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Topological Duality for Diagonalizable Algebras

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Introduction Diagonalizable Algebras have been introduced by Magari in [9] to provide an algebraic treatment of logical incompleteness phenomena. We recall that a *Diagonalizable Algebra* (briefly a *DA*) is a structure $\mathfrak{A} = \langle A;+,\cdot,\nu,0,1,\tau\rangle$ where $\langle A;+,\cdot,\nu,0,1\rangle$ is a Boolean Algebra,[1] and τ is a unary operation such that the following identities hold:

$$\tau 1 = 1; \qquad \tau(p\cdot q) = \tau p\cdot\tau q; \qquad \tau(\nu\tau p + p) = p.$$

Sometimes it is more convenient to consider the operation $\sigma = \nu\tau\nu$ instead of τ, because σ is a hemimorphism in the sense of Halmos ([8]).

If a theory T possesses a formula Theor(v) which numerates the set of theorems and satisfies the usual derivability conditions, we get the *DA* of T endowing the Lindenbaum Algebra of T with the operation τ defined as follows: $\tau[p] = [\text{Theor}(\bar{p})]$. In this way, many logical features of T can be discussed in purely algebraic terms. See [4] and [14] for general surveys about *DA*'s and the corresponding modal logic *GL*.

A representation theorem for *DA*'s has been obtained in [10] by applying Halmos' duality for hemimorphisms. More precisely, starting from a *DA* $\langle\mathfrak{A};\tau\rangle$, a Boolean relation $\mathcal{R}$ can be defined on the Stone space X of $\mathfrak{A}$ as follows:

$$x\,\mathcal{R}\,y \quad \text{iff} \quad \tau p \in x \Rightarrow p \in y \qquad \forall p \in \mathfrak{A} \quad [\text{iff } \sigma y \subseteq x];$$

and $\mathcal{R}$ can be proved to be transitive and relatively reverse well-founded (that is, every nonempty clopen set in X has a maximal element). Conversely, let a Boolean relation $\mathcal{R}$ be given on the Stone space X of a Boolean Algebra $\mathfrak{A}$; if $\mathcal{R}$ is transitive and relatively reverse well-founded, the operation τ defined on $\mathfrak{A}$ as follows:

(1) $\tau p = \{x/x\,\mathcal{R}\,y \text{ for no } y \in \nu p\}$ (that is, $\sigma p = \{x/x\,\mathcal{R}\,y \text{ for some } y \in p\}$)

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satisfies the identities of DA's. $(X,\mathcal{R})$ is sometimes called the dual space of the DA $\mathfrak{A}$.

On the other hand, the Stone duality provides a topological representation for Boolean Algebras. In this direction, some recent papers ([6],[5]) suggest *a topological representation of the operation* σ: namely, another topology is introduced in a dense subset of X and σ is interpreted as the derived set operator with regard to the new topology. In [6] the key topological concept is the following: a topological space S is said to be *scattered* if every nonempty subset of S contains an isolated point. (Note that every subspace of a scattered space is in turn scattered.)

However, as we shall see in Section 2, both the topologies discussed in [6] and [5] are not defined on the whole Stone space X; moreover, a topological characterization of DA's is not supplied.

It is just in order to characterize DA's in a topological way that we slightly modify the topology defined in [5] and introduce the notion of *relatively scattered bi-topological space* (Definition 2). In fact, this notion allows us to construct a *topological duality* for DA's, that is, to translate into topological terms the algebraic concepts of homomorphism (Corollary 3), τ-filter and quotient algebra, subalgebra, and direct product (Section 6).

Some applications of the topological representation are given in Section 7. One of these is concerned with the topological concept of sheaf (Theorems 17 and 18). In this regard, we add that a similar technique could be useful in other situations—in particular to get a topological characterization of Hyperdiagonalizable Algebras (see [3] for definition).

Among other things, some results are proved about *reflexive points* (Theorems 4, 7, 8, 10 and Corollary 1).

Since we refer to two topologies defined on the same set (the Stone topology and the one which induces σ), when using a topological term we usually specify which topology we are considering. However, a clopen set always denotes a clopen set in the Stone topology, and we identify it with the corresponding element of the algebra.

We conclude this section by recalling three known examples of DA's, to which we will often refer in the following.

Example 1 (see [10]): Let $Y = \{x_n/n \in \omega\} \cup \{y_n/n \in \omega\}$. Define:

$$x_n \,\mathcal{R}\, x_m \quad \text{iff } n > m; \qquad y_n \,\mathcal{R}\, y_m \quad \text{iff } n < m;$$
$$y_n \,\mathcal{R}\, x_m \quad \text{for every } n,m; \qquad x_n \,\mathcal{R}\, y_m \quad \text{for no } n,m.$$

Let $\mathfrak{A}$ be the Boolean Algebra of finite and cofinite subsets of Y. We get a DA $\mathfrak{A}$ defining the operation τ as in (1). Note that $\mathcal{R}$ is not well-founded on Y.

Example 2 (see [10]): Consider the Boolean Algebra $\mathcal{P}(\omega)$. Let u be a fixed nonprincipal ultrafilter and u_0 be the principal ultrafilter generated by $\{0\}$. Define $\mathcal{R}$ to be just $\{(u_0,u)\}$ and call $\mathfrak{B}$ the corresponding DA.

Example 3: Let $Y = \{x_n/n \in \omega\} \cup \{y_n/n \in \omega\}$. Define:

$$x_n \,\mathcal{R}\, x_m \quad \text{iff } n > m; \qquad y_n \,\mathcal{R}\, x_m \quad \text{for every } n,m;$$
$$x_n \,\mathcal{R}\, y_m \quad \text{for no } n,m; \qquad y_n \,\mathcal{R}\, y_m \quad \text{for no } n,m.$$

We call $\mathfrak{C}$ the *DA* which is obtained introducing in $\mathcal{P}(Y)$ the operation τ defined as in (1).

1 σ as a derived set operator In this section we examine some previous topological results about *DA*'s.

