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Simplicity

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1 Introduction A fairly common practice in today's mathematics is to introduce a theory by 'giving' a first-order language (with a denumerable set of individual variables) $L = L(\mathbf{F},\mathbf{R},r)$ (where $\mathbf{F}$ and $\mathbf{R}$ are two finite collections of symbols, the former called *function-* or *operation*-symbols, the latter *predicate-* or *relation*-symbols; $r: \mathbf{F} \cup \mathbf{R} \to \mathbb{N}$ is a function assigning to each symbol a natural number, its 'arity') and a set Σ of sentences in L. This is done with the aim of introducing special structures for L, called *models.* A *structure* $\mathfrak{A}$ for L has the following components ([55]):

(i) a nonvoid set $u(\mathfrak{A})$, called the universe of $\mathfrak{A}$
(ii) for all $f \in \mathbf{F}$ an $r(f)$-ary function $f_{\mathfrak{A}}: u(\mathfrak{A})^{r(f)} \to u(\mathfrak{A})$
(iii) for all $R \in \mathbf{R}$ an $r(R)$-ary relation $R_{\mathfrak{A}}$ on $u(\mathfrak{A})$, i.e. $R_{\mathfrak{A}} \subseteq u(\mathfrak{A})^{r(R)}$.

A structure $\mathfrak{A}$ for L is called a *model* of a set Σ of sentences in L iff all $\alpha \in \Sigma$ are true in $\mathfrak{A}$ (see [73], [75] for the meaning of 'true').

For example, in *Group Theory*, $\mathcal{G}$, $\mathbf{F} = \{\mu\}$, $\mathbf{R} = \varnothing$, $r(\mu) = 2$, $\Sigma = \{A,B\}$, where

A $\forall xyz\ \mu(x\mu(yz)) = \mu(\mu(xy)z)$
B $\exists e \forall x \exists x'\ \mu(ex) = x \wedge \mu(x'x) = e.$

In most cases Σ is required to be either finite or recursive (see [24] for a definition of this term) and is referred to as an axiom system (AS).

The reasons for accepting these sentences as 'axioms' have been an issue ever since. The conservative view is that of Aristotle, who claims that the axioms must be known by an infallible intuition (*Analytica posteriora* II. 29, 100^b6). From this point of view, these days, both mathematicians and philosophers attack (sometimes, [14], most violently) the formalist point of view, that the AS—as well as the language, i.e. the collections $\mathbf{F}$ and $\mathbf{R}$—may be freely chosen, subject to the modest requirement of consistency, i.e. noncontradiction. If this sort of criticism comes from an intuitionist, to whom a theory is not constructed within a logical system, but by a creative cognitive process (close to what

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is called (e.g. in [12]) the *genetic method*), we should accept his point of view just because it corresponds to the facts, and thus is true, according to Tarski's ([73], [75]) definition of truth. Let's get down to facts: mathematics was built up by the genetic method, so its historical roots have nothing to do with the 'axiomatic method' (see [12], [17], [61]-[64]). At the origin of geometry, we do not find axioms; understanding space does not mean producing axioms (see [30]). Moreover, it lies in the very nature of the foundations that they don't come first, but last. "*Denn es liegt im Wesen der Fundamente, daß sie nur im Rückschauen von dem aus, was auf ihnen beruht, sichtbar werden können*", writes Hartmann [21].[1] We have no example of any interesting mathematical theory that came into being axiomatically "*out of nothing*".

To the intuitionist, AS's may have no (or, [25], a very restricted) meaning; we do not polemize with him, since he doesn't accept classical logic as well. Moreover, for Brouwer ([6]), Euclidean geometry, as it stands, is contradictory.

Asking for the 'right' axioms, i.e. not for freely chosen axioms but for axioms 'revealed' by some 'infallible intuition', in order to deduce a theory from that AS with the aid of classical logic, implies that one does not know the consequences of (at least some of) the axioms, thus maintaining the fiction of axioms miraculously producing a very interesting theory.

Since there can be no such thing as 'the right AS', one may wonder whether there is any requirement – besides consistency – imposed on an AS. For Tarski ([74]), "*No fundamentally theoretical considerations decide upon the choice of a system determined by primitive terms and axioms among all equivalent systems: the reasons are rather of a practical, educational and even aesthetical nature*". We shall insist, in the present article, only on those of an aesthetical nature, being concerned with the simplicity of the AS's and of the languages (the "primitive terms") in which these are expressed.

Some will think this a bogus problem, because a 'simpler' AS does the same job as a more 'complicated' one; moreover, for its 'classroom user' it most often becomes much more difficult to arrive at the 'results' of that theory if one prefers a 'simpler' AS to a more 'complicated' one. However, there are about one hundred papers and books (over a period of eighty years) which deal with 'simplifications' of Hilbert's ([27]) AS for Euclidean geometry, therefore some mathematicians do look at the problem of simplicity seriously. In all these papers the concept of 'simplicity' is a *relative* one, the proposed AS's being simpler than another AS; in any such case it is easy to establish that one is simpler than the other (their authors either 'weaken' – by replacing an axiom by a particular instance of it – one or more axioms or prove that one is superfluous). What is the aim of these simplifications? That is, where does one stop? When is an AS most simple, not significantly simplifiable any longer? In order to provide an answer to these questions, we must have an *absolute* criterion of simplicity.

For clarity's sake, we shall restrict our attention to the case of first-order AS's, for which we shall provide several simplicity criteria, tested on *Euclidean Geometry* (EG) and *Group Theory* (GT).

