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On the Logic of Continuous Algebras

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Introduction For continuous algebras, i.e., ordered algebras with specified joins which all operations preserve, the following analogue of the Birkhoff variety theorem has been proved by Adámek, Nelson, and Reiterman ([3]): a class of algebras can be described by inequalities between terms iff it is an *HSP* class. The terms here are more complicated than those used in universal algebra because they contain, besides variables and operation symbols, formal join signs.

This paper deals with equational logic (or rather, the logic of inequalities) appropriate for continuous algebras. We present deduction rules for inequalities between terms, and, for finitary algebras, we prove that these rules are complete, i.e., an inequality can be deduced from a collection E of inequalities iff it holds in each model of E. We also discuss infinitary algebras; the completeness theorem holds, e.g., for ω-continuous algebras, but it does not hold in general, and counterexamples are given.

The terms used in [3] were simpler than those introduced below: they were "small", i.e., restricted in size, and "regular", i.e., the formal join did not appear inside operation symbols. For finitary algebras, we prove that each term is syntactically equal to a small, regular term. However, the natural formulation of the deduction rules requires more complex terms, and they are also needed for the infinitary case.

The presence of the formal join sign makes terms only partially defined: for a term t in certain variables, the interpretation of the variables in a given algebra does not necessarily lead to a computation of t in the algebra.

Consequently, our deduction rules use two kinds of statements: $Def(t)$, the definability of t (the semantics of which is that t can be computed under each

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interpretation of the variables), and $t \le s$ (which entails the definability of both t and s and then has the obvious semantics).

Our proof of the completeness resembles Henkin's proof for first-order logic since we work with terms inductively, as opposed to Birkhoff's proof for equational logic (which was based on free algebras and worked with all terms at once). This is a natural consequence of the partial definability of terms.

We use classes of terms and of deduction rules below. Although our concept of set and class is naive, there would be no difficulty in formalizing our results in von Neumann–Gödel–Bernays set theory, or, in case the subset system Z below is definable, in Zermelo-Fraenkel set theory.

1 Separately continuous algebras We recall that a subset system Z assigns to each poset P a collection $Z(P)$ of subsets of P, such that for each order-preserving map f: $P \to Q$, $X \in Z(P)$ implies $f(X) \in Z(Q)$. Examples are $Z = \omega$ ($\omega(P)$ consists of all ω-chains in P) and $Z = \Delta(\Delta(P)$ consists of all directed subsets of P). A poset P is *Z-complete* if every Z-set $X \in Z(P)$ has a join in P, and an order-preserving function $f{:}P \to Q$ is *Z-continuous* provided it preserves all existing Z-joins.

For any finitary type Σ of algebras, a *separately Z-continuous Σ-algebra* is a Σ-algebra A whose underlying set is a Z-complete poset such that for each operation σ of arity n, σ is order-preserving and for each $i < n$ and each $\varnothing \neq X \in Z(A)$

$$\sigma\big(a_1, \dots, a_{i-1}, \bigvee X, a_{i+1}, \dots\big) = \bigvee_{x \in X} \sigma(a_1, \dots a_{i-1}, x, a_{i+1}, \dots).$$

A subset system Z is *normal* if $P \in Z(Q)$ implies $P \in Z(P^T)$, where P^T is P with a new top element T added. It is proved in [3] that every subset system Z is equivalent to a normal one, enabling us to restrict our attention to normal Z. (Z_1 is equivalent to Z_2 iff $Z_1 \le Z_2$ and $Z_2 \le Z_1$, where $Z_2 \le Z_1$ means every Z_1-complete poset is Z_2-complete and every Z_1-continuous map is Z_2-continuous.) Further, if $\perp \le Z$ (i.e., $\varnothing \in Z(P)$ for each P) then each Z-complete poset has a smallest element $\perp = \bigvee \varnothing$ and Z-continuous maps are strict, i.e., preserve $\perp$. We may, and will, assume without loss of generality, that $\underset{\sim}{2} \in Z(\underset{\sim}{2})$, for $\underset{\sim}{2} = \{0,1\}$ with $0 < 1$.

2 Terms Let Z be a normal subset system and Σ a finitary signature.

The class $T^*(V)$ of all *terms* over the set V (we assume $\perp \notin V$) is defined by induction: $T^*(V) = \bigcup T^*_\alpha(V)$ for ordinals α where

(i) $T^*_0(V)$ is the free (nonordered) algebra of type Σ over V, if $\perp \not\le Z$, or over $V \cup \{\perp\}$, if $\perp \le Z$

(ii) $T^*_{\alpha+1}(V)$ is the class of all formal expressions

 (a) $\bigvee_{p \in P} t_p$ where P is a nonempty poset with $P \in Z(P^T)$ and $t_p \in T^*_\alpha(V)$ for each $p \in P$

 (b) $\sigma(t_1, \dots, t_n)$ where $\sigma \in \Sigma$ has arity n and $t_i \in T^*_\alpha(V)$ for each $1 \le i \le n$

(iii) $T^*_\alpha(V) = \bigcup_{\beta < \alpha} T^*_\beta(V)$ for limit ordinals α.

Terms which differ only in an order-isomorphic renaming of the index poset are considered equal.

A term is *regular* if it is built from terms in $T_0^*(V)$ by iterating joins only. More precisely, the class $R(V)$ of all *regular terms over the set* V is defined in exactly the same way, with (iib) omitted.