The first step is due to H. Simmons who in [13] studies logical problems in topological terms. Actually, he does not consider *DA*'s (which were not defined until the following year), or topological spaces, either. Simmons introduces pseudo-topological spaces $\langle A;+,\cdot,\nu,0,1,d\rangle$, that is, Boolean Algebras which are endowed with a unary operation d that satisfies the following identities: $d0 = 0$; $d(p+q) = dp + dq$; $ddp \leq dp$.

If we interpret the operation d as the derived set operator, then we can still call dp the "set" of all accumulation points of an element p of $\mathfrak{A}$. Similarly, considering $is(p) = p - dp$, we obtain the "set" of all isolated points of p. In this sense, if $is(p) \neq 0$ for every $p \in \mathfrak{A}$ (with $p \neq 0$) then $\langle\mathfrak{A}, d\rangle$ is called scattered.

If $\mathfrak{A}$ is the Lindenbaum Algebra of a suitable theory T and dp is defined to be $\neg Theor(\overline{\neg p})$, then $\langle\mathfrak{A}, d\rangle$ is a pseudo-topological space. Simmons does not consider algebraic aspects at all. The logical properties of the theory T are directly translated into topological properties of the associated pseudo-topological space. For instance, if the theory satisfies Löb's property (if $T \vdash Theor(\bar{\alpha}) \rightarrow \alpha$ then $T \vdash \alpha$) then the associated pseudo-topological space is scattered. Therefore, using Simmons' terminology, we could say that a *DA* is a scattered pseudo-topological space.

Some years later, Russian and Polish mathematicians studied *DA*'s from a topological point of view, obtaining results somewhat similar to the above mentioned results by Simmons.

The link between *DA*'s and scattered spaces is pointed out by Esakia in [6], for the first time in explicit terms. The operation σ defined in a *DA* is interpreted as the derived set operator in a suitable topological space. The main result is the following:

Proposition 1 *Let (X, T) be a scattered topological space; if d is the derived set operator, then $\langle\mathcal{P}(X); d\rangle$ is a DA. Conversely, if X is a topological space such that $\langle\mathcal{P}(X); d\rangle$ is a DA, then X is scattered.*

Of course, the previous property characterizes only the *DA*'s of the form $\mathcal{P}(X)$ (in fact, X is not the dual space of the algebra $\mathcal{P}(X)$). Actually, as we shall see in Section 3, this result cannot be generalized to all *DA*'s.

The idea of interpreting the operation σ as a derived set operator has been developed by Busrkowski and Prucnal in [5]. Their construction applies to all *DA*'s, but it does not refer to scattered spaces. The principal result is the following

Proposition 2 *For every DA $\mathfrak{A}$ there exists a topological space X such that $\mathfrak{A}$ is embeddable in $\mathcal{P}(X)$ and $dp = \sigma p$ for every $p \in \mathfrak{A}$.*

The proof depends on two lemmas.

Lemma 1 *Every DA $\mathfrak{A}$ is embeddable in an atomic DA $\mathfrak{D}$.*

Lemma 2 *For every atomic DA $\mathfrak{D}$ there exists a topological space (Y, T^*) such that $\mathfrak{D}$ is embeddable in $\mathcal{P}(Y)$ and $\sigma p = dp$ for every $p \in \mathfrak{D}$.*

The set Y that is mentioned in Lemma 2 is the set of all atoms of $\mathfrak{D}$. An open basis for T^* is the set of all subsets of the form $p \cdot \tau p$. We can extend the topology T^* to the whole dual space X of any *DA* $\mathfrak{A}$ considering the same basis. In this way, every open set H is "$\mathcal{R}$-*hereditary*", in the following sense: if $x \in H$ and $x\, \mathcal{R}\, y$ then $y \in H$. (If the algebra is finite this property characterizes the open sets of T^*.)

In [11] an interior operator I is defined on a *DA* $\mathfrak{A}$ in the following way: $Ip = p \cdot \tau p$ for every $p \in \mathfrak{A}$. The operator I is not defined on the whole dual space X of $\mathfrak{A}$, but only on the clopen sets of X. It is of interest to notice that if we extend the operator I to the whole dual space X putting $IH = \bigcup\{p \cdot \tau p / p \cdot \tau p \subseteq H\}$ for every $H \subseteq X$, then the operator I induces the same topology T^*.

2 The topology $\bar{T}$ In this section, we introduce and discuss a topology $\bar{T}$, defined on the set of ultrafilters of a *DA* and similar to T^*, but handier in various respects.

Definition 1 Let $\mathfrak{A}$ be a *DA* and let $(X, \mathcal{R})$ be its dual space. A subset H of X is open in $\bar{T}$ if H is $\mathcal{R}$-hereditary, that is $x\, \mathcal{R}\, y$ and $x \in H$ imply $y \in H$. (It is trival that $\bar{T}$ is a topology.)

Therefore the open sets in $(X, \bar{T})$ are exactly the subsets of X of the form $H \cup \mathcal{R}H$ for some $H \subseteq X$; similarly, the closed sets in $(X, \bar{T})$ are exactly the subsets of X of the form $H \cup \mathcal{R}^{-1}H$.

Topologies induced by relations in this sense have been considered in different contexts, in particular to represent pseudo-Boolean Algebras (see, for example [16]).

First of all, we show that the operation σ can be represented also by the topology $\bar{T}$.

Theorem 1 *In each DA $\mathfrak{A}$ we have $\sigma p = dp$ for every $p \in \mathfrak{A}$, where d is the derived set operator in $\bar{T}$.*

Proof: Let x be an element of σp; by Lemma 7 of [2] there exists a maximal element y in p (that is, an element y such that $y\, \mathcal{R}\, z$ for no $z \in p$) such that $x\, \mathcal{R}\, y$. Each open set H of $\bar{T}$ containing x must contain y too; then $x \in dp$ (note that $x \neq y$ because $y \not\mathcal{R}\, y$). Conversely, let $x \notin \sigma p$. The set $H = \mathcal{R}\{x\} \cup \{x\}$ is open in $\bar{T}$ and $x \in H$, but $(H - \{x\}) \cap p = \emptyset$; hence $x \notin dp$.