Hilbert's ([27]) second-order AS for Euclidean space-geometry intends to describe a unique model, namely the Cartesian three-dimensional space constructed over the field of real numbers, $\mathfrak{C}_3(\mathbb{R})$. By a curious property of first-order logic, called the *Löwenheim-Skolem theorem* (a first-order theory having an infinite model, has models in all infinite powers), the above 'categoricity' ideal is unattainable within any first-order language. The first first-order axiomatization of Euclidean geometry was given by Tarski ([79]) and will be dealt with in the next paragraph.

2 L_{BD}-axiomatization of Cartesian spaces The n-dimensional Cartesian space over an ordered field F is the following structure:

$$\mathfrak{C}_n(F) = (F^n, B_F, D_F)$$

where

$$B_F = \{(a,b,c) \in (F^n)^3 \mid \exists t \in F,\ 0 \leq t \leq 1,\ a - b = t(a-c)\}$$

and

$$D_F = \left\{ (a,b,c,d) \in (F^n)^4 \,\middle|\, \sum_{i=1}^{n} (a_i - b_i)^2 = \sum_{i=1}^{n} (c_i - d_i)^2 \right\}.$$

First-order n-dimensional Euclidean geometry (or simply "*elementary geometry*" [79]) may be naturally defined as $\mathcal{E}_n'' = \mathrm{Th}_{L_{BD}}\mathfrak{C}_n(\mathbb{R})$, where $L_{BD} = L(\mathbf{F},\mathbf{R},r)$, with $\mathbf{F} = \emptyset$, $\mathbf{R} = \{B, D\}$, $r(B) = 3$, $r(D) = 4$, and '$B(abc)$', '$D(abcd)$' may be read '*b lies between a and c*' and '*a is as distant from b as c is from d*' (or alternatively '*the segment ab is equal in length to the segment cd*') respectively. By $\mathrm{Th}_{L_{BD}}\mathfrak{C}_n(\mathbb{R})$ we mean 'the theory containing all the L_{BD}-sentences true in $\mathfrak{C}_n(\mathbb{R})$'. An AS for $\mathcal{E}_2''$ is the following (a variant of the AS in [79], to be found in [69]):

A1 $\forall abc\ B(abc) \rightarrow B(cba)$
A2 $\forall abcd\ B(abd) \wedge B(bcd) \rightarrow B(abc)$
A3 $\forall ab\ D(abba)$
A4 $\forall abc\ D(abcc) \rightarrow (a = b)$
A5 $\forall abcdef\ D(abcd) \wedge D(abef) \rightarrow D(cdef)$
A6 $\forall abca'b'c'pp'\ \neg(a = b) \wedge B(abc) \wedge B(a'b'c') \wedge D(aba'b') \wedge D(bcb'c') \wedge D(pap'a') \wedge D(pbp'b') \rightarrow D(pcp'c')$
A7 $\forall pacd \exists b\ B(pab) \wedge D(abcd)$
A8 $\forall ac \exists b\ B(abc) \wedge D(abbc)$
A9 $\forall abcde \exists f\ B(bcd) \wedge B(cea) \rightarrow B(fed) \wedge B(bfa)$
A10 $\exists abc\ \neg(B(abc) \vee B(bca) \vee B(cab))$
A11 $\forall abcpq\ \neg(p = q) \wedge D(apaq) \wedge D(bpbq) \wedge D(cpcq) \rightarrow B(abc) \vee B(bca) \vee B(cab)$
A12 $\forall abc \exists p\ \neg(B(abc) \vee B(bca) \vee B(cab)) \rightarrow D(papb) \wedge D(papc)$
A13 All sentences of the form

$$\forall vw \ldots \{\exists z \forall xy [\phi \wedge \psi \rightarrow B(zxy)] \rightarrow \exists u \forall xy [\phi \wedge \psi \rightarrow B(xuy)]\}$$

where ϕ stands for any formula in which the variables $x, v, w, \ldots$ but neither y nor z nor u, occur free, and similarly for ψ, with x and y interchanged.

A13 is an axiom schema, containing infinitely many axioms. Tarski ([79]) and Montague ([43]) proved that $\mathcal{E}_n''$ ($n \geq 2$) is not finitely axiomatizable. An AS for $\mathcal{E}_n''$ is obtained by a suitable replacement of the dimension axioms A10 and A11. Tarski also states ([79]) and proves ([57]) the following

Representation theorem $\mathfrak{M} \in \mathrm{Mod}(\mathcal{E}_n'')$ *iff* $\mathfrak{M} \simeq \mathfrak{C}_n(F)$, *where F is a real closed field* (see [80] for a definition).

A great deal of traditional Euclidean geometry can already be done in $\mathcal{E}_2 = \mathrm{Cn}(\mathrm{A1}\text{–}\mathrm{A12})$; however, the sentence '*the circle, drawn with radius greater than the distance from its center to a given line, intersects that line*' does not belong to $\mathcal{E}_2$. A theory in which all elementary (straightedge and compass) constructions can be performed is $\mathcal{E}_2' = \mathrm{Cn}(\mathrm{A1}\text{–}\mathrm{A8}, \mathrm{A9}_1, \mathrm{A10}\text{–}\mathrm{A12})$, with

A9$_1$ $\forall abcd \exists e\, B(abc) \rightarrow B(dbe) \wedge D(aeac)$.