3 Definability For each separately Z-continuous algebra A and each map h: $V \to A$, we define the computation $h^{\#}(t)$ of terms $t \in T^*(V)$ in A as a partial map $h^{\#}$: $T^*(V) \to A$. We put $h^{\#} = \bigcup h_\alpha$ (α ordinal) where h_α: $T_\alpha^*(V) \to A$ is a partial map defined as follows:

(1) h_0: $T_0^*(V) \to A$ is the unique (total) Σ-homomorphism extending h (with $h_0(\perp) = \perp$ if $\perp \leq Z$)

(2a) For $t = \bigvee_{p \in P} t_p \in T_{\alpha+1}^*(V)$, $h_{\alpha+1}(t)$ is defined iff

(i) all $h_\alpha(t_p)$ are defined

(ii) $h_\alpha(t_p) \leq h_\alpha(t_q)$ whenever $p \leq q$ in P

(iii) $\{h_\alpha(t_p) | p \in P\} \in Z(A)$ (equivalently, by the normality of Z and the Z-completeness of A, $\{h_\alpha(t_p) | p \in P\}$ has an upper bound in A)

and in this case

$$h_{\alpha+1}(t) = \bigvee_{p \in P} h_\alpha(t_p)$$

(2b) For $t = \sigma(t_1, \ldots, t_n)$ in $T_{\alpha+1}^*(V)$, $h_{\alpha+1}(t)$ is defined iff all $h_\alpha(t_i)$ are defined, and then $h_{\alpha+1}(t) = \sigma(h_\alpha(t_1), \ldots, h_\alpha(t_n))$

(3) For a limit ordinal α, $h_\alpha = \bigcup_{\beta < \alpha} h_\beta$.

A term $t \in T^*(V)$ is said to be *definable in* A if $h^{\#}(t)$ is defined for all h: $V \to A$. An inequality $s \leq t$ (for $s, t \in T^*(V)$) is *satisfied* by A iff $h^{\#}(s)$ and $h^{\#}(t)$ *are defined, and* $h^{\#}(s) \leq h^{\#}(t)$, for all h: $V \to A$. The notion of an equation $s \approx t$ being satisfied by A is defined analogously.

Remark: An inequality $t_0 \leq t_1$, is satisfied by a Z-continuous algebra A iff the term $t = \bigvee_{i \in 2} t_i$ is definable in A or, equivalently, the equation $t \approx t$ is satisfied by A. Thus, classes presented by the definability of a collection of terms are just the same as classes presented by a collection of inequalities or a collection of equations.

4 Rules of deduction In this section, we establish rules of deduction which permit us to deduce an inequality between terms, or the definability of a term, from a collection of inequalities and (already) definable terms. Our aim is to prove a completeness theorem for these rules.

Since an inequality is semantically equivalent, and so will also be syntactically equivalent, to the definability of an appropriate term (see the preceding section), we could have written all deduction rules for definability of terms only. We have chosen to have deduction rules involving both notions in order to enhance the intuition behind them.

We formulate three groups of rules; the first establishes rules of definability, the second establishes the order and Z-completeness, and the third deals with the continuity of the operations.

The very last rule involves the notion of substitution of terms into terms, which is defined inductively (on the complexity) as follows:

Definition For $s \in T^*(W)$ and h: $W \to T^*(V)$, the term $s(x \to h(x)) \in T^*(V)$ is defined as follows:

(i) if $s = y \in W$ then $s(x \to h(x)) = h(y)$
(ii) if $s = \sigma(s_1, \ldots, s_n)$ then $s(x \to h(x)) = \sigma(s_1(x \to h(x)), \ldots, s_n(x \to h(x)))$
(iii) if $s = \bigvee_{p \in P} s_p$ then $s(x \to h(x)) = \bigvee_{p \in P} s_p(x \to h(x))$.

Next, we state the rules of deduction:

Definability Rules.

(D1) $\vdash Def(x)$ for each variable x
$\vdash Def(c)$ for each constant c
(D2) $s \leq t \vdash Def(s)$ and $Def(t)$
(D3) $Def(\sigma(t_1, \ldots, t_n)) \vdash Def(t_i)$ for each operation σ of arity n and each $i \leq n$.

Order and Completeness Rules.
(C1) $Def(t) \vdash t \leq t$
(C2) $s \leq t$ and $t \leq u \vdash s \leq u$
(C3) For each poset Q and each $P \in Z(Q)$

$$t_q \leq t_r \text{ for all } q \leq r \text{ in } Q \vdash Def\left(\bigvee_{p \in P} t_p\right)$$

(C4) $Def\left(\bigvee_{p \in P} t_p\right) \vdash t_q \leq \bigvee_{p \in P} t_p$ for each $q \in P$

(C5) $Def\left(\bigvee_{p \in P} t_p\right)$ and $t_p \leq u$ for all $p \in P \vdash \bigvee_{p \in P} t_p \leq u$

(C6) $Def\left(\bigvee_{p \in P} t_p\right) \vdash t_p \leq t_q$ for all $p \leq q$ in P.

In the case that $\perp \leq Z$ we also have the following deduction rule:

(C7) $Def(t) \vdash \perp \leq t$.

Algebraic Rules.
(A1) $s_i \leq t_i$ for $1 \leq i \leq n \vdash \sigma(s_1, \ldots, s_n) \leq \sigma(t_1, \ldots, t_n)$
(A2) If $s_k = \bigvee_{p \in P} t_p$, then

$$Def(s_i) \text{ for } 1 \leq i \leq n \vdash \sigma(s_1, \ldots, s_n) \leq \bigvee_{p \in P} \sigma(s_1, \ldots, s_{k-1}, t_p, s_{k+1}, \ldots, s_n)$$

(A3) For any map h: $W \to T^*(V)$ and any $s \in T^*(W)$, $Def(h(x))$ for all $x \in W$ and $Def(s) \vdash Def(s(x \to h(x)))$.