The previous statement can be expressed by the equality $dp = \mathcal{R}^{-1}p$. However, dH does not equal $\mathcal{R}^{-1}H$ for each $H \subseteq X$: as an example, consider the singleton of a point y such that $y\, \mathcal{R}\, y$. On the other hand, called c the closure operator in $\bar{T}$ the equality $cH = H \cup dH = H \cup \mathcal{R}^{-1}H$ holds for every $H \subseteq X$.

Unlike Lemma 2 of [5], it is not necessary to require that the algebra is atomic. From this point of view, the proof of Theorem 1 is much simpler than the analogous proof in [5] which refers to the topology T^*.

As we have seen, the two topologies $\bar{T}$ and T^* play a similar role for a topological representation of the operation σ. Actually, there are in general many topological spaces such that $\sigma p = dp$ for every $p \in \mathfrak{A}$. Let us discuss the situation. First, although $(X, \bar{T})$ induces σ, if we define the topology T^* on X, as at the end of Section 1, it does not work in general: consider the *DA* $\mathfrak{B}$ introduced in Example 2, where $u \in d1$, but $u \notin \sigma 1 = \{u_0\}$. In fact, in this case the topologies $\bar{T}$ and T^* are quite different, even if restricted to the subset Y of all atoms of $\mathfrak{B}$. Indeed, $\bar{T}_{|Y}$ is discrete (and therefore it does not induce the operation σ), while in $T^*_{|Y}$ the singleton $\{u_0\}$ is not open. It is worth noticing that $T^*_{|Y}$ coincides with the topology whose existence is guaranteed by Esakia's theorem, and that it induces σ.

In Remark 2 below we shall examine the connection between $\bar{T}$ and Esakia's topology in a general framework.

As another example, consider the *DA* $\mathfrak{C}$ (see Example 3). In this case, both T^* and $\bar{T}$, restricted to the set of all atoms of $\mathfrak{C}$, induce the operation σ (that is, the derived set operators of T^* and $\bar{T}$ coincide when applied to the subsets which correspond to the elements of $\mathfrak{C}$). On the other hand, $\bar{T}_{|Y}$ and $T^*_{|Y}$ are different from each other: the subset $\{x_n / n \in \omega\} \cup \{y_0\}$ is open in $\bar{T}_{|Y}$ but not in $T^*_{|Y}$.

In general, since every open set in T^* is $\mathfrak{R}$-hereditary (see Section 1), T^* is less fine than $\bar{T}$. In fact, T^* is less fine than any topology whose derived set operator d satisfies the equality $dp = \sigma p$ for every $p \in \mathfrak{A}$; this is because the set $p \cdot \tau p = \nu(\nu p + \nu \tau p) = \nu(\nu p + \sigma \nu p) = \nu(\nu p + d \nu p)$ is open.

Though T^* and $\bar{T}$ are different, we can define $\bar{T}$ starting from T^* in the following way. Let T^{**} be the topology generated by T^* and in which the intersection of open sets is always open: we claim that $\bar{T} = T^{**}$. It is trivial that $T^{**} \leq \bar{T}$. Conversely, it suffices to show that each open set in $\bar{T}$ of the form $\{x\} \cup \mathfrak{R}\{x\}$ is also open in T^{**}. For every $x \in X$ the set $\{x\} \cup \mathfrak{R}\{x\}$ is closed in the Stone topology T; hence $\{x\} \cup \mathfrak{R}\{x\} = \bigcap_{i \in I} p_i$ where each p_i is a clopen set of T. Now it is easily seen that $\{x\} \cup \mathfrak{R}\{x\} = \bigcap p_i = \bigcap (p_i \cdot \tau p_i)$; hence $\{x\} \cup \mathfrak{R}\{x\}$ is open also in T^{**}.

Therefore $\bar{T}$ is the least fine topology satisfying the following two conditions: $\sigma p = dp$ and the intersection of arbitrarily many open sets is in turn open. This characterization of $\bar{T}$ is deeply related to the equations:

$$c\{y\} = \left(\bigcap_{y \in q} \sigma q \right) \cup \{y\} \tag{1}$$

$$cH = \bigcup_{y \in H} c\{y\}. \tag{2}$$

Indeed, the latter equation is equivalent to the fact that the union of arbitrarily many closed sets is closed (the proof is straightforward), while the former says that the closure of a singleton is as large as possible ($c\{y\}$ is always contained in $\left(\bigcap_{y \in q} \sigma q \right) \cup \{y\}$ in every topology such that $\sigma p = dp$). The equations (1) and (2) are of some interest, because, as it is readily seen, they allow $\bar{T}$ to be defined directly from σ with no reference to the relation $\mathfrak{R}$.

Remark 1: We can compare $\bar{T}$ with the Stone topology T. Actually, the following conditions can be easily proved to be equivalent: (1) $T \subseteq \bar{T}$; (2) $\mathfrak{R} = \varnothing$; (3) $\bar{T}$

is discrete; (4) $\tau p = 1$ for every $p \in \mathfrak{A}$; (5) $p \cdot \tau p = p$ for every $p \in \mathfrak{A}$; (6) $\bar{T}$ is $\mathfrak{I}_1$; (7) $\bar{T}$ is $\mathfrak{I}_2$.

Remark 2: Of course, not every topology T in a set Y is induced by a relation on Y in the previous sense. However, if (Y, T) is a topological space, then we can obtain another topological space $(Z, \bar{T})$ which is strictly related to the given one, but such that $\bar{T}$ is induced by a relation. More precisely: consider the Boolean algebra $\mathcal{P}(Y)$ and let Z be its dual space. We define a relation $\mathcal{R}$ on Z in the following way: $x \,\mathcal{R}\, y$ iff for every $A \in \mathcal{P}(Y)$ if $A \in y$ then $dA \in x$ (where d is the derived set operator in T). Since d is a hemimorphism, the relation $\mathcal{R}$ is Boolean (see [8]), and some topological properties of T are translated into properties of the relation $\mathcal{R}$. In particular, $\mathcal{R}$ is transitive iff $ddH \subseteq dH$ for every $H \subseteq Y$.