The representation theorems for $\mathcal{E}_2$ and $\mathcal{E}_2'$ are

(i) $\mathfrak{M} \in \mathrm{Mod}(\mathcal{E}_2)$ *iff* $\mathfrak{M} \simeq \mathfrak{C}_2(F)$, *where F is a Pythagorean ordered field*
(ii) $\mathfrak{M} \in \mathrm{Mod}(\mathcal{E}_2')$ *iff* $\mathfrak{M} \simeq \mathfrak{C}_2(F)$, *where F is a Euclidean ordered field.*

An ordered field is called *Pythagorean* if

$$\forall xy \exists z\, x^2 + y^2 = z^2$$

and *Euclidean* if

$$\forall x \exists y\, x \geq 0 \rightarrow x = y^2.$$

There is, of course, no specific reason for accepting 'points' as individual variables and the two relations 'Betweenness' and 'Equidistance' as undefined notions for EG. One may put forward other collections **F** and **R** and the interpretation of the individual variables may be other than 'points'. How does one decide that those AS's axiomatize EG?

Suppose that, given two theories, $\mathfrak{T}_1$ and $\mathfrak{T}_2$, we want to decide whether they describe the 'same' phenomena or not, to put it differently, whether they are 'equivalent'. If the intended interpretation of the individual variables is the same (e.g. 'points') both in $\mathfrak{T}_1$ and $\mathfrak{T}_2$, then their 'equivalence' is called "synonymity"; it was defined by de Bouvère ([9], [10]); if different (e.g., 'points' and 'lines'), then their 'equivalence' is called "mutual interpretability", as defined by Szczerba ([67]).

Given any AS Σ in a language L, we should, in principle, be able to tell whether $\mathrm{Cn}_L(\Sigma)$ is either synonymous or else mutually interpretable with any of the $\mathcal{E}$'s (or with $\mathcal{G}$), therefore whether it might be regarded as an AS for EG (or for GT). There also is, as pointed out by Prażmowski ([51]), a group-theoretical way to see that a theory is equivalent to one of the $\mathcal{E}$'s.

3 Simplicity criteria A most natural requirement—going back to William of Ockham's "razor" ("*Frustra fit per plura quod potest fieri per pauciora*")—is that each axiom in a finite AS is independent of all the remaining ones.[2] Such an AS will be called completely independent. Completely independent AS's may often be 'simplified' by replacing an axiom with a special instance of it. We do not know whether the AS proposed for $\mathcal{E}_2$ is completely independent or not (completely independent AS's for $\mathcal{E}_2$, $\mathcal{E}_2'$, and $\mathcal{E}_2''$ may be found in [54], [47], [20]), but A7, although independent of the remaining axioms, may be replaced (see e.g. [46]) by its 'special instance'

A7′ $\forall pad\, \exists b\; B(pab) \wedge D(abad)$.

So complete independence is a necessary yet insufficient criterion for simplicity.

Given as AS $\Sigma = \{A_1, A_2, \ldots\}$, how can we check that Σ is completely independent, or equivalently, how can we check whether an axiom $A_i \in \Sigma$ is (or isn't) independent of $\Sigma_{A_i} := \Sigma \setminus \{A_i\}$? A classical way to do this is to provide an independence model for A_i, i.e. a model for Σ_{A_i} and $\neg A_i$. Unfortunately, there is no general method for constructing independence models, therefore their construction relies largely on how lucky we are in our obscure search for them. Even worse, if A_i happens to depend on Σ_{A_i}, the search is doomed to futility from the very beginning. If Σ is infinite, there is no issue. If, however, $\Sigma = \{A_1, A_2, \ldots, A_n\}$, then A_i is independent of Σ_{A_i} iff $\bigwedge_{j \neq i} A_j \to A_i$ is not a tautology. The algorithmic solution of such problems is dealt with in [4], [7], [13], [38], [82], but in most cases of practical interest the number of steps to be performed in order to get an answer is discouragingly large.

(a) Syntactical criteria

The first two criteria apply only to finitely axiomatizable (f.a.) theories. The first, due to Weaver ([81]), is:

An f.a. theory $\mathfrak{T}$ (in L) has *(1)-simplicity degree m* if *m* is the least integer for which there exists an AS for $\mathfrak{T}$, all of whose axioms contain no more than *m* individual variables. Such an AS will be called *(1)-simple.*

$\mathcal{E}_n'$ has (1)-simplicity degree 5 for n = 2 and n + 2 for all n ≥ 3. $\mathcal{E}_2$ has (1)-simplicity degree 6. The corresponding (1)-simple AS's were proposed in [46]. Any theory with $\mathbf{F} = \varnothing$, which is synonymous with $\mathcal{E}_n'$ (n ≥ 3), must necessarily have (1)-simplicity degree ≥n + 2 (see [59], [20]), but there is a theory, synonymous with $\mathcal{E}_2$, with (1)-simplicity degree 5 (see [46]). For an f.a. theory $\mathfrak{T}$ in L with $\mathbf{F} = \varnothing$ one can (at least in principle) find its (1)-simplicity degree by listing all *i*-variables sentences in $\mathfrak{T}$ (there are finitely many such sentences because of $\mathbf{F} = \varnothing$) for $i = 1, 2, \ldots$. The first *i* for which these constitute an AS for $\mathfrak{T}$ is its (1)-simplicity degree and those sentences constitute (after eliminating redundant ones) a (1)-simple AS for $\mathfrak{T}$.

