Notre Dame Journal of Formal Logic
Volume 29, Number 4, Fall 1988

A Basic Free Logic

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1 Introduction In [5] Lambert and van Fraassen propose a first-order predicate logic with identity using Fitch's method of subordinate proofs [3]. The system is universally free, that is, valid whether or not the domain is empty and whether or not all terms are assumed to refer to existents. For those cases in which the domain is assumed nonempty, they provide a special rule of vacuous quantifier elimination. With this rule the logic is still free, but not universally.

In [7] I construct a universally free logic with identity, also using Fitch's method, but one simpler and more intuitive than Lambert and van Fraassen's. I begin with a nonfree or "standard" logic, that is, one valid only if all terms are assumed to refer to existents. From this system, I form a free logic valid for only nonempty domains by placing a restriction on just two rules: the rule of universal quantifier elimination and the rule of existential quantifier introduction. This restriction limits nonvacuous universal quantifier elimination and existential quantifier introduction to general subordinate proofs with respect to the instantial term. With a stronger restriction on those same two rules, a universally free logic results. The stronger restriction limits both nonvacuous and vacuous universal quantifier elimination and existential quantifier introduction to general subordinate proofs.

My purposes in this paper are (i) to construct a universally free logic that is simpler and more intuitive than that in [7] and (ii) from it to generate a free logic for only nonempty domains and then a nonfree logic. The universally free system, which I call $S1$, is proposed as "a basic free logic" because of (i) the simplicity of its language and rules and (ii) the intuitively obvious and natural way in which the other two systems develop from it.

The language L of $S1$ is without identity '=' and without the existence symbol '$\exists !$'. The proof technique follows Fitch's with the major exception of a novel

*Some results reported in this paper are contained in an earlier paper entitled "Simplified quantifier rules for natural deduction," presented at the annual meeting of the Association for Symbolic Logic, 29 December 1985, Washington, D.C. I wish to thank Hugues Leblanc and an anonymous referee for helpful comments and suggestions.

Received February 10, 1986; revised November 3, 1986

technique I call "prefixed proofs". Both main and subordinate proofs may be prefixed. For simpler quantifier rules, subordinate prefixed proofs are used instead of Fitch's general subordinate proofs. A non-universally free system results from adding to $S1$ a rule of existence assumption. This rule directly reflects the assumption that the domain is nonempty and can be defined in a language that excludes vacuous quantifiers. For those cases in which all terms are assumed to refer to existents, I modify $S1$ by counting last items (formulas) of not only regular but also prefixed categorical main proofs as theorems. As a prefixed term is assumed to refer to an existent, enlarging the class of theorems in this way to yield a nonfree logic seems quite natural.

2 The language The rules to be presented are intended for a language L having the following characteristics. The primitive signs are: atomic formulas; n-place predicates; term variables; terms; the usual logical connectives '$\sim$' and '$\supset$'; the quantifier '$\exists$'; and the two parentheses '(' and ')'. The connectives '&', '$\vee$', and '$\equiv$' are defined as usual. As syntactical notation, 'x', 'y', and 'z' are used to refer to any term variable; 'a', 'b', and 'c' are used to refer to any term; and the logical constants and parentheses are used to refer to themselves. The letters 'A', 'B', and 'C' are used to refer to pseudo-formulas. The pseudo-formulas are: atomic formulas, n-place predicates followed, respectively, by n term variables or terms; $\sim A$; $(A \supset B)$; $(x)A$ and $(\exists x)A$. An occurrence of a term variable x in A is bound in A if it occurs in a part $(x)B$ or $(\exists x)C$ of A; if an occurrence of a term variable is not bound in A, it is free. Notation like $(a/x)A$ is used to refer to any result of replacing every free occurrence of a term variable x in A by some term a. If x does not occur free in A, then $(a/x)A$ is A. The formulas are a subset of the pseudo-formulas; they are the pseudo-formulas in which no term variable occurs free. The universal and existential formulas are, respectively, formulas like $(x)A$ and $(\exists x)A$. An instance of a universal formula $(x)A$ or an existential formula $(\exists x)A$ is $(a/x)A$ with a being the instantial term.