Now, let $\bar{T}$ be the topology induced by the relation $\mathcal{R}$, that is, the topology in which a subset H of Z is open if it is $\mathcal{R}$-hereditary. First of all, T is less fine than $\bar{T}_{|Y}$. Indeed, let K be a closed set of T; from the definition of $\mathcal{R}$ it is trivial that $\mathcal{R}^{-1}\tilde{K} \subseteq \tilde{K}$ (where $\tilde{K}$ is the subset of Z constituted by all ultrafilters to which K belongs) and hence $\tilde{K}$ is closed in $(Z, \bar{T})$.

Even if in general $\bar{T}_{|Y}$ is different from T, these two topologies are "equivalent" in the sense that for every $H \subseteq Y$ we have $dH = \bar{d}\tilde{H} \cap Y$, that is, the derived set operators d and $\bar{d}$ (of T and $\bar{T}$, respectively) essentially coincide. To show that $dH \subseteq \bar{d}\tilde{H} \cap Y$, consider an element x of dH. Let $\mathcal{G} = \{G \cap H - \{x\}/G$ is an open neighborhood of x in $T\}$. The set $\mathcal{G}$ has the f.i.p.; then there exists an ultrafilter y of $\mathcal{P}(Y)$ such that $\mathcal{G} \subseteq y$. Of course, H is an element of y; let us show that $dA \in x$ for every $A \in y$, that is, $x \,\mathcal{R}\, y$. Let A be an element of y and let G be an open neighborhood of x in T; then $(G \cap H - \{x\}) \cap A \in y$. Hence, $G \cap A - \{x\}$ belongs to y and therefore it is not empty; we can conclude $x \in dA$. Thus we have proved that $x \in \bar{d}\tilde{H} \cap Y$. On the other hand, $\bar{d}\tilde{H} \cap Y \subseteq dH$. Indeed, let $x \in \bar{d}\tilde{H} \cap Y$ and let G be an open neighborhood of x in T. The set $\tilde{G}$ is open in $\bar{T}$ and hence $(\tilde{G} - \{x\}) \cap \tilde{H} \neq \emptyset$. Let $y \in (\tilde{G} - \{x\}) \cap \tilde{H}$; we have $(G - \{x\}) \cap H \in y$ and so $(G - \{x\}) \cap H \neq \emptyset$. Therefore x belongs to dH.

3 Relatively scattered bi-topological spaces Our aim is now to determine topological properties which provide a characterization of DA's.

Notation: A *bi-topological space* $(X; T, \bar{T})$ is a set X on which two topologies T and $\bar{T}$ are defined.

Definition 2 A bi-topological space $(X; T, \bar{T})$ is said to be *relatively scattered* if:
(1) each nonempty clopen set p of T is not dense in itself in $\bar{T}$
(2) if p is a clopen set of T, then dp is in turn a clopen set of T (where d is the derived set operator of $\bar{T}$)
(3) $ddp \subseteq dp$ for every clopen set p of T.

We notice that Condition (3) appears as the direct translation of an identity of DA's; on the other hand, from a topological point of view, this requirement is reasonable because many topological spaces satisfy it (for example, all $\mathfrak{I}_1$ spaces).

Theorem 2 (First characterization) *Let $\mathfrak{A}$ be a DA; let $(X;T)$ be the Stone space of $\mathfrak{A}$ and let $\bar{T}$ be the topology induced by the relation $\mathcal{R}$ on X. Then the bi-topological space $(X;T,\bar{T})$ is relatively scattered.*

Conversely, let $(X;T)$ be a Stone space and let $\bar{T}$ be any topology on X such that $(X;T,\bar{T})$ is relatively scattered. Then the algebra $\mathfrak{A}$ of all clopen sets of T endowed with the derived set operator d of $\bar{T}$ is a DA.

Proof: (⇒) Let p be a clopen set of T. In view of Theorem 1 we have that dp is equal to σp. Now the proofs of (1), (2), (3) are trivial.

(⇐) We have to show that the derived set operator d of the topology $\bar{T}$ satisfies the following properties (see [10]): (a) $d\varnothing = \varnothing$; (b) $d(p+q) = dp + dq$; (c) $ddp \subseteq dp$; (d) if $p \subseteq dp$ then $p = \varnothing$ (for all p,q clopen sets of T). Every derived set operator satisfies (a) and (b). By our hypothesis $(X;T,\bar{T})$ is relatively scattered: so, (c) and (d) are trivial consequences of (3) and (1), respectively.

Theorem 3 (Second characterization) *Let $\mathfrak{A}$ be a DA; then the bi-topological space $(X;T,\bar{T})$ (defined as in Theorem 2) satisfies the following properties:*

(i) *if p is a clopen set of T, then dp is in turn a clopen set of T (where d is the derived set operator in $\bar{T}$)*

(ii) *for each clopen set p of T and for each open set H of $\bar{T}$, the set $p \cap H$ (if nonempty) contains an element x which is isolated in $cp = p \cup dp$.*

Conversely, let $(X;T)$ be a Stone space and let $\bar{T}$ be any topology on X such that $(X;T,\bar{T})$ satisfies (i) and (ii). Then the Boolean Algebra of clopen sets of T endowed with the derived set operator d of $\bar{T}$ is a DA. In other words, (i) and (ii) characterize the relatively scattered bi-topological spaces.

Proof: (⇒) The proof of (i) is trivial. Then let x be an element of $p \cap H$, where p is a clopen set of T and H is an open set of $\bar{T}$. If x is maximal in p, then x is the only element of the intersection of the open set $\mathcal{R}\{x\} \cup \{x\}$ and p; hence x is isolated in cp. If x is not maximal in p, there exists a maximal element y of p (see Lemma 7 of [2]) such that $x \,\mathcal{R}\, y$. The subset H is open in $\bar{T}$ and $x \in H$: thus $y \in H$. In this case $(\mathcal{R}\{y\} \cup \{y\}) \cap p = \{y\}$; hence y is isolated in cp.