The (1)-simplicity degree of $\mathcal{G}$ is 3 (a (1)-simple AS being {A,B}), since all the 2-variables sentences in $\mathcal{G}$ are true in $\mathfrak{M} = (u(\mathfrak{M}), \mu_{\mathfrak{M}})$, where $u(\mathfrak{M}) = \mathbb{N}$ and $\mu_{\mathfrak{M}}(ab) = |a - b|$, and $\mathfrak{M}$ is not a group. GT may also be axiomatized by a ternary relation S instead of μ (with $S(xyz) \leftrightarrow \mu(xy) = z$), i.e. $\mathbf{R} = \{S\}$, $\mathbf{F} = \varnothing$ and $r(S) = 3$, with the following AS:

S1 $\forall xy \exists z \forall z'\; S(xyz) \wedge (S(xyz') \to (z' = z))$

S2 $\forall xyztuv\; S(xyt) \wedge S(yzv) \wedge S(tzu) \to S(xvu)$

S3 $\exists e \forall x \exists x'\; S(exx) \wedge S(x'xe)$.

$\mathcal{G}' = \mathrm{Cn}(\mathrm{S1–S3})$, which is synonymous with $\mathcal{G}$, has (1)-simplicity degree 6, since all the 5-variables sentences in $\mathcal{G}'$ are true in $\mathfrak{M}' = (u(\mathfrak{M}'), S_{\mathfrak{M}'})$, where $u(\mathfrak{M}') = \mathbb{N}$, $S_{\mathfrak{M}'} = \{(x,y,z) \mid x,y,z \in \mathbb{N},\, z = |x - y|\}$. An f.a. theory may also be axiomatized by a single sentence (obtained, e.g., by forming the conjunction of all the axioms of one of its finite AS's); hence the second criterion:

An f.a. theory $\mathfrak{T}$ is said to have *(2)-simplicity degree m* if *m* is the least integer for which there is a sentence σ, containing *m* variables, such that $\mathrm{Cn}(\sigma) = \mathfrak{T}$.

The (2)-simplicity degree of $\mathcal{E}_2'$ and $\mathcal{E}_2$ is 6 and (2)-simple sentences axiomatizing them can be found in [46]. To prove that there is no single 5-variables sentence to axiomatize $\mathcal{E}_2'$ we made use of a model-theoretic characterization of those theories which are axiomatizable by a single *m*-variables sentence (to be found in [71] and [41], p. 325). The (2)-simplicity degree of $\mathcal{G}$ is 4, the (2)-simple sentence being

$$\gamma.\ \forall a \exists b \forall cd\ \mu(a\mu(ba)) = a \wedge (\mu(a\mu(ca)) = a \rightarrow (c = b)) \wedge \mu(a\mu(cd)) = \mu(\mu(ac)d).$$

For $\mathrm{Cn}(\gamma) = \mathcal{G}$, see [22]. No single 3-variables sentence can axiomatize $\mathcal{G}$; for, if the 3-variables sentence contains at least one $\exists$-quantifier, it is true in $\mathfrak{M}$; if it contains no $\exists$-quantifier, it is true in $\mathfrak{M}_1 = (\mathrm{u}(\mathfrak{M}_1), \mu_{\mathfrak{M}_1})$, $\mathrm{u}(\mathfrak{M}_1) = \mathbb{N}$, $\mu_{\mathfrak{M}_1}(xy) = x + y$, both not groups.

These two simplicity criteria apply only to f.a. theories, so they are of no use to those which are known to be non-f.a., e.g. $\mathcal{E}_2''$. According to Kleene's ([33]) result (improved in [78] and [8]), however, any axiomatizable theory $\mathfrak{T}$ having infinite models is *finitely axiomatizable using additional predicates* (*f.a.*$^+$). A theory $\mathfrak{T}$ in L is called *f.a.*$^+$ if there is a theory $\mathfrak{T}'$ in $\mathrm{L}' = \mathrm{L}'(\mathbf{F},\mathbf{R}',\mathrm{r}')$, $\mathbf{R}' \supseteq \mathbf{R}$, $\mathrm{r}'|_{\mathbf{R}\cup\mathbf{F}} = \mathrm{r}$, such that $\mathfrak{T}'$ is f.a. and an arbitrary sentence in L is in $\mathfrak{T}$ iff it is in $\mathfrak{T}'$. It is always possible (see [78], [8]) to take $\mathbf{R}' = \mathbf{R} \cup \{P\}$, $\mathrm{r}'(P) = 2$. Since the proof of this theorem is 'nonconstructive', we do not know any finite AS for $(\mathcal{E}_2'')'$, nor what the interpretation of the binary predicate *P* should be in this case.

We obtain another syntactic 'beauty-criterion' by stipulating that all the axioms should have—when written in prenex form (which means 'quantifiers first')—a specific arrangement of the $\forall$- and $\exists$-quantifiers. One may, for example, be interested in avoiding "*ontological commitments*" (i.e. the $\exists$-quantifier), therefore demanding that the AS should contain only *universal* axioms. For theories $\mathfrak{T}$ in L (with $\mathbf{F} = \emptyset$) Tarski ([77]) gave the following model-theoretic characterization:

$\mathfrak{T}$ *is axiomatizable by a universal AS iff* $\mathrm{Mod}(\mathfrak{T})$ *is closed under substructures* (i.e., iff $(\mathfrak{A} \in \mathrm{Mod}(\mathfrak{T}) \wedge \mathfrak{B} \subseteq \mathfrak{A}) \rightarrow \mathfrak{B} \in \mathrm{Mod}(\mathfrak{T})$). If $\mathfrak{T}$ happens to be f.a., then it is axiomatizable by a single universal sentence under the same conditions as those above.