Definition For any collection E of definability statements and inequalities between terms, a term t (respectively, an inequality $s \le t$) is said to be a *consequence of* E, denoted

$$E \vdash \mathit{Def}(t) \text{ or } E \vdash s \le t$$

iff there is a sequence ϕ_i $(i \le \alpha)$ for some ordinal α, of definability statements and inequalities, such that $\phi_\alpha = \mathit{Def}(t)$ (or $s \le t$ respectively) and for each $i \le \alpha$, ϕ_i can be deduced from $E \cup \{\phi_j \mid j < i\}$ using one of the deduction rules above.

By a model of E is meant a separately Z-continuous algebra which satisfies all the definability statements and all inequalities in E.

Notation: We write $E \vdash s \approx t$ provided $E \vdash s \le t$ and $t \le s$.

Remark 1: By Rules (C3) and (D4), we have for any terms t_0 and t_1, $t_0 \le t_1 \vdash \mathit{Def}\left(\bigvee_{i\in 2} t_i\right)$ where $2 = \{0,1\}$ is the two-element chain, and conversely $\mathit{Def}\left(\bigvee_{i\in 2} t_i\right) \vdash t_0 \le t_1$. As a result, for any map h: $W \to T^*(V)$ and any $t_0, t_1 \in T^*(W)$,

$$[\mathit{Def}(h(x)) \text{ for all } x \in W \text{ and } t_0 \le t_1] \vdash t_0\ (x \to h(x)) \le t_1(x \to h(x)).$$

Remark 2: Using Rules (C1) and (A1), we have the "converse" of (D3), namely $\mathit{Def}(t_i)$ for $1 \le i \le n \vdash \mathit{Def}\ \sigma(t_1, \ldots, t_n)$.

Remark 3: We also obtain the reverse of the inequality deduced in (A2), namely, if $s_k = \bigvee_{p\in p} t_p$ then

$$\mathit{Def}(s_i) \text{ for } 1 \le i \le n \vdash \bigvee_{p\in P} \sigma(s, \ldots, s_{k-1}, t_p, s_{k+1}, \ldots, s_n) \le \sigma(s_1, \ldots, s_n).$$

We see this as follows:

$$\mathit{Def}(s_i) \text{ for } 1 \le i \le n \vdash \mathit{Def}\left(\bigvee_{p\in P} \sigma(s_1, \ldots, s_{k-1}, t_p, s_{k+1}, \ldots, s_n)\right)$$
by (A2) and (D2)

and we deduce $\sigma(s_1, \ldots, s_{k-1}, t_p, s_{k+1}, \ldots, s_n) \le \sigma(s_1, \ldots, s_n)$ for all $p \in P$ by (C1), (C4), and (A1), and hence the result follows from (C5). Thus, if $s_k = \bigvee_{p\in P} t_p$ then

$$\mathit{Def}(s_i) \text{ for } 1 \le i \le n \vdash \sigma(s_1, \ldots, s_n) \approx \bigvee_{p\in P} \sigma(s_1, \ldots, s_{k-1}, t_p, s_{k+1}, \ldots, s_n).$$

Remark 4: For any $P \in Z(P^T)$ and any terms s_p, t_p $(p \in P)$ we have

$$s_p \approx t_p \text{ for all } p \in P \text{ and } \mathit{Def}\left(\bigvee_{p\in P} s_p\right) \vdash \bigvee_{p\in P} s_p \approx \bigvee_{p\in P} t_p.$$

We see this as follows: for each $q \in P$

$$s_q \approx t_q \text{ and } Def\left(\bigvee_{p\in P} s_p\right) \vdash t_q \leq \bigvee_{p\in P} s_p \text{ by (C4) and (C2)}$$

and for any $q \leq r$ in P we have

$$s_q \approx t_q \text{ and } s_r \approx t_r \text{ and } Def\left(\bigvee_{p\in P} s_p\right) \vdash t_q \leq t_r \text{ by (C6) and (C2)}$$

and hence, applying (C3) where $Q = P^T$ and $t_T = \bigvee_{p\in P} s_p$ we obtain

$$s_p \approx t_p \text{ for all } p \in P \text{ and } Def\left(\bigvee_{p\in P} s_p\right) \vdash Def\left(\bigvee_{p\in P} t_p\right).$$

But now applying (C5) with $u = \bigvee_{p\in P} s_p$ we obtain

$$s_p \approx t_p \text{ for all } p \in P \text{ and } Def\left(\bigvee_{p\in P} s_p\right) \vdash \bigvee_{p\in P} t_p \leq \bigvee_{p\in P} s_p;$$

the reverse inequality is obtained symmetrically.

5 *The completeness theorem*

Proposition 1 *For every term s there is a regular term $\bar{s}$ in the same variables such that $Def(s) \vdash s \approx \bar{s}$.*

Proof: We prove the proposition in case $s = \sigma(s_1, \ldots, s_n)$ for regular terms $s_1, \ldots, s_n$; the result then follows from this special case by induction on the complexity of s.

We proceed by showing, by induction on k, that, if $s_1, \ldots, s_k \in R(V)$ and $s_{k+1}, \ldots, s_n \in T_0^*(V)$ then there is a regular term $\bar{s}$ with $Def(s) \vdash s \approx \bar{s}$; the case $k = n$ is the desired result.

For $k = 0$, we have $s_1, \ldots, s_n \in T_0^*(V)$ and hence $s \in T_0^*(V)$ and so s is itself regular, so that we simply apply (C1).

For the induction step $k \to k + 1$, we proceed by induction on the complexity of s_k.

(a) If $s_k \in T_0^*(V)$ then we use the induction hypothesis on k.