(⇐) Let $(X;T,\bar{T})$ be a bi-topological space which satisfies the properties (i) and (ii). Obviously, $d\varnothing = \varnothing$ and $d(p+q) = dp + dq$ for all clopen sets p,q of T. Let p be a nonempty clopen set of T. Consider the open set $H = X$; by Condition (ii) there exists an isolated element y of p in cp. Therefore y is isolated in p. As a consequence, we have that $p \nsubseteq dp$. Finally, let us prove that $ddp \subseteq dp$ for every clopen set p of T. Consider an element y of ddp; for every open neighborhood G of y in $\bar{T}$ there exists an element z different from y such that $z \in G \cap dp$. By Condition (ii) there is an element $w \in G \cap p$ which is isolated in cp. Then y must be different from w and so $y \in dp$.

In general, the topological space $(X;\bar{T})$ of a DA is not scattered. Actually, consider the DA $\mathfrak{A}$ in Example 1 and call X the set of all ultrafilters of $\mathfrak{A}$. *There exists no topology $\tilde{T}$ such that $(X;\tilde{T})$ is scattered and $\tilde{d}p = \sigma p$ for every clopen* *set p in $\mathfrak{A}$* (where $\tilde{d}$ is the derived set operator in $\tilde{T}$). Indeed, for every $m \in \omega$ let p_m denote the subset $\{y_0, \ldots, y_m\}$ of Y. Each p_m is an element of $\mathfrak{A}$ and $\tilde{d}p_{m+1} = \sigma p_{m+1} = p_m$. Define H to be $\bigcup_{m \in \omega} p_m$ (where each p_m is regarded as a subset of X); then $\tilde{d}H = \tilde{d} \bigcup_{m \in \omega} p_m \supseteq \bigcup_{m \in \omega} \tilde{d}p_m = H$. Hence H is dense in itself. We can conclude that the space $(X; \tilde{T})$ is not scattered.

4 Reflexive elements

In this section, given a *DA* $\mathfrak{A}$, we shall study a subset Z of the dual space $(X;\mathcal{R})$ of $\mathfrak{A}$ which is of some interest.

Definition 3 A subset W of X is said to be *representative* for the *DA* $\mathfrak{A}$ with respect to the *relation* $\mathcal{R}$ if:

(1) W is dense in X

(2) $\mathcal{R}$ and $\mathcal{R}_{|W \times W}$ define the same operation τ on $\mathfrak{A}$, that is, $\tau p \cap W = \{y \in W / y\ \mathcal{R}_{|W \times W}\ x$ for no $x \in \nu p \cap W\}$ for every $p \in \mathfrak{A}$.

If the *DA* $\mathfrak{A}$ is atomic, the set Y of all atoms of $\mathfrak{A}$ is a dense subset of the Stone space, but, in general, Y does not satisfy Condition (2)—see the *DA* $\mathfrak{B}$ of the Example 2 in which $\mathcal{R}_{|Y \times Y} = \varnothing$, but σ is not trivial.

Theorem 4 *Let Z be the set of all irreflexive elements of X, that is, $Z = \{x \in X / x\ \not\mathcal{R}\ x\}$. The set Z is representative for the DA $\mathfrak{A}$ with respect to $\mathcal{R}$.*

Proof: Let p be an element of $\mathfrak{A}$ (different from 0); p must contain an irreflexive element because the relation $\mathcal{R}$ is relatively reverse well-founded. Then $p \cap Z \neq \varnothing$.

Let τ^* be the operation defined by $\mathcal{R}_{|Z \times Z} = \mathcal{R}^*$, that is, $\tau^* p_Z = \{y \in Z / y\ \mathcal{R}^*\ x$ for no $x \in \nu p_Z\}$, where p_Z and νp_Z stand for $p \cap Z$ and $\nu p \cap Z$ respectively. We have to prove that $\tau^* p_Z = \tau p \cap Z$ for every $p \in \mathfrak{A}$. Assume that $y \notin \tau p \cap Z$; hence $R\{y\} \cap \nu p \neq \varnothing$. By Lemma 7 in [2] there exists a maximal element x of νp such that $y\ \mathcal{R}\ x$. By the maximality of x we have that $x \in Z$; hence $x \in Z \cap \nu p = \nu p_Z$. This implies $y \notin \tau^* p_Z$. The other inclusion is obvious.

We can conclude that if we remove all reflexive elements from the space X (which is convenient in several cases), we modify neither the Boolean structure nor the operation τ. More generally, reflexive points can be eliminated from any representative subspace W of X; in other words, if we take them away from W, we obtain again a representative subspace of X. The property is expressed by the following theorem, whose proof is quite similar to the proof of Theorem 4.

Theorem 5 *If W is representative with respect to $\mathcal{R}$, so is $W \cap Z$.*

Slightly modifying Definition 3 we get a topological concept.

Definition 4 A subspace W of X is said to be *representative* for the *DA* $\mathfrak{A}$ with respect to the *topology* $\bar{T}$ if:

(1) W is dense in X

(2) $\bar{T}$ restricted to W (which is denoted by $\bar{T}_{|W}$) and $\bar{T}$ induce the same operation σ on $\mathfrak{A}$, that is, $dp \cap W$ equals the derived set of $p \cap W$ in the topology $\bar{T}_{|W}$.

As may be expected, Theorems 4 and 5 can be translated into topological terms. More precisely, we have

Lemma 3 *W is representative with respect to $\mathcal{R}$ iff it is representative with respect to $\bar{T}$.*

Corollary 1
(a) *The set Z of all irreflexive points is representative for $\mathfrak{A}$ with respect to $\bar{T}$.*
(b) *If W is a representative subspace of X, so is $W \cap Z$.*

Proof: We omit the proofs, which are not difficult.