3 Prefixed proofs The proof technique I use follows Fitch's with the major exception of prefixed proofs. Proofs as well as formulas may be items of proofs, but pseudo-formulas that are not formulas may not. Thus, no term variable occurs free in any item of a proof. Only prefixed proofs are described below. For a more complete account of the proof method, see the Appendix.

A main proof, as well as a subordinate proof, may be either regular or prefixed by a term. In a proof prefixed by a term, the term appears to the left and near the top of the vertical line bordering the column of items of the proof. No formula preceding and no direct consequence of a proof prefixed by a term may contain that term.[1] Thus, in constructing a subordinate proof prefixed by a term, one must take care (i) to pick a term which does not occur in any formula preceding the subproof and (ii) not to claim that a formula which contains the term is a direct consequence of the proof. In constructing a main proof prefixed by a term, such caution is not needed as no main proof is an item of any other proof.

If a term is prefixed to either a main or a subordinate proof, the term is assumed, within that proof and any proof subordinate to it, to refer to an existent. If a term prefixed to a main proof is assigned a referent, it is understood to refer to that referent within that proof and any proof subordinate to it. The situation is somewhat different for subordinate proofs. No formula preceding a subordinate proof prefixed by a term may contain the term. Because of this restriction, the prefixed term may be replaced by any term which refers to an existent and the subproof be valid. A term prefixed to a subordinate proof, unlike a term prefixed to a main proof, stands in for any term that refers to an existent. Thus, the referent (if any) of a term prefixed to a subproof is irrelevant just as the referent (if any) of 'John Doe' is irrelevant in informal reasoning. Once the function of terms prefixed to subordinate proofs is understood, the restriction on any formula in which a term occurs being a direct consequence of a subproof prefixed by the term seems quite natural.

4 Quantifier rules The rules for the logical connective '$\sim$' and '$\supset$' and the rule of reiteration are given in the Appendix. The quantifier rules are given below.

Rule of Universal Quantifier Elimination ("u q elim") An instance of a universal formula is a direct consequence of the universal formula in a proof prefixed by the instantial term or subordinate to one that is. This rule may be represented graphically as follows:

$$\begin{array}{r|l} a & \underline{(x)A} \\ & (a/x)A. \end{array}$$

Rule of Universal Quantifier Introduction ("u q int") A universal formula is a direct consequence of a categorical subproof which contains one of its instances and is prefixed by the instantial term. This rule may be expressed graphically as follows:

$$\begin{array}{|l} \begin{array}{r|l} a & . \\ & . \\ & . \\ & (a/x)A \end{array} \\ (x)A. \end{array}$$

Rule of Existential Quantifier Elimination ("e q elim") A formula B is a direct consequence of an existential formula and a subproof prefixed by a term not occurring in B (i) which has as its only hypothesis an instance of the existential formula with the prefixed term as its instantial term and (ii) which contains the formula B. This rule may be represented graphically as follows:

$$\begin{array}{|l} \underline{(\exists x)A} \\ \begin{array}{r|l} a & \underline{(a/x)A} \\ & . \\ & . \\ & . \\ & B \end{array} \\ B. \end{array}$$

Rule of Existential Quantifier Introduction ("e q int") An existential formula is a direct consequence of an instance of the existential formula in a proof prefixed by the instantial term or in a proof subordinate to one that is. This rule may be represented graphically as follows:

$$\begin{array}{l|l} a & (a/x)A \\ \hline & (\exists x)A. \end{array}$$

The rules for universal quantifier elimination and existential quantifier introduction should be obvious; in the other two rules, however, a formula is a direct consequence, not just of a formula in a prefixed proof or a formula in a proof subordinate to one, but of a prefixed subproof (as in u q int) or of a formula and a prefixed subproof (as in e q elim). By the rule of reiteration, a subproof may have as items the preceding formulas of the proof to which it is subordinate. By the definition of a prefixed proof, none of these formulas and no direct consequence of the prefixed proof can contain the prefixed term. From this, it follows that if $(a/x)A$ is obtainable in a categorical subproof prefixed by a, the result of replacing a in $(a/x)A$ by any term that refers to an existent (if there are any) is obtainable in the proof to which the prefixed proof is subordinate. Thus, the rule of universal quantifier introduction says, in effect, that $(x)A$ may be entered as an item of a proof if it is shown that $(a/x)A$ is obtainable in the proof for any term a that refers to an existent (if there are any). And the rule of existential quantifier elimination says, in effect, that B may be entered as an item of a proof if: (i) it is shown that B would be obtainable in the proof if $(a/x)A$ were an item of the proof for any term a that refers to an existent (if there are any) and (ii) $(\exists x)A$ is a preceding item in the proof.

Unlike many formalizations of free logic, the system formed, by the rules above, does not contain either '=' or '$\exists !$' as a primitive sign. Thus, 'term a refers to an existent', usually rendered as '$(\exists x)(x = a)$' or '$\exists ! a$', is not expressible in the language. Yet, prefixing a term a to a proof indicates that within the proof a is assumed to refer to an existent and thus within the proof serves the same purpose as $(\exists x)(x = a)$. In a proof prefixed by the term a, $(a/x)A$ is a direct consequence of $(x)A$ and $(\exists x)A$ is a direct consequence of $(a/x)A$. In systems which use identity to express existence, to reach the same conclusions proofs at least as complicated as the following are required:

$$\begin{array}{r|l} 1 & (x)A \\ 2 & (\exists x)(x = a) \\ \hline 3 & (a/x)A \end{array} \qquad \begin{array}{r|l} 1 & (a/x)A \\ 2 & (\exists x)(x = a) \\ \hline 3 & (\exists x)A. \end{array}$$

*5 **A universally free logic*** The language of $S1$ is the language L described in Section 2. The proof technique is that described in the Appendix and Section 3. The natural deduction rules for $S1$ are the rules for the logical connectives and the rule of reiteration in the Appendix and the quantifier rules in Section 4. A formula is a theorem of $S1$ if and only if it is the last item of a regular main proof that is categorical. The following will be proven as theorems of $S1$:

T1 $A \supset (x)A$

T2 $(x)(A \supset B) \supset ((x)A \supset (x)B)$

T3 $(y)((x)A \supset (y/x)A)$

T4 $(x)(y)A \supset (y)(x)A$.

Proofs:

T1	1	$\mid\ \mid\ A$	hyp
	2	$\mid\ \mid\ a \mid\ A$	1, reit
	3	$\mid\ \mid\ (x)A$	2, u q int
	4	$\mid\ A \supset (x)A$	1–3, cond int

Item 2 is $(a/x)A$ as x does not occur free in A.

T2	1	$\mid\ \mid\ (x)(A \supset B)$	hyp
	2	$\mid\ \mid\ \mid\ (x)A$	hyp
	3	$\mid\ \mid\ \mid\ a \mid\ (x)(A \supset B)$	1, reit
	4	$\mid\ \mid\ \mid\ \ \ \mid\ (x)A$	2, reit
	5	$\mid\ \mid\ \mid\ \ \ \mid\ (a/x)A \supset (a/x)B$	3, u q elim
	6	$\mid\ \mid\ \mid\ \ \ \mid\ (a/x)A$	4, u q elim
	7	$\mid\ \mid\ \mid\ \ \ \mid\ (a/x)B$	5, 6, mp
	8	$\mid\ \mid\ \mid\ (x)B$	3–7, u q int
	9	$\mid\ \mid\ (x)A \supset (x)B$	2–8, cond int
	10	$\mid\ (x)(A \supset B) \supset ((x)A \supset (x)B)$	1–9, cond int

Item 5 is $(a/x)(A \supset B)$.