(b) If $s_k = \bigvee_{p\in P} t_p$ and for each term

$$u_p = \sigma(s_1, \ldots, s_{k-1}, t_p, s_{k+1}, \ldots, s_n)$$

there is a regular term $\bar{u}_p$ such that

$$Def(u_p) \vdash u_p \approx \bar{u}_p$$

then we put $\bar{s} = \bigvee_{p \in P} \bar{u}_p$. Then $\bar{s}$ is a regular term and, by Remark 3 above,

$$Def(s) \vdash s \approx \bigvee_{p \in P} u_p$$

and

$$Def(s) \vdash u_p \approx \bar{u}_p$$

and hence by Remark 4,

$$Def(s) \vdash \bigvee_{p \in P} u_p \approx \bigvee_{p \in P} \bar{u}_p \approx \bar{s}$$

which yields the desired result.

Definition For any class E of terms, let $T_E^*(V)$ be the class of all terms $t \in T^*(V)$ such that $Def(E) \vdash Def(t)$. Define a binary relation $\equiv_E$ on $T_E^* V$ by:

$$s \equiv_E t \text{ iff } Def(E) \vdash s \approx t.$$

Proposition 2 *For any class E of terms and any set V, there are at most* $2^{card\, T_0^*(V)}$ *equivalence classes of* $T_E^*(V)$ *modulo the relation* $\equiv_E$.

Proof: It is enough, by the preceding proposition, to prove the analogous result for $R_E(V)$, the class of all regular terms $t \in R(V)$ such that $Def(E) \vdash Def(t)$.

For each $t \in R_E(V)$, define a subset $S_t \subseteq T_0^*(V)$ as follows:

(i) for $t \in T_0^* V$, $S_t = \{t\}$
(ii) for $t = \bigvee_{p \in P} t_p$, $S_t = \bigcup_{p \in P} S_{t_p}$.

Then the desired result follows immediately from the following claim:

(*) for $s, t \in R_E(V)$, if $S_s \subseteq S_t$ then $Def(E) \vdash s \leq t$.

The latter is proved by induction on the complexity of s:

(i) $s \dot{\in} T_0^*(V)$. Proof by induction on the complexity of t: if $t \in T_0^*(V)$ it is trivial, and if $t = \bigvee_{p \in P} t_p$ then $S_s = \{s\} \subseteq \bigcup_{p \in P} S_{t_p}$ implies that $s \in S_{t_p}$ for some p and hence (by the induction hypothesis on the second component), $Def(E) \vdash s \leq t_p$ and so $Def(E) \vdash s \leq t$ by Rules (C2) and (C4).

(ii) $s = \bigvee_{p \in P} s_p$. Then $Def(E) \vdash s_p \leq t$ for each p by the induction hypothesis, and hence by (C5), $Def(E) \vdash s \leq t$.

Completeness Theorem *For any class E of inequalities, an inequality* $s \leq t$ *is a consequence of E iff it holds in each model of E.*

Proof: Since an inequality $t_0 \leq t_1$ is equivalent, both syntactically and semantically, to the definability of the term $\bigvee_{i \in 2} t_i$, it is enough to prove that for any class E of terms and any term t, $Def(E) \vdash Def(t)$ iff t is definable in each model of E, i.e., in each separately Z-continuous algebra in which E is definable.

The "only if" part is clear. For the converse, we construct, for each set V of variables, a model A_V of E such that

$$t \in T^*(V) \text{ is definable in } A \text{ iff } Def(E) \vdash Def(t), \text{ for all } t \in T^*(V).$$

Let $A_V = T_E^*(V)/{\equiv_E}$; then by Proposition 2, A_V is a set, of cardinality at most $2^{card\, T_0^*(V)}$. For $t \in T_E^*(V)$, let $[t]$ be the equivalence class of t modulo $\equiv_E$. We define operations $\sigma \in \Sigma$ on A_V as follows:

$$\sigma([t_1], \dots, [t_n]) = [\sigma(t_1, \dots, t_n)] \text{ for all } t_1, \dots, t_n \in T_E^* V.$$

Note that $\sigma(t_1, \dots, t_n) \in T_E^*(V)$ and by (A1) our equivalence $\equiv_E$ is a congruence on $T_E^*(V)$.

A_V is ordered by: $[s] \leq [t]$ iff $Def(E) \vdash s \leq t$.

Claim 1 A_V is a Z-continuous algebra.

In fact, we prove the following: for each $P \in Z(A_V)$ and each choice of elements $t_q \in T_E^*(V)$ for $q \in A_V$ such that $[t_q] = q$, we have $Def(E) \vdash Def\left(\bigvee_{p \in P} t_p\right)$ and moreover $\left[\bigvee_{p \in P} t_p\right]$ is the join in A_V of $P = \{[t_p] | p \in P\}$.

For each $p \leq q$ in A_V we have $Def(E) \vdash t_p \leq t_q$ and hence by (C3) we obtain $Def(E) \vdash Def\left(\bigvee_{p \in P} t_p\right)$. Thus $\bigvee_{p \in P} t_p \in T_E^*(V)$. Moreover, Rules (C4) and (C5) ensure that $\left[\bigvee_{p \in P} t_p\right] = \bigvee_{p \in P} [t_p]$, where the latter join is in A_V. Thus A_V is Z-complete.

The fact that the operations preserve nonempty Z-joins in each component is an immediate consequence of Remark 3.

Claim 2 Every $t \in E$ is definable in A_V.

Suppose $t \in E \cap T^*(W)$, and let h: $W \to A_V$ be any map; we must prove that $h^{\#}(t)$ is defined.

Choose a map k: $W \to T_E^*(V)$ such that $h(x) = [k(x)]$ for each $x \in W$.

Note that for each subterm s of t, $Def(t) \vdash Def(s)$ by Rules D3, D4, and D2, and hence by (A3), $Def(E) \vdash Def(s(x \to k(x)))$. Therefore, $s(x \to k(x)) \in T_E^*(V)$.