This result does not apply to $\mathcal{G}$, which, nevertheless is not axiomatizable by a universal AS, all the universal sentences in $\mathcal{G}$ being true in $\mathfrak{M}_1$. None of $\mathcal{G}', \mathcal{E}_n, \mathcal{E}_n', \mathcal{E}_n''$ ($n \geq 2$) are closed under substructures, so they are not axiomatizable by universal AS's. Nevertheless, EG is axiomatizable by a universal AS, i.e. there are two theories $\mathcal{KE}_2$ and $\mathcal{KE}_2'$, synonymous with $\mathcal{E}_2$ and $\mathcal{E}_2'$ respectively, which are axiomatizable—in languages without predicate symbols—by a finite number of universal axioms. The news was reported in 1968 by Moler and Suppes ([42]). Some minor mistakes in [42] were corrected by Seeland ([60]), who also gave the first universal AS for $\mathcal{KE}_2'$. The language for $\mathcal{KE}_2'$ has $\mathbf{R} = \emptyset$, $\mathbf{F} = \{a_0, a_1, a_2, S, I, C\}$, $\mathrm{r}(a_0) = \mathrm{r}(a_1) = \mathrm{r}(a_2) = 0$ (i.e., a_0, a_1, a_2 are individual constants), $r(S) = r(I) = r(C) = 4$, the one for $\mathcal{KE}_2$ is the same, but without 'C'. The 'intuitive meaning' of them is:

for $I(xyuv) = w$, '*w is the point of intersection of the lines xy and uv, provided that* $\neg((u = v) \vee (L(xyu) \wedge L(xyv)) \vee Par(xyuv))$, *otherwise arbitrary*' (here $L(abc)$ stands for $B(abc) \vee B(bca) \vee B(cab)$ and $Par(xyuv)$ means that the lines xy and uv are parallel);

for $S(xyuv) = w$, '*the point w is as distant from u on the ray uv as y is from x, provided that* $\neg(u = v) \vee (u = v \wedge x = y)$, *otherwise arbitrary*';

for $C(xyuv) = w$, '*w is the point of intersection of the circle with center x that passes through y with the segment uv, provided that* $x \neq y$ *and u lies inside and v outside that circle, otherwise arbitrary*'.

GT, too, is axiomatizable, in a language with $\mathbf{R} = \emptyset$, $\mathbf{F} = \{/\}$, $r(/) = 2$, by a universal AS. This news was reported earlier, in 1952 (in [26]; see also [36] for a like-minded result). The axioms are:

HN1 $\forall xyz (x/z)/(y/z) = x/y$
HN2 $\forall xy (x/x)/((y/y)/y) = y$,

where x/y stands for $/(xy)$ and 'means' (in the notations below) $\mu(xf(y))$.

Any axiomatizable theory may be axiomatized, using some additional function symbols—via skolemization—by a universal AS. $\mathcal{KE}_2$ and $\mathcal{KE}_2'$ have the advantage—over the skolemized $\mathcal{E}_2$ and $\mathcal{E}_2'$—of being expressed in a language with $\mathbf{R} = \emptyset$. The theory obtained by skolemizing $\mathcal{G}$ is expressed in a language with $\mathbf{R} = \emptyset$, $\mathbf{F} = \{e, f, \mu\}$, with $r(e) = 0$, $r(f) = 1$, $r(\mu) = 2$, with A and

$\mathbf{B}_s$ $\forall x\, \mu(ex) = x \wedge \mu(f(x)x) = e$

as axioms.

Another kind of AS we may be interested in is the AS all of whose axioms are $\forall\exists$-sentences, i.e., all $\forall$-quantifiers (if any) precede all $\exists$-quantifiers (if any). A theory will be called *inductive* if it is axiomatizable by such an AS. For languages with $\mathbf{F} = \emptyset$, we have the following model-theoretic characterization of inductive theories (see [37]):

$\mathfrak{T}$ *is inductive iff* Mod($\mathfrak{T}$) *is closed under monotone countable unions* (i.e., $(\forall n \in \mathbb{N}\ \mathfrak{A}_n \in \text{Mod}(\mathfrak{T}) \wedge \mathfrak{A}_n \subseteq \mathfrak{A}_{n+1}) \rightarrow \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n \in \text{Mod}(\mathfrak{T})$).

The axioms we stated for $\mathcal{E}_2$ and $\mathcal{E}_2'$ are all $\forall\exists$-axioms, but A13 is not an $\forall\exists$-axiom. However, Mod($\mathcal{E}_2''$) is closed under monotone countable unions, therefore $\mathcal{E}_2''$ is inductive; it would be of interest to find a $\forall\exists$-AS for $\mathcal{E}_2''$.[3]

Both $\mathcal{G}$ and $\mathcal{G}'$ admit $\forall\exists$-AS's, which can be easily obtained by conveniently expressing the axioms proposed in [22].

Szmielew in [70] (see also [48]) gave an example of a noninductive theory, which is synonymous with an inductive one.

What if—perhaps influenced to some extent by [19]—we want to have a *positive* AS, i.e. an AS all of whose axioms are '*positive*' (that is, they contain the logical symbols $\wedge, \vee, \forall, \exists$, but not $\neg, \rightarrow, \leftrightarrow$)? The answer was given by Lyndon ([39]):

$\mathfrak{I}$ *is axiomatizable by a positive AS iff* Mod($\mathfrak{I}$) *is closed under homomorphic images* (i.e., ($\mathfrak{A} \in$ Mod($\mathfrak{I}$) $\wedge$ h: $\mathfrak{A} \to \mathfrak{B}$ onto homomorphism) $\to$ $\mathfrak{B} \in$ Mod($\mathfrak{I}$)).