T3	1	$\mid\ a \mid\ \mid\ (x)A$	hyp
	2	$\mid\ \ \ \mid\ \mid\ (a/x)A$	1, u q elim
	3	$\mid\ \ \ \mid\ (x)A \supset (a/x)A$	1–2, cond int
	4	$\mid\ (y)((x)A \supset (y/x)A)$	1–3, u q int

As y does not occur free in $(x)A$, $(a/y)(x)A$ is $(x)A$ and $(a/y)((y/x)A)$ is $(a/x)A$. Thus, item 3 is $(a/y)((x)A \supset (y/x)A)$.

T4	1	$\mid\ \mid\ (x)(y)A$	hyp
	2	$\mid\ \mid\ a \mid\ (x)(y)A$	1, reit
	3	$\mid\ \mid\ \ \ \mid\ b \mid\ (x)(y)A$	2, reit
	4	$\mid\ \mid\ \ \ \mid\ \ \ \mid\ (b/x)(y)A$	3, u q elim
	5	$\mid\ \mid\ \ \ \mid\ \ \ \mid\ (a/y)(b/x)A$	4, u q elim
	6	$\mid\ \mid\ \ \ \mid\ (a/y)(x)A$	3–5, u q int
	7	$\mid\ \mid\ (y)(x)A$	2–6, u q int
	8	$\mid\ (x)(y)A \supset (y)(x)A$	1–7, cond int

Item 4 is $(y)(b/x)A$, item 5 is $(b/x)(a/y)A$, and item 6 is $(x)(a/y)A$.

The theorems T1–T4 of $S1$ that have just been proven are axioms A1–A4 of the universally free logic of Lambert in [4].[2] Following Leblanc, I call this system QC^*. Like that of $S1$, the language of QC^* contains neither identity nor the existence symbol '$\exists!$'. In addition to A1–A4, QC^* has axioms A0 and A5 and the rule of *modus ponens*. The axioms of QC^* in virtue of

A0 A, if A is a tautology

are theorems of $S1$ in virtue of the definitions of connectives in $S1$ and the rules for conditional and indirect proof. The axioms of QC^* in virtue of

A5 $(x)A$, if $(a/x)A$ is an axiom (where a does not occur in A)

are also theorems of $S1$ in that: (i) axioms of QC^* in virtue of A0–A4 are theorems of $S1$; (ii) $(x)A$ is an axiom of QC^* in virtue of A5 if $(a/x)A$ is an axiom of QC^*, where a does not occur free in A; and (iii) $(x)A$ is a theorem of $S1$ by the rule of universal quantifier introduction if $(a/x)A$ is a theorem of $S1$, where a does not occur free in A.

As all of the axioms of QC^* are theorems of $S1$ and the only rule of QC^* is modus ponens (which is also a rule of $S1$), any formula provable as a theorem of QC^* is provable as a theorem of $S1$. In short, it follows that all theorems of QC^* are theorems of $S1$.

In this paper, I do not describe a semantics for $S1$ or for either the non-universally free or the nonfree systems generated from it. I save a detailed discussion of semantics and proofs of completeness and soundness of these systems for another occasion. However, because the language of $S1$ and the language of QC^* are the same and the theorems of QC^* are theorems of $S1$, it should be noted that the semantics provided for QC^* by either Leblanc and Meyer in [6] or Bencivenga in [1] (which differ) could be adopted as a semantics for $S1$. That $S1$ can be proven complete relative to either of these semantics should be obvious given Leblanc and Meyer's weak completeness proof of QC^* in [6] and Bencivenga's strong completeness proof of QC^* in [1].

6 A free logic for only nonempty domains A free logic valid for only nonempty domains (or a non-universally free logic) may be constructed by adding any one of three rules to $S1$: a rule of vacuous universal quantifier elimination, a rule of vacuous existential quantifier introduction, or a novel rule of existence assumption.[3]

Rule of Vacuous Universal Quantifier Elimination ("v u q elim") A is a direct consequence of $(x)A$, where x is not free in A.

Rule of Vacuous Existential Quantifier Introduction ("v e q int") $(\exists x)A$ is a direct consequence of A. (As pseudo-formulas that are not formulas cannot be items of proofs, the condition that x does not occur free in A is not needed as a restriction on the rule of vacuous existential quantifier introduction.)