We will prove inductively that for each subterm s of t, $h^{\#}(s)$ is defined, and equals $[s(x \to k(x))]$.

If $s \in W$ then $[s(x \to k(x))] = [k(s)] = h(s)$, as required.

If $s = \sigma(s_1, \dots, s_n)$ with $h^{\#}(s_i) = [s_i(x \to k(x))]$ then by definition of $h^{\#}$, we have

$$\begin{aligned} h^{\#}(s) &= \sigma(h^{\#}(s_1), \dots, h^{\#}(s_n)) \\ &= \sigma([s_1(x \to k(x))], \dots, [s_n(x \to k(x))]) \\ &= [\sigma(s_1(x \to k(x)), \dots, s_n(x \to k(x)))] \\ &= [s(x \to k(x))]. \end{aligned}$$

Let $s = \bigvee_{p \in P} s_p$ with $h^{\#}(s_p) = [s_p(x \to k(x))]$ for each $p \in P$. Since $Def(E) \vdash Def(s)$ and hence also $s_p \leq s_q \leq s$ for each $p \leq q$ in P, we conclude that $Def(E) \vdash s_p(x \to k(x)) \le s_q(x \to k(x)) \le s(x \to k(x))$ for all $p \le q$ in P and thus in A_V we have

$$[s_p(x \to k(x))] \le [s_q(x \to k(x))] \le [s(x \to k(x))];$$

i.e., $h^\#(s_p) \le h^\#(s_q) \le [s(x \to k(x))]$ for all $p \le q \in P$. Since $P \in Z(P^T)$, it follows that

$$\{h^\#(s_p) | p \in P\} \in Z(A_V)$$

and hence $h^\#(s)$ is defined and equals

$$\bigvee_{p\in P} h^\#(s_p) = \bigvee_{p\in P} [s_p(x \to k(x))] = \left[\bigvee_{p\in P} s_p(x \to k(x))\right] = [s(x \to k(x))]$$

as required.

Claim 3 If $t \in T^*(V)$ is definable in A_V then $Def(E) \vdash Def(t)$.

To prove this, we show that for the map h: $V \to A_V$ given by $h(x) = [x]$, if $h^\#(t)$ is defined then $t \in T_E^*(V)$ and $h^\#(t) = [t]$. The proof is by induction on the complexity of t. This is clear for $t \in T_0^*(V)$.

If $t = \sigma(t_1, \ldots, t_n)$, $t_i \in T_E^*(V)$ and $h^\#(t_i) = [t_i]$ for each i then $t \in T_E^*(V)$ and $h^\#(t) = \sigma([t_1], \ldots, [t_n]) = [\sigma(t_1, \ldots, t_n)] = [t]$.

If $t = \bigvee_{p\in p} t_p$ and $h^\#(t)$ is defined then, by the definition of $h^\#$, $p \le q$ in P implies $h^\#(t_p) \le h^\#(t_q)$ and hence $[t_p] \le [t_q]$, and $\{h^\#(t_p) | p \in P\} \in Z(A_V)$, and $h^\#(t) = \bigvee_{p\in P} h^\#(t_p)$. Now the same argument as in Claim 1 shows that

$$\bigvee_{p\in P} h^\#(t_p) = \bigvee_{p\in P} [t_p] = \left[\bigvee_{p\in P} t_p\right]$$

and hence $h^\#(t) = [t]$, as required.

6 Small terms and the small completeness theorem We have used, so far, terms, the complexity of which is an arbitrary ordinal and a proper class of deduction rules. It is, however, possible to restrict the concept of term in such a way that both the deduction rules and terms over a given set of variables form sets only. The basic idea is that any continuous algebra generated by a set V has at most $\|V\|$ points where

$$\|V\| = 2^{card\, T_0^*(V)} = 2^{card(V \cup \Sigma) + \aleph_0}$$

as proved in [6]. This makes it possible to restrict terms to the small terms defined as follows:

Definition The class $S(V)$ of all *small terms* over the set V is defined by the same induction as in Section 2 above except that in (iia), the poset P is required to have cardinality at most $\|V\|$.

Remark: All small terms clearly have complexity smaller or equal to the first regular cardinal larger than $\|V\|$. Therefore, $S(V)$ is a set. (Recall that changing the index poset to an isomorphic one does not change the term.)

In [3], we used the set $T(V) = S(V) \cap R(V)$ of small regular terms (called just terms). These are "sufficient" as the following proposition shows:

Proposition 3 *For each term t there is a small regular term $\bar{t}$ in the same variables such that $Def(t) \vdash t \approx \bar{t}$.*

Proof: It is enough to prove the claim for regular terms (by Proposition 1), and this is done by induction on the complexity of t. If $t \in R_0(V)$ then t itself is small.

If $t = \bigvee_{p \in P} t_p$ then by the induction hypothesis there are small regular terms $\bar{t}_p$, such that

$$Def(t_p) \vdash t_p \approx \bar{t}_p, \text{ and hence } Def(t) \vdash t_p \approx \bar{t}_p \text{ for all } p.$$

Define a quasi-order $\sqsubseteq$ on P by:

$$p \sqsubseteq q \text{ iff } Def(t) \vdash \bar{t}_p \leq \bar{t}_q.$$

Note that $p \leq q$ implies $p \sqsubseteq q$.
Define a binary relation $\equiv$ on P by:

$$p \equiv q \text{ iff } p \sqsubseteq q \text{ and } q \sqsubseteq p,$$

then applying Proposition 2 to $E = \{t\}$, we obtain that $Q = P/{\equiv}$ is a set, with at most $\|V\|$ elements. Let Q have the order induced from the quasi-order $\sqsubseteq$, so that $[p] \leq [q]$ iff $p \sqsubseteq q$; then the quotient map $P \to Q$ is order preserving, and hence so is its extension $P^T \to Q^T$ which maps T to T. Since $P \in Z(P^T)$ it follows that $Q \in Z(Q^T)$.