Unlike $\mathcal{G}$, none of $\mathcal{E}_n,\mathcal{E}'_n,\mathcal{E}''_n,\mathcal{G}'$ are axiomatizable by positive AS's.

A *Horn sentence* (see [28], [18]) is a conjunction of sentences of the form

$$\theta_1 \vee \theta_2 \vee \ldots \vee \theta_n$$

where all the θ_i (except at most one, which may be an atomic formula) are negated atomic formulas. Galvin ([18]) proved the following model-theoretic characterization of theories admitting a finite AS, all of whose axioms are Horn sentences:

$\mathfrak{I}$ *is axiomatizable by a Horn sentence* (*or equivalently, by finitely many Horn sentences*) *iff* Mod($\mathfrak{I}$) *is closed under reduced products* (i.e., ($\forall$i $\in$ I $\mathfrak{A}_i \in$ Mod($\mathfrak{I}$) $\wedge$ D filter over I) $\to \prod_{i \in D} \mathfrak{A}_i \in$ Mod($\mathfrak{I}$), in the notation of [15]).

Since the classes of models of $\mathcal{E}_n,\mathcal{E}'_n$ are not closed under direct products, they are not axiomatizable by finitely many Horn sentences, whereas the AS's proposed for both $\mathcal{G}$ and $\mathcal{G}'$ consist of Horn sentences.

(b) Language-simplicity criteria

The problem we are interested in regards the simplicity of primitive notions, i.e., of the collections **F** and **R**. Let $\mathfrak{I}$ be a first-order theory in a language L with **F** $= \varnothing$ and let [$\mathfrak{I}$] be the class of those theories—in languages with **F** $= \varnothing$—which are synonymous with $\mathfrak{I}$. A finite collection **R** consisting of the predicate symbols of the language L(**R**,r) for a theory in [$\mathfrak{I}$] will be called *minimal* (according to Lindenbaum [35]) if, for any other language L(**R**$'$,r$'$) for a theory in [$\mathfrak{I}$], we have

$$\max_{R \in \mathbf{R}} \mathrm{r}(R) \leq \max_{R' \in \mathbf{R}'} \mathrm{r}'(R')$$

and, in case of equality, also $|\mathbf{R}| \leq |\mathbf{R}'|$.

For $\mathfrak{I} = \mathcal{E}''_n$ (or $\mathcal{E}_n$, or $\mathcal{E}'_n$, with n $\geq$ 2), there is no theory in [$\mathfrak{I}$], with collection of predicates consisting of binary predicates only, since both '*B*' and '*D*', unlike nontrivial binary predicates (see [66], [53]), are invariant to similarities, therefore '*B*' and '*D*' cannot be defined in terms of binary predicates. Robinson ([53]) proved the stronger result: "There is no theory, synonymous with $\mathcal{U}\mathcal{E}''_n$ (n $\geq$ 2), written in a language with binary predicates only; the language for $\mathcal{U}\mathcal{E}''_n$ has **F** $= \varnothing$, **R** $= \{B,D,U\}$ and $\mathcal{U}\mathcal{E}''_n$ = Cn($\mathcal{E}''_n$, U1 – U3), where

U1 $\forall xy\ U(xy)$
U2 $\forall xyzt\ U(xy) \wedge U(zt) \to D(xyzt)$
U3 $\forall xyzt\ U(xy) \wedge D(xyzt) \to U(zt)$."

Unlike Euclidean and hyperbolic geometry, elliptic geometry is axiomatizable using a single binary predicate with 'points' as individual variables (see [53]).

Each of $\{I\},\{J\},\{H\},\{S\}$ are minimal collections of predicate symbols for both [$\mathcal{E}''_n$] and [$\mathcal{E}'_n$], whereas $\{J\}$ is also a minimal collection for [$\mathcal{E}_n$] (with n $\geq$ 2). '*I*' was introduced by Pieri [49] (see also [53]) with $I(xyz) \leftrightarrow D(xyyz)$; '*J*' and '*H*' by Tarski ([76]) (see also [58]) with '$J(xyz)$' iff '*the distance between x and y is less than the distance between y and z*' and $H(xyz) \leftrightarrow (I(xyz) \vee$

$I(yzx) \vee I(zxy)) \wedge \neg((x = y) \vee (y = z) \vee (z = x))$; '$S$' by Scott ([58]), with '$S(xyz)$' iff '*x, y, z are (in any order) the vertices of a rectangular triangle*'. $\{E\}$ is a minimal collection for both $[\mathcal{E}_n'']$ and $[\mathcal{E}_n']$ (with $n \geq 3$), where $E(xyz) \leftrightarrow D(xyyz) \wedge D(yzzx) \wedge \neg((x = y) \vee (y = z) \vee (z = x))$; it was introduced in [3].

Although axiomatizable by ternary predicates among 'points', first-order EG is—unlike the elementary theory of real closed fields, which is mutually interpretable with the $\mathcal{E}''$'s—not axiomatizable by individual constants and binary operations on 'points', as shown in [45] (see also [16], pp. 77, 80 and [57], p. 343).

One recalls that the concept of 'synonymity' is too narrow to express the fact that two theories axiomatize the 'same' mathematical structure. Hilbert's ([27]) EG has three kinds of individual variables (for 'points', 'lines' and 'planes') and therefore, if expressed in a first-order language (as in [45]), is not synonymous with $\mathcal{E}_3''$.