Rule of Existence Assumption ("ex a") A formula which is an item of a categorical subordinate proof prefixed by a term is a direct consequence of the proof so long as the prefixed term does not occur in the formula. This rule may be represented graphically as follows:

a	.
	.
	B
$B.$	

The following shows that the rule of vacuous universal quantifier elimination is derivable from the rule of vacuous existential quantifier introduction. It is assumed that x is not free in A. Thus, item 4 is $(a/x)A$.

1		$(x)A$	
2		$(\exists y)(x)A$	1, v e q int
3	a	$(x)A$	hyp
4		A	3, u q elim
5		A	2, 3-4, e q elim

The following shows that the rule of existence assumption is derivable from the rule of vacuous universal quantifier elimination. It is assumed that a does occur in B. As x does not occur free in B, item m is $(a/x)B$.

1	a	.	
		.	
m		B	
n		$(x)B$	1-m, u q int
o		B	n, v u q elim

The rule of vacuous existential quantifier introduction is derivable, in turn, from the rule of existence assumption. As x does not occur free in A, item 2 is $(a/x)A$. As 2-3 is a proof prefixed by a, a does not occur in A.

1		A	
2	a	A	1, reit
3		$(\exists x)A$	2, e q int
4		$(\exists x)A$	2-3, ex a

Of these three rules, I prefer the rule of existence assumption. Unlike the two rules for vacuous quantifiers, the rule of existence assumption can be defined in a language which excludes vacuous quantifiers. It also makes the assumption of existence explicit and more intuitive. The system which results from adding the rule of existence assumption to $S1$ I call $S2$. The following formulas which are characteristic theorems of non-universally free logic are provable as theorems of $S2$:

T5 $(x)A \supset A$

T6 $(\exists x)(A \vee \sim A)$, where x is free in A.

Theorem T5 would not be provable, of course, if the language of $S2$ excluded vacuous quantifiers.

Instead of adding one of the three rules discussed above to $S1$ in order to get a free logic valid for only nonempty domains, I could adopt the following convention: In a system in which something is assumed to exist, the last item of a categorical main proof prefixed by exactly one term is a theorem so long as the prefixed term does not occur in that item. This convention is reliable for reasons analogous to those justifying the rule of existence assumption. Within a proof prefixed by a term, the term is assumed to refer to an existent; but if no item in which the term occurs is taken as a theorem, the assumption that the term refers to an existent is limited to the proof. If an item which does not contain the term is taken as a theorem, at most the assumption that something exists is taken out of the proof into the system. As this assumption is already in the system, however, no harm is done.

A logic is free if not all terms are assumed to refer to existents and nonfree if all are. So far, I have formulated two free logics: a universally free logic and a free logic valid for only nonempty domains. Although, perhaps, of little interest, there are, nevertheless, an infinite number of other non-universally free logics which are valid only if some specific term or terms refer to existents, but not all. These free logics may be formed from $S1$ by allowing as theorems last items of categorical main proofs prefixed by the term or terms assumed to refer to existents.

7 A nonfree logic In $S1$, only the last item of a regular, categorical main proof is a theorem. By also allowing the last item of a prefixed, categorical main proof to be a theorem, we have a nonfree logic. This system I call $S3$. To illustrate how this works, I shall prove the following as theorems of $S3$:

T7 $(x)A \supset (a/x)A$

T8 $(a/x)A \supset (\exists x)A$.

Proofs:

T7	a	1	$\mid\_$ $(x)A$	hyp
		2	$\mid$ $(a/x)A$	1, u q elim
		3	$(x)A \supset (a/x)A$	1-2, cond int
T8	a	1	$\mid\_$ $(a/x)A$	hyp
		2	$\mid$ $(\exists x)A$	1, e q int
		3	$(a/x)A \supset (\exists x)A$	1-2, cond int

T7 is *Specification* and T8 is *Particularization.* Both are characteristic theorems of nonfree logic.