From a topological point of view, the space Z with the topology $\bar{T}_{|Z}$ enjoys a good property. Note that the topological space $(X; \bar{T})$ is not usually $\mathfrak{I}_0$. (Consider the *DA* $\langle \mathcal{P}(\omega); \sigma \rangle$ where σ is induced by the usual relation $>$ on ω. Every pair of nonprincipal ultrafilters x and y are such that $x \mathcal{R} y$ and $y \mathcal{R} x$. Therefore x and y have the same neighborhoods.)

Theorem 6 *The subspace $(Z; \bar{T}_{|Z})$ is $\mathfrak{I}_0$ (but in general it is not $\mathfrak{I}_1$).*

Proof: For every x and y of Z, we cannot have both $x \mathcal{R} y$ and $y \mathcal{R} x$. If $y \not\mathcal{R} x$, the open subset $\mathcal{R}\{y\} \cup \{y\}$ does not contain x.

The following theorems characterize the reflexive elements (and hence the subspace Z) in purely topological terms. The first one regards the topology $\bar{T}$, while the other one is more general because it applies to every topology in which the derived set operator equals the operation σ.

Theorem 7 *An element x of X is reflexive iff there exists an open subset H of $(X; \bar{T})$ such that $x \in c\{y\}$ for every $y \in H$ and $x \in \bar{H} - H$ (where c is the closure operator in $\bar{T}$, while $\bar{H}$ is the closure of H in the Stone topology).*

Proof: ($\Rightarrow$) Let x be reflexive; the subset $H = \mathcal{R}\{x\} - \{x\}$ is nonempty because x is not a maximal element of X. It is trivial that $x \in c\{y\}$ for every $y \in H$. Now, let us prove that $x \in \bar{H} - H$. Since $x \mathcal{R} x$, for every clopen set p of T containing x we have that $x \in \sigma p$; hence $x \notin \tau \nu p$. By Lemma 7 of [2] there exists a maximal element z of p such that $x \mathcal{R} z$; z is an element of H, too. Therefore x is an accumulation point of H.

($\Leftarrow$) We must show that for every clopen set p of T from $x \in p$ it follows $x \in \sigma p$. By the hypothesis, there exists an element y of $H \cap p$ such that $x \mathcal{R} y$. Then x belongs to σp.

Theorem 8 *Let $\tilde{T}$ be a topology on X such that $\tilde{T}$ represents the operation σ. An element x of X is reflexive iff $x \in \overline{G - \{x\}}$ for every open neighborhood G of x in $\tilde{T}$ (where $\overline{G - \{x\}}$ is the closure of $G - \{x\}$ in the Stone topology T).*

Proof: Straightforward.

5 Homomorphisms between DA's Throughout this section, we consider two *DA*'s $\langle \mathfrak{A}; \tau_{\mathfrak{A}} \rangle$ and $\langle \mathfrak{B}; \tau_{\mathfrak{B}} \rangle$. We denote the corresponding bi-topological spaces by $(X; T_{\mathfrak{A}}, \bar{T}_{\mathfrak{A}})$ and $(Y; T_{\mathfrak{B}}, \bar{T}_{\mathfrak{B}})$, while $\mathcal{R}_{\mathfrak{A}}$ and $\mathcal{R}_{\mathfrak{B}}$ are the relations induced by $\tau_{\mathfrak{A}}$ and $\tau_{\mathfrak{B}}$ on X and Y, respectively.

Given a homomorphism $f: \mathfrak{A} \to \mathfrak{B}$, define a function f_* from Y to X in the usual way: $f_* y = \{p \in \mathfrak{A}/fp \in y\} = f^{-1}y$ for every $y \in Y$. By the Stone duality f_* is continuous with respect to the topologies $T_{\mathfrak{A}}$ and $T_{\mathfrak{B}}$. First, let us show that the following implication holds: if $x \mathcal{R}_{\mathfrak{B}} y$ then $f_* x \mathcal{R}_{\mathfrak{A}} f_* y$ for every $x, y \in Y$. Assume to the contrary that there exist two elements x and y of Y such that $x \mathcal{R}_{\mathfrak{B}} y$ and not $f_* x \mathcal{R}_{\mathfrak{A}} f_* y$. As a consequence, there exists a clopen set p of $(X; T_{\mathfrak{A}})$ such that $f_* x \in \tau_{\mathfrak{A}} p$ but $f_* y \notin p$. Therefore $x \in f\tau_{\mathfrak{A}} p = \tau_{\mathfrak{B}} fp$ and $y \notin fp$. Hence it is not true that $x \mathcal{R}_{\mathfrak{B}} y$.

Now, we are in a position to give a first characterization of the homomorphisms between DA's.

Theorem 9 *If f: $\langle \mathfrak{A}, \tau_{\mathfrak{A}} \rangle \to \langle \mathfrak{B}, \tau_{\mathfrak{B}} \rangle$ is a homomorphism, then the dual function f_*: $Y \to X$ satisfies the following conditions*:
(1) *f_* is continuous with respect to the Stone topologies $T_{\mathfrak{B}}$ and $T_{\mathfrak{A}}$*
(2) *f_* is an open function with respect to the topologies $\bar{T}_{\mathfrak{B}}$ and $\bar{T}_{\mathfrak{A}}$*
(3) *if $x \mathcal{R}_{\mathfrak{B}} y$ then $f_* x \mathcal{R}_{\mathfrak{A}} f_* y$ for every $x, y \in Y$.*

Conversely, if h: $Y \to X$ is a function which satisfies (1), (2), (3) *then the dual function h_*: $\mathfrak{A} \to \mathfrak{B}$ (defined in the usual way: $h_* p = h^{-1} p$) is a homomorphism.*

First, let us prove three lemmas.