For each $r \in Q$, choose $p \in P$ with $r = [p]$, and let $u_r = \bar{t}_p$. Note that if $r \leq s$ in Q then $Def(t) \vdash u_r \leq u_s$, since $Def(t) \vdash t_p \leq t_q$ for the chosen p, q with $r = [p]$, $s = [q]$.

Now, $u = \bigvee_{r \in Q} u_r$ is a small regular term, and since $Def(t) \vdash u_r \leq u_s \leq t$ for all $r \leq s$ in Q we have $Def(t) \vdash Def(u)$ by Deduction Rule (C3), because $Q \in Z(Q^T)$. It remains to show that $Def(t) \vdash t \approx u$.

The inequality $u \leq t$ comes from the fact that $Def(t) \vdash t_p \leq t$ and $\bar{t}_p \approx t_p$ for all $p \in P$ and hence $Def(t) \vdash u_r \leq t$ for all $r \in Q$. By Rule (C5), $Def(t) \vdash u \leq t$. Conversely, we have $Def(t) \vdash t_p \approx \bar{t}_p$ and $u_r \leq u$ for all $p \in P$, $r \in Q$ and hence $t_p \leq u$ which, again by Rule (C5), yields $t \leq u$.

Definition We define the notion $\vdash_s$ of "small consequence" for small terms analogously to $\vdash$ introduced above, with the only change that in (C3) the poset Q is restricted to have cardinality $\leq \|V\|$, and the set P appearing in rules (D4), (C4), (C5), and (A2) will also have cardinality $\leq \|V\|$.

Small Completeness Theorem *For any class E of inequalities between small terms, an inequality $s \leq t$ between small terms is a small consequence of E iff it holds in all models of E.*

Proof: The proof is the same as the proof of the preceding theorem, with $\vdash_s$ replacing $\vdash$; the only time the Deduction Rule (C3) is used in proving that A_V is Z-complete, we only need $Q = A_V$, which has cardinality $\leq \|V\|$ as required for small deduction.

7 Jointly continuous algebras The preceding sections have dealt with separately Z-continuous algebras. Here, we deal with the analogous results for jointly continuous algebras. Recall that for any type Σ, a (jointly) *Z-continuous Σ-algebra* is a Σ-algebra A whose underlying set is a Z-complete poset such that for each operation σ of arity n, $\sigma\colon A^n \to A$ is order-preserving and preserves joins of nonempty Z-sets in A^n, ordered component-wise. Note that every Z-continuous algebra is separately Z-continuous, and that the converse is true iff Σ is finitary and all Z-sets are directed (see [6]). For jointly continuous algebras, the appropriate terms are the same, and the deduction rules are also the same as before, with the exception that Rule (A2) must be replaced with

(A2j) If $s_i = \bigvee_{p\in P} t_{pi}$, then

$$Def(s_i) \text{ for } 1 \le i \le n \vdash \sigma(s_1,\ldots,s_n) = \bigvee_{p\in P} \sigma(t_{p1},\ldots,t_{pn}).$$

Note that the Rule (A2j), in the presence of the other rules, implies the Rule (A2) since we can always use the fact that $Def(s) \vdash Def\left(\bigvee_{p\in P} t_p\right)$ and $s \approx \bigvee_{p\in P} t_p$ where $t_p = s$ for all $p \in P$, for any Z-set P.

With this change in the rules of deduction, all of the results of the preceding sections, including the Completeness Theorems, hold.

8 Infinitary algebras—Positive results Logic for infinitary continuous algebras is the same as in the finitary case but the question of completeness is more delicate. The appropriate terms are defined just as for the finitary case in Section 2, although in (iib) σ may now be of infinite rank. Also, the notion of regular term is defined as before. The deduction rules can also be formulated analogously to the finitary case.

When trying to extend our proof of the Completeness Theorem to infinitary algebras, the basic difficulty is that Proposition 2 may not be valid: there exist sets V and classes E of inequalities such that $T_E^*(V)/{\equiv_E}$ is a proper class. If, however, $T_E^*(V)/{\equiv_E}$ *is* a set, then the proof of the Completeness Theorem proceeds just as in the finitary case. Moreover, the algebra A_V constructed there will be the relatively free algebra over V in the variety of all algebras which are models of E. The converse also holds: given a class E of inequalities, if the completeness theorem holds for E (i.e., $E \vdash s \le t$ iff every model of E is a model of $s \le t$) and if the relatively free algebra over V in the class of all models of E exists, then $T_E^*(V)/{\equiv_E}$ is a set.

It has been proved in [1] that whenever $\Delta \le Z$, i.e., each directed set is a Z-set, then jointly Z-continuous algebras have bounded generation. By inspecting the proof, we easily see that the hypothesis $\Delta \le Z$ entails, for each term t, the existence of a regular term $\bar{t}$ with $Def(t) \vdash t \approx \bar{t}$. Therefore, in this case, Proposition 2 is valid, and hence by the above remarks we have the following.

Theorem *If $\Delta \le Z$, then for each class E of inequalities for (jointly) Z-continuous algebras, an inequality $s \le t$ is a consequence of E iff it holds in all models of E.*

Nevertheless, we will see in the next section that the assumption $\Delta \le Z$ is essential: the completeness theorem does not hold in general for infinitary algebras.