Huntington ([29]) proved that second-order EG of dimension ≥ 3 is axiomatizable in a language with a single binary predicate R, the individual variables and '$R(ab)$' having the interpretation '*closed balls*' ("*solid spheres*") and '*a inside of b*', respectively. The same result, for first-order EG, was proved by Tarski ([72], [75]), with individuals being interpreted as '*open balls*' and by Jaśkowski [31] for '*closed balls*'. '*Lines*' as individual variables and $\mathbf{R} = \{\perp, \times\}$, $\mathbf{F} = \emptyset$, $r(\perp) = r(\times) = 2$, with '$\perp(ab)$' and '$\times(ab)$' meaning '*a and b are perpendicular and they intersect*' and '*a intersects b*' respectively, can axiomatize n-dimensional EG over Euclidean fields, with $n \geq 3$ (cf. [56]), each of '$\perp$' and '$\times$' being undefinable from the other for $n = 3$ (as shown in [34]; see also [2]), whereas '$\times$' is superfluous (being definable from '$\perp$') for $n \geq 4$, so the single binary predicate '$\perp$' suffices for $n \geq 4$ (cf. [56]). Plane EG with a universe of '*lines*' cannot be axiomatized using only binary relations; a ternary relation on '*lines*' is necessary ([56]); it is axiomatizable with $\mathbf{R} = \{\perp, \dot{\times}\}$, $\mathbf{F} = \emptyset$, $r(\dot{\times}) = 3$, with '$\dot{\times}(abc)$' meaning '*a, b, and c intersect at a single point*'. Makowiecka [40] proved that EG (of any dimension ≥ 1) over real closed fields may be axiomatized using a single symmetric binary relation between individuals to be interpreted as '*segments*'. Prażmowski ([50]) proved that plane EG over Euclidean fields can be axiomatized with a single binary predicate 'T' among individuals to be interpreted as '*circles*', '$T(ab)$' standing for '*a is tangent to b*'.

One sees that the results obtained for our $\mathcal{E}$'s can be improved by considering theories where individual variables have other 'intended interpretations'; instead of minimal collections consisting of one ternary predicate symbol, we get minimal collections containing one binary predicate symbol. This is the best result one may ever expect to obtain, since there is no nontrivial unary predicate.

'Circles' as individual variables may appear quite an artificial choice, but, according to Zeuthen ([83], p. 123), to the Greek they were *primitive* figures.

$\{S\}$ is a minimal collection for $\mathcal{G}'$, for, according to Lindenbaum ([35]) even Abelian groups cannot be axiomatized by binary relations.

(c) Semantical criteria

Unlike the syntactical criteria, which were thoroughly studied, no work has been done on the semantical ones; therefore this paragraph consists of nothing but open problems.

One of the first simplicity criteria for AS's (which we shall state in a modified [45] version) was proposed in 1935 by Helmer ([23]). According to this one, each axiom should have *"as little content as possible"*. Let Σ be an AS for $\mathfrak{T}$, A an axiom in Σ, written in prenex form $Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \alpha$, the Q_i's being either $\forall$ or $\exists$ and α quantifier-free, with free variables $x_1, x_2, \ldots, x_n$. A will be called *primary* iff for all sentences B and C,

$$(\mathrm{Cn}(\Sigma) = \mathrm{Cn}(\Sigma_A, B \wedge C) \wedge C \notin \mathrm{Cn}(\Sigma_A, B) \wedge B \notin \mathrm{Cn}(\Sigma_A, C)) \rightarrow \exists P_p((B \leftrightarrow A \vee P) \wedge (C \leftrightarrow A \vee \neg P)) \vee ((B \leftrightarrow Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \alpha \vee p) \wedge (C \leftrightarrow Q_1 x_1 Q_2 x_2 \ldots Q_n x_n \alpha \vee \neg p)),$$

where P is a sentence and p is a quantifier-free formula, with free variables among the x_i's. if $A \wedge A'$ is a primary axiom in $(\Sigma \backslash \{A, A'\}) \cup \{A \wedge A'\}$, where $A \in \Sigma$, $A' \in \Sigma$, then the two will be called *conjugate* axioms. A finite AS will be called *(3)-simple* if all its axioms are primary and there is no couple of conjugate axioms among them.

We do not know whether our AS for $\mathcal{E}_2$ is (3)-simple or not; the one for $\mathcal{E}_2'$ is not (3)-simple because $A9_1$ is not primary in that AS (see [69]). Nor is the AS for $\mathcal{G}$ (or $\mathcal{G}'$) (3)-simple, since B has a nontrivial decomposition in (see [80]):

B1 $\forall ab \exists x\ \mu(xa) = b$
B2 $\forall ab \exists x\ \mu(ax) = b$,

i.e., Cn(A,B1,B2) = Cn(A,B), B1 $\notin$ Cn(A,B2), B2 $\notin$ Cn(A,B1). I conjecture that {A,B1,B2} is (3)-simple. The AS for the betweenness relation on a line, proposed in [70] is an example of a (3)-simple AS.

The criterion we are going to propose applies only to *complete* theories. $\mathcal{E}_n''$, for example, is complete (see [79], [57]) and $\mathrm{Cn}(\Sigma_{A12})$, where $\Sigma = \{A1\text{–}A13\}$ has just two possible complete extensions (*completions*): $\mathcal{E}_2''$ and $\mathcal{H}_2''$ (complete hyperbolic two-dimensional geometry) (see [68], [79]). To ask the same phenomenon to happen to all axioms seems quite restrictive, since $\mathrm{Cn}(\Sigma_{A11})$ obviously has infinitely many completions, although we may replace A10 and A11 with two axioms A10′ and A11′, stating that the dimension is one or two, respectively two or three, in order to have both $\mathrm{Cn}(\Sigma_{A10'})$ and $\mathrm{Cn}(\Sigma_{A11'})$ with only two possible completions; but A10′ and A11′ could hardly be called simpler than A10 and A11.