Lemma 4 *Let f be a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$ and let $x \in X$. The following conditions are equivalent:* (i) *$x \in f_* Y$*; (ii) *if $x \in p$ then $fp \neq 0$ for every $p \in \mathfrak{A}$*; (iii) *fx has the f.i.p.*

Proof: (i) $\Rightarrow$ (ii) Let $x = f_* y = f^{-1} y$ where $y \in Y$, that is, $fx \subseteq y$. If there were a p in x such that $fp = 0$, it would follow $0 \in y$, a contradiction.

(ii) $\Rightarrow$ (iii) Let $p_1, \ldots, p_n \in x$ be such that $fp_1 \cdot \ldots \cdot fp_n = 0$. Since f is a homomorphism, we have $fp_1 \cdot \ldots \cdot fp_n = f(p_1 \cdot \ldots \cdot p_n) = 0$. The element $p_1 \cdot \ldots \cdot p_n$ belongs to x but $f(p_1 \cdot \ldots \cdot p_n) = 0$.

(iii) $\Rightarrow$ (i) Let z be an ultrafilter of $\mathfrak{B}$ such that $fx \subseteq z$. It is readily seen that $f_* z = x$.

Lemma 5 *Let $x \in X$ and $y \in Y$. If $f_* y \mathcal{R}_{\mathfrak{A}} x$ then $x \in f_* Y$. In other words, $f_* Y$ is open in $\bar{T}_{\mathfrak{A}}$.*

Proof: If $x \notin f_* Y$, by Lemma 4 there exists an element p of $\mathfrak{A}$ such that $p \in x$ and $fp = 0$. Since $f_* y \mathcal{R}_{\mathfrak{A}} x$, from $x \in p$ it follows that $y \in f\sigma_{\mathfrak{A}} p = \sigma_{\mathfrak{B}} fp$, which is absurd.

Lemma 6 *Let $x \in X$ and $y \in Y$. If $f_* y \mathcal{R}_{\mathfrak{A}} x$ then the set $fx \cup \tau_{\mathfrak{B}}^{-1} y$ has the f.i.p.*

Proof: First of all, it is trivial that both the subsets fx and $\tau_{\mathfrak{B}}^{-1} y$ are closed under finite intersection. Suppose that the subset $fx \cup \tau_{\mathfrak{B}}^{-1} y$ does not have the f.i.p., that is, there exist a $p \in x$ and a $q \in \tau_{\mathfrak{B}}^{-1} y$ such that $q \cdot fp = 0$; hence $fp \leq \nu q$. Since $q \in \tau_{\mathfrak{B}}^{-1} y$, we have $y \notin \sigma_{\mathfrak{B}} \nu q$; besides, in view of the hypothesis $f_* y \mathcal{R}_{\mathfrak{A}} x$ and of the fact that $p \in x$, the element y belongs to $\sigma_{\mathfrak{B}} fp$. We conclude that $y \in \sigma_{\mathfrak{B}} \nu q$, a contradiction.

Proof of Theorem 9: ($\Rightarrow$) Condition (1) is well-known. Consider an open set H of $\bar{T}_{\mathfrak{B}}$. Let $f_* y \mathcal{R}_{\mathfrak{A}} x$, where $y \in H$ and $x \in X$: we have to prove that $x \in f_* H$. By Lemma 6, there exists an ultrafilter z such that $fx \cup \tau_{\mathfrak{B}}^{-1} y \subseteq z$; hence $fx \subseteq$

z, that is equivalent to $f_* z = x$. Moreover, $\tau_{\mathfrak{B}}^{-1} y \subseteq z$ and so, for every $q \in \mathfrak{B}$, if $y \in \tau_{\mathfrak{B}} q$ then $z \in q$, that is, $y \,\mathcal{R}_{\mathfrak{B}}\, z$. From our hypothesis it follows that $z \in H$. This concludes the proof of Condition (2). We have already proved Condition (3).

($\Leftarrow$) By Condition (1) we have that h_* preserves the Boolean structure. Let us prove that $h_* \sigma_{\mathfrak{A}} p = \sigma_{\mathfrak{B}} h_* p$ for every $p \in \mathfrak{A}$. If y is an element of $h_* \sigma_{\mathfrak{A}} p$, that is, $hy \in \sigma_{\mathfrak{A}} p$, then there exists an element x of p such that $hy \,\mathcal{R}_{\mathfrak{A}}\, x$. By Lemma 7 of [2] we can choose x to be maximal in p, so that $x \not\mathcal{R}\, x$. Consider the open set $G = \mathcal{R}\{y\} \cup \{y\}$ of $\bar{T}_{\mathfrak{B}}$; by Condition (2) hG is open in $\bar{T}_{\mathfrak{A}}$ and then $x \in hG$. Let z be an element of G such that $x = hz$; from $x \in p$ it follows that $z \in h^{-1} p = h_* p$; hence $y \in \sigma_{\mathfrak{B}} h_* p$.

Conversely, let $y \in \sigma_{\mathfrak{B}} h_* p$; then there exists an element $x \in h_* p = h^{-1} p$ such that $y \,\mathcal{R}_{\mathfrak{B}}\, x$. By Condition (3) $hy \,\mathcal{R}_{\mathfrak{A}}\, hx$; hence $hy \in \sigma_{\mathfrak{A}} p$, that is, $y \in h^{-1} \sigma_{\mathfrak{A}} p = h_* \sigma_{\mathfrak{A}} p$.

Our purpose is to find a purely *topological* characterization of homomorphisms between *DA*'s. From this point of view the previous statement is not completely adequate, because of Condition (3). So, we need a topological translation of this condition.

First of all, we have the following

Lemma 7 *A function h: $Y \to X$ is continuous with respect to the topologies $\bar{T}_{\mathfrak{B}}$ and $\bar{T}_{\mathfrak{A}}$ iff from $y \,\mathcal{R}_{\mathfrak{B}}\, x$ it follows either $hy \,\mathcal{R}_{\mathfrak{A}}\, hx$ or $hy = hx$.*

Proof: We omit the straightforward proof.