For separately continuous algebras, the situation is even more subtle. The completeness theorem does hold for $Z = \omega$, or, more generally, for any Z with a bound on cardinalities of Z-sets, since in such cases, for any set V, $T^*(V)$ is a set. On the other hand, even for $Z = \Delta$, the completeness theorem is false for infinitary types; this will be seen in the next section.

9 Infinitary algebras — Counterexamples Here, we present examples of classes E of inequalities for which every model has only one element, and yet the equation $x = y$ cannot be derived from E by our rules of deduction. In fact the result is, to a certain extent, independent of the choice of deduction rules: we shall present a proper class model of E. Hence the completeness theorem will fail for any choice of deduction rules which (like ours) are valid not only for continuous algebras, but also for their proper-class relatives.

The notion of a "large" or proper-class continuous algebra is defined just as is that of a continuous algebra, except that the universe of such a "large" algebra may be a proper class rather than a set. An ordered class C is said to be Z-complete if it has joins of all Z-sets, i.e., all sets $X \subseteq C$ for which there is a set $C_0 \subseteq C$ with $X \in Z(C_0)$. A large algebra is (separately or jointly) continuous if it is Z-complete and the operations, as usual, preserve (either jointly or separately) joins of all nonempty Z-sets. The concepts of definability of terms and satisfiability of equations are defined just as for continuous algebras, and it is straightforward to see that, for any class E of inequalities, if $E \vdash s \leq t$ then any proper class model of E also satisfies $s \leq t$.

The examples below depend on the following set-theoretical hypothesis:

(*) There exists a long binary tree (i.e. going through all ordinal levels) with a leaf at each infinite level, i.e., the tree has a maximal branch of every infinite height.

Remark 1: (*) is equivalent to

(**) There are no subtle cardinals and *Ord* is not subtle where *Ord* is the class of all ordinals. (For the definition of subtle see [4] or [1].)

The proof is analogous to arguments appearing in [1], Theorem 1.

Remark 2: It follows from [4], Proposition 2.5, that every subtle cardinal is the limit of the strongly inaccessibles below it. In the presence of a "global" choice function, i.e. a choice function for the class of all sets, the same techniques prove that if *Ord* is subtle then there is a proper class of strongly inaccessibles. Hence, the existence of a global choice function and the nonexistence of strongly inaccessible cardinals together imply (*). Thus (*) is consistent with the axioms of set theory.

Remark 3: It is possible to use α-nary trees instead of binary trees. The (*) could be weakened to the following assumption: There is only a set of subtle cardinals, and *Ord* is not subtle. (This is certainly true if there is only a set of strongly inaccessibles, in the presence of a global choice function.)

We omit the details of this generalization because they are quite analogous to such considerations in [1].

9.1 Separate continuity Here, we present a counterexample to the completeness theorem for separately Δ-complete infinitary algebras. We present a class E of inequalities which has a large model, but does not have any set model with more than one element.

Let Σ be the type consisting of an ω-ary operation σ, one binary operation δ, and countably many constants c_n, $n \in \omega$, and let $Z = \Delta$.

Assume that S is a long binary tree with leaves on all infinite levels. Using the techniques of [1], Proposition 4, we may assume that S has 2^α leaves at level α for all $\alpha \geq \omega$. Moreover, we may assume that each chain in S has a join, else embed S in the uniform long binary tree and add the missing joins.

Let $S^+ = S \cup \{\infty\}$, with $s \leq \infty$ for each $s \in S$; then S^+ is a Δ-complete (even complete) p.o. class.

For each ordinal α, let $S(\alpha)$ be the αth level of S, and let $L(\alpha) \subseteq S(\alpha)$ be the set of leaves at level α.

For each $\alpha \geq \omega$, let $f_\alpha \colon L(\alpha)^\omega \to S(\alpha + 1)$ be a surjective map such that

$$f_\alpha((a_n)_{n\in\omega}) = f_\alpha((b_n)_{n\in\omega}) \text{ whenever } a_n = b_n \text{ for almost all } n,$$

where "almost all" means all but a finite number.

Now, define the constants c_n in S^+ so that they cover the finite levels, i.e.

$$\bigcup \{S(n) \mid n \in \omega\} = \{c_n \mid n \in \omega\}.$$

Define σ on S^+ as follows:

$$\sigma((b_n)_{n\in\omega}) = \begin{cases} f_\alpha((a_n)_{n\in\omega}) & \text{if there exist } \alpha \text{ and } a_n \in L(\alpha) \text{ with} \\ & \quad a_n = b_n \text{ for almost all } n. \\ \infty & \text{if } b_n = \infty \text{ for almost all } n \\ \perp & \text{else.} \end{cases}$$

Then it is straightforward to check that σ is order-preserving, and the fact that $\sigma((a_n)_{n\in\omega}) = \sigma((b_n)_{n\in\omega})$ whenever $a_n = b_n$ for almost all n ensures that σ is separately continuous.

Define δ on S^+ as follows

$$\delta(a,b) = \begin{cases} \infty & \text{if } a = \infty \text{ or } b = \infty \text{ or } a \text{ and } b \text{ are incomparable} \\ \perp & \text{else.} \end{cases}$$

Then it is straightforward to check that δ is order-preserving and separately Δ-continuous.

Now, all of S is generated from the constants c_n by the operation σ and the formulation of directed joins. Hence for each $a \in S$ there is a term t_a (with no variables) whose interpretation in S^+ is a. Hence, if $a, b \in S$ are different leaves then S^+ satisfies

E: $\delta(t_a, t_b) \geq z$ and $\delta(t_a, t_a) = \perp$, for all leaves $a \neq b \in S$, where z is a variable.

Finally, if a separately Δ-continuous algebra A is a model of E, then we prove that A is trivial. Since S has a proper class of leaves, there exist leaves

$a, b \in S$ for which the terms t_a and t_b have the same interpretation in A. Thus, in A, we have $\perp = \delta(t_a, t_a) = \delta(t_a, t_b) \geq y$, so A is trivial.

