k_0} \)), \( Q \) is countable and hence \( Q \) can be written as \( Q = \{q_n: n \in \mathbb{N}\} \). By Theorem 2, \( Q \) contains a countable incompatible set, \( I \).

If \( Q = \bigcup_{x \in I} c(x) \) then \( I \) is a maximal countable incompatible set.

Suppose that \( Q \neq \bigcup_{x \in I} c(x) \). Let \( d(1) \) be the least natural number such that \( q_{d(1)} \in Q - \bigcup_{x \in I} c(x) \), and let \( I_1 = I \cup \{q_{d(1)}\} \). For \( n > 1 \) define, by induction, \( d(n) \) and \( I_n \) as follows: Let \( d(n) \) be the least natural number such that
\(q_d(n) \in Q - \bigcup_{x \in I_{n-1}} c(x)\) (if \(Q \neq \bigcup_{x \in I_{n-1}} c(x)\)) and let \(I_n = I_{n-1} \cup \{q_d(n)\}\). Either
\[Q = \bigcup_{x \in I_k} c(x)\] for some \(k\) and then \(I_k\) is a maximal countable incompatible set; or \(Q \neq \bigcup_{x \in I_k} c(x)\) for all \(k\) and then \(\bigcup_{k \in \mathbb{N}} I_k\) is a maximal countable incompatible set.

Therefore \(QM_{K_0}^f\) is a theorem of \(ZF \cup \{AC_{K_0}\}\).

By an argument similar to that of Claim 1 it follows that \(QM_{K_0}^f\) is equivalent to \(PM_{K_0}^f\).

A summary of the implications is that in \(ZF\):

\[
\begin{align*}
PU_{K_0}^f & \iff QU_{K_0}^f \iff AC_{K_0}^f \iff QM_{K_0}^f \iff PM_{K_0}^f \\
ACS_{K_0}^f & \iff U_{K_0}^f \iff Q_{K_0} \iff P_{K_0}.
\end{align*}
\]

Note that all of the arrows except possibly \(AC_{K_0}^f \Rightarrow Q_{K_0}\) are reversible. I do not know if in \(ZF\) \(Q_{K_0}\) is equivalent to \(AC_{K_0}^f\).

Some of the ideas for families of finite sets extend to families of countable sets; but some do not, and some of the implications are not known. A summary of the implications is that in \(ZF\):

\[
\begin{align*}
PU_{K_0}^f & \iff QU_{K_0}^f \iff AC_{K_0}^f \iff QM_{K_0}^f \iff PM_{K_0}^f \\
ACS_{K_0}^f & \iff U_{K_0}^f.
\end{align*}
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REFERENCES