The class of theorems provable in $S3$ will *not* be reduced, if the following requirement is adopted:

Prefixing Requirement A main proof is prefixed by all terms which occur in items of the main proof or items of proofs subordinate to it, with the exception of those terms which are prefixed to subordinate proofs.

Once the prefixing requirement is adopted, the phrase 'in a proof prefixed by the instantial term or subordinate to one that is' which occurs in the rules of universal quantifier elimination and existential quantifier introduction may be omitted. With this simplification, the quantifier rules, with the exception of using prefixed subproofs instead of general subordinate proofs, are similar to those found in other Fitch-style formulations of nonfree logic. As the last item of a categorical main proof is a theorem of $S3$ regardless of whether the proof is regular or prefixed, the actual listing of required prefixes may be left as understood. With the listing of prefixes left as understood, proofs in $S3$ are similar to those of other Fitch-style formulations of nonfree logic and indistinguishable from those using the unrestricted rules in [7].

Appendix A proof is a column of items bordered on the left by a vertical line extending the length of the column. Each item is either a formula or a subordinate proof. A proof is subordinate to any proof in which it is an item, or an item of an item, and so on, but is directly subordinate only to the proof in which it is itself an item. A formula which is an item of a proof is: (i) a hypothesis, (ii) a direct consequence of preceding items of the proof by one of the rules of direct consequence, or (iii) in case the proof is a subordinate proof, a reiterate by the rule of reiteration of a preceding formula in the proof to which it is subordinate. The hypotheses, if any, are the first items of the column and are separated from the others by a short horizontal line extending out to the right of the vertical line which borders the column. A proof with hypotheses is a hypothetical proof; a proof without, is categorical. A main proof, as well as a subordinate proof, may be either a regular proof, as described above, or a prefixed proof, as described in Section 3.

The rules for the logical connectives '$\sim$' and '$\supset$' and the rule of reiteration are the following:

Rule of Modus Ponens ("m p") The consequent of a conditional is a direct consequence of the conditional and its antecedent.

Rule of Conditional Introduction ("cond int") A conditional is a direct consequence of a (regular) subordinate proof which has the antecedent of the conditional as its only hypothesis and the consequent of the conditional as one of its items.

Rule of Indirect Proof ("ind pr") A formula is a direct consequence of a (regular) subordinate proof which has the negate of that formula as its only hypothesis and a formula and its negate as items.

Rule of Reiteration ("reit") Any item of a proof may be reiterated in a proof directly subordinate to it.

The rules of modus ponens and reiteration should be obvious; in the other two rules, however, a formula is a direct consequence, not of just a pair of formulas but of a regular subordinate proof with one hypothesis. By the rule of reiteration, a subproof with a hypothesis may have as additional items the preceding formulas of the proof to which it is directly subordinate. From this, it follows that what is obtainable in a regular subproof with a hypothesis is what would be obtainable in the proof to which it is directly subordinate if the hypothesis were an item of that proof rather than of the subproof. Thus, the rule of conditional introduction, in effect, says: a conditional may be entered as an item of a proof if it has been shown previously that the consequent of the conditional would be obtainable in that proof if its antecedent were an item of the proof. And the rule of indirect proof, in effect, says: a formula may be entered as an item of a proof if it has been shown previously that a formula and its negate would be obtainable in that proof if the negate of the formula to be entered were an item of the proof.

NOTES

1. Subproofs prefixed by a term are similar in appearance to and serve the same purpose as Fitch's general subproofs with respect to y ([3], pp. 130–131). However, as it is stipulated from the start that no formula preceding and no direct consequence of a prefixed proof can contain the prefixed term, quantifier rules formulated in terms of prefixed proofs do not require the various restrictions on reiterates, hypotheses, and direct consequences that accompany formulations in terms of general subordinate proofs. In short, prefixed proofs are more efficient.
2. Until Kit Fine showed that $(x)(y)A \supset (y)(x)A$ is independent of the other axioms of QC^* ([2], pp. 335–337), A4 was presumed derivable from them.
3. A rule for existential formulas analogous to the rule of universal quantifier introduction could also be used.

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