According to a theorem by Lindenbaum (see [11]), every consistent theory has a completion. A complete theory $\mathfrak{T}'$ will be called a *finite completion* of $\mathfrak{T}$ if $\mathfrak{T}' = \mathrm{Cn}(\mathfrak{T} \cup \{\sigma\})$, for some sentence σ. Let $\alpha(\mathfrak{T})$ be the cardinal number of the set of all finite completions of $\mathfrak{T}$ and $n(\mathfrak{T})$ the cardinal number of all other completions of $\mathfrak{T}$. There are four distinct categories of consistent theories (see [11], [44]):

(a) *essentially incomplete* ($\alpha(\mathfrak{T}) = 0$, $n(\mathfrak{T}) = 2^{\aleph_0}$);
(b) *almost essentially incomplete* ($0 < \alpha(\mathfrak{T}) \leq \aleph_0$, $n(\mathfrak{T}) = 2^{\aleph_0}$);
(c) *virtually complete* ($0 < \alpha(\mathfrak{T}) < \aleph_0$, $n(\mathfrak{T}) = 0$);
(d) $\aleph_0$*-incomplete* ($\alpha(\mathfrak{T}) = \aleph_0$, $0 \leq n(\mathfrak{T}) \leq \aleph_0$).

Our new simplicity criterion is (see [45]):

An AS Σ for a complete theory $\mathfrak{T}$ will be called *(4)-simple* if $\mathrm{Cn}(\Sigma_A)$ is either virtually complete or $\aleph_0$-incomplete, for all $A \in \Sigma$ (i.e., $n(\mathrm{Cn}(\Sigma_A)) \leq \aleph_0$).

$\Sigma = \{A1\text{–}A13\}$ is not a (4)-simple AS for $\mathcal{E}_2''$. For, although $\alpha(Cn(\Sigma_{A12})) = 2$, $n(Cn(\Sigma_{A12})) = 0$, $\alpha(Cn(\Sigma_{A10})) = 3$, $n(Cn(\Sigma_{A10})) = 0$, $\alpha(Cn(\Sigma_{A11})) = \aleph_0$, $n(Cn(\Sigma_{A11})) = 1$ (see [59]), we have, according to Szczerba [65], $n(Cn(\Sigma_{A9})) = 2^{\aleph_0}$.

A last category of semantical simplicity criteria will be dealt with in the following lines.

According to a theorem of Robinson ([52]) $\mathcal{E}_2''$ is the 'model-completion' of $\mathcal{E}_2$, therefore, in a certain (metamathematical) sense, we may say that A13 was not 'freely chosen', but that its choice was 'forced' upon us by the other axioms and a metamathematical operation on them. Could this happen to all the axioms of an AS? Although we do not have any example of such an AS, we propose four (related) simplicity criteria.

An AS Σ for a theory $\mathfrak{T}$ will be called *(5ᵢ)-simple* if $(Cn(\Sigma_A))^{\alpha_i} = \mathfrak{T}$, for all $A \in \Sigma$; here i may be assigned precisely one of the values 1,2,3,4; for all theories $\mathfrak{T}$, $\mathfrak{T}^{\alpha_i}$ means 'the *model-completion* of $\mathfrak{T}$' (see [52] for a definition) for $i = 1$, 'the *model-companion* of $\mathfrak{T}$' (see [1] for a definition) for $i = 2$, 'the *forcing-companion* of $\mathfrak{T}$' (*see* [1]) *for* $i = 3$ *and* '*the inductive hull* of $\mathfrak{T}$' (see [32], [1]) for $i = 4$.

4 *Conclusions* Out of the eight papers presented in the Section of Logic at the 1935 International Congress of Scientific Philosophy, three dealt with simplicity problems for AS's. Today the subject is no longer encountered. As pointed out throughout this paper, these problems are far from being settled. So what should be the reason for this lack of interest in 'Axiomatology' (if we may give this name to that part of metamathematics which studies the aesthetical problems of AS's)? Is it due to the fact that most mathematicians would still answer with a quotation from Bolzano:

> *Diese Aufgabe däucht mir jedoch so schwer, und die Versuche, die man bis jetzt zu ihrer Lösung gemacht, scheinen mir so mißlungen, daß ich nicht Lust habe, noch einen zu wagen*? [5]

NOTES

1. A like-minded statement can be found in Saint Thomas Aquinas' *Commentaries on Aristotles Metaphysica* (English translation by J. Rowan, Chicago, 1961), Book X, II: "*Simpliciora autem quae sunt priora et notiora secundum naturam, cadunt in cognitionem nostram per posterius.*"

2. This requirement cannot be imposed upon an infinite AS, where it loses its economic significance. To exemplify, take the AS for real-closed fields that contain all the axioms for ordered fields and the axioms A_n, stating that any polynomial of degree $2n + 1$ has a root. No subsystem of this AS is completely independent, since $A_n \to A_m$ for all $n > m$.

3. A13 is the geometric counterpart of the first-order continuity axiom schema for real-closed fields, which can be replaced by the A_n, which are $\forall\exists$-statements.

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