From Lemma 7 and Condition (3) of Theorem 9 it follows that the dual function of a homomorphism between two *DA*'s is *continuous* also with respect to the topologies $\bar{T}_{\mathfrak{B}}$ and $\bar{T}_{\mathfrak{A}}$. In fact, we are in a position to characterize a particular class of homomorphisms just from a topological point of view.

Corollary 2 *A function f: $\mathfrak{A} \to \mathfrak{B}$ is a surjective homomorphism, that is, $\mathfrak{B}$ is a quotient DA of $\mathfrak{A}$, iff the dual function f_*: $Y \to X$ is one-one, open with respect to the topologies $\bar{T}_{\mathfrak{B}}$ and $\bar{T}_{\mathfrak{A}}$, and continuous with respect to both the topologies $T_{\mathfrak{B}}, T_{\mathfrak{A}}$ and $\bar{T}_{\mathfrak{B}}, \bar{T}_{\mathfrak{A}}$.*

Proof: Trivial.

However, in the statement of Theorem 9, Condition (3) cannot in general be replaced by a simple topological requirement, as the following Examples show.

Example 4: The function h: $Y \to X$ represented in Figure 1 is both open and continuous in respect to *any* topologies, but the dual function h_* is not a homomorphism. (In the diagrams we intend that a lower point x is joined with an upper point y by a climbing up line segment iff $y \,\mathcal{R}\, x$.)

Example 5: The function h: $Y' \to X'$ represented in Figure 2 is a homeomorphism with respect to the Stone topologies, and an open function with respect to the topologies $\bar{T}_{\mathfrak{B}}$ and $\bar{T}_{\mathfrak{A}}$. But the dual function h_* is not a homomorphism.

On the other hand, Condition (2) of Theorem 9 can be expressed in terms of the relation $\mathcal{R}$ as follows: *a function h of Y into X is open iff the image hY*

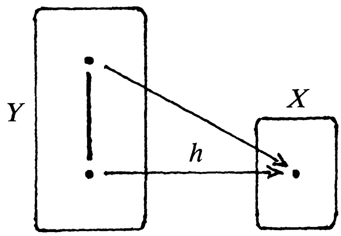

Figure 1

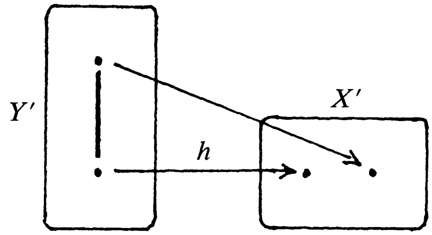

Figure 2

is open and the following property holds: if $hy\ \mathcal{R}_{\mathfrak{A}}\ hx$ (where $hy \neq hx$) then there exists an element z of Y such that $y\ \mathcal{R}_{\mathfrak{B}}\ z$ and $hz = hx$. We omit the proof.

Similar conditions are often studied in modal logic, where the name *p-morphism* is used. However, in these contexts Boolean Algebras of the form $\mathcal{P}(X)$ (represented by the set X) are mostly considered.

In our case, difficulties arise because in Condition (3) of Theorem 9 we have to take into account the possibility that f_*x coincides with f_*y (in which case it is a nonprincipal ultrafilter). In the next section we shall see a topological translation of Condition (3) which applies to irreflexive points, and to reflexive points as well.

Regarding reflexive points, from Condition (3) it follows that if y is a reflexive element of Y then f_*y is a reflexive element of X. On the other hand, f_*y can be reflexive without y being reflexive. Indeed, consider the following

Example 6: Let $\langle \mathfrak{A}; \sigma_{\mathfrak{A}} \rangle$ be the *DA* of finite and cofinite subsets of ω, where $\sigma_{\mathfrak{A}}$ is defined by the usual relation $>$ on ω. Let $\langle \mathfrak{B}; \sigma_{\mathfrak{B}} \rangle$ be the *DA* of finite and cofinite subsets of the set $\omega^* = \omega \cup \{a\}$, where $\sigma_{\mathfrak{B}}$ is defined by the following relation $>^*$ on ω^*: $a > n$ for every $n \in \omega$ and, if $n, m \in \omega$, then $n >^* m$ iff $n > m$. Let X, Y be the sets of all ultrafilters of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Consider the function h: $Y \to X$ such that $hn = n$ for every $n \in \omega$, and $hu' = ha = u$, where u and u' are the nonprincipal ultrafilters of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. By Theorem 9 h induces a homomorphism h_*. The element a of Y is irreflexive, while $ha = u$ is reflexive. On the other hand, there exists a reflexive element u' of Y such that $hu' = ha$. This property is satisfied in every *DA*, as the following theorem says.

Theorem 10 *Let f: $\mathfrak{A} \to \mathfrak{B}$ be a homomorphism. If $f_*y\ \mathcal{R}_{\mathfrak{A}}\ f_*y$, where $y \in Y$, then there exists an $x \in Y$ such that $f_*x = f_*y$ and $x\ \mathcal{R}_{\mathfrak{B}}\ x$.*

To prove the theorem we need the following lemma.

Lemma 8 *Let $(X; T)$ be the Stone space of a DA $\mathfrak{A}$. Every nonempty closed set C contains a "semimaximal" element z, that is, an element z such that, if $z\ \mathcal{R}\ x$ for some $x \in C$, then $x\ \mathcal{R}\ z$ (and hence $z\ \mathcal{R}\ z$).*

Proof: Consider the following partially ordered set $(\{\mathcal{R}\{x\} \cap C / x \in C\}, \supseteq)$. Since C is compact and every $\mathcal{R}\{x\}$ is closed, the previous set is inductive. By Zorn's Lemma there exists an element z of C such that $\mathcal{R}\{z\} \supset \mathcal{R}\{x\}$ for no

$x \in C$. If z is maximal, then it is also semimaximal. If not, there is $x \in C$ such that $z \mathcal{R} x$ and $\mathcal{R}\{x\} = \mathcal{R}\{z\}$, so that $x \mathcal{R} x$. It is immediate that x is semimaximal.

Proof of Theorem 10: Let F be