9.2 Joint continuity Let Σ consist of an ω-ary operation σ, a binary operation δ, and a constant c. Here, we present a class of inequalities for jointly Δ_ω-continuous Σ-algebras which has a proper class model but no nontrivial set model, again under the hypothesis (*). We recall that Δ_ω is the subset system consisting of all countably directed sets; i.e., $X \in \Delta_\omega(P)$ iff every countable subset of X has an upper bound in X.

Suppose S is a long binary tree with 2^α leaves at level α for all $\alpha \geq \omega$, which is chain complete.

Construct a proper class algebra A from S as in [1], Section 4. That is, $A = \{x, y, \perp\} \cup C$ where C consists of all nonconstant decreasing ω-sequences

$$s_0 \geq s_1 \geq s_2 \geq \dots$$

of elements of S such that $s_i = s_{i+1}$ implies s_i is a leaf or $\perp$. For $s = (s_n)_{n \in \omega} \in C$, the first k such that s_k is not a leaf is called the width of s, and denoted $w(s)$. Denote $C(i) = \{s \in C | s_0 \in S(i)\}$ for each i.

The ordering on A is defined as follows: for $s, s' \in C$, $s' \leq s$ iff $s'_k \leq s_k$ for $k \leq w(s)$ and $s'_k = s_k$ for all $k > w(s)$. Further, s, x, and y are pairwise incomparable for each $s \in C$, and $\perp$ is the smallest element.

We define σ: $A^\omega \to A$ using auxiliary maps $f_n (\omega \leq n < \beta,\ cof\, n \leq \omega)$ chosen as follows. First, we choose a bijection

$$f_\omega: \{(x, y)\}^\omega - \{x\}^\omega \to \bigcup_{n \leq \omega} C(n).$$

For each isolated ordinal $n > \omega$ we choose a surjective map f_n: $L(n-1)^\omega \to C(n)$ which merges almost identical sequences. Finally, for each limit ordinal n with cofinality ω we choose an ω-sequence $n_k < n$ with $n = \Sigma_{k<\omega} n_k$, and we choose a surjective map

$$f_n: L(n_0) \times L(n_1) \times L(n_2) \times \dots \to C(n)$$

which merges almost identical sequences. By abuse of language, we evaluate $f_n(s_0, s_1, s_2, \dots)(n \neq \omega)$ also in the case that all but finitely many s_k are in $L(n-1)$ (for n isolated) or in $L(n_k)$ (for $cof\, n = \omega$).

Now we define σ as follows: for sequences in $\{x, y\}^\omega$ we put

$$\sigma(x, x, x, \dots) = y;$$
$$\sigma(z_0, z_1, z_2, \dots) = f_\omega(z_0, z_1, z_2, \dots) \text{ if } z_i \neq x \text{ for some } i.$$

For sequences in C^ω we put

$$\sigma(s^0, s^1, s^2, \dots) = f_n(s^0_0, s^1_0, s^2_0, \dots)$$

provided that

(i) the widths of the s^k grow beyond all bonds, i.e., $lim_{k \to \infty} w(s^k) = \infty$,

and

(ii) for almost all $k < \omega$ we have $s^k_0 \in L(n-1)$ for n isolated, $n > \omega$, or $s^k_0 \in L(n_k)$ for $cof\, n = \omega$.

All the remaining sequences in A^ω are mapped by σ to $\perp$.

A proof analogous to the one in [1] then shows that A is a Δ_ω-continuous proper class σ-algebra, generated by $\{x\}$.

Now let $A^+ = A \cup \{\infty\}$, ordered so that $\infty > a$ for all $a \in A$. Define $\sigma(a_0, a_1, \ldots) = \infty$ whenever one of the a_i's is equal to ∞; then σ is Δ_ω-continuous on A^+.

Define δ on A^+ as follows:

$$\delta(a,b) = \begin{cases} \infty & \text{if } a = \infty \text{ or } b = \infty \text{ or } a_0 \text{ is incomparable with } b_0 \\ \perp & \text{else} \end{cases}$$

where a_0 is the first member of a if a is an ω-sequence, and is just a if a is x, y, or $\perp$.

Then it is straightforward to check that δ is order-preserving and separately Δ_ω-continuous. Hence, since δ is binary and $\Delta_\omega \leq \Delta$, it follows that δ is jointly Δ_ω-continuous.

Let the constant c be interpreted as the generator x.

Now, there is a proper class $D \subseteq A$ such that $\delta(a,b) = \infty$ for all $a \neq b$ in D. For each $a \in D$ there is a term t_a, with no variables, whose interpretation in A^+ is a. Thus A^+ satisfies

E: $\delta(t_a, t_a) = \perp$ and $\delta(t_a, t_b) \geq y$ for all $a \neq b$ in D.

However, just as in the preceding example, any set model of this class E of inequalities is trivial.

Open problems:

1. The preceding examples provide classes E of inequalities which have a proper class model but no nontrivial set model. Is there such an example, either for jointly or separately continuous algebras, where E is a *set* of inequalities?
2. Two interesting special cases of infinitary separately continuous algebras are:

   (a) distributive lattices with arbitrary joins and countable meets (i.e., Σ has one ω-ary operation which is interpreted as meet, and every set is a Z-set)

   (b) σ-frames (that is complete lattices satisfying $x \wedge \bigvee Y = \bigvee \{x \wedge y \mid y \in Y\}$) with countable meet as an operation.

For both these examples, there are no relatively free algebras, and there is a proper class of terms over a countable set (see [5]). Does the completeness theorem fail in these settings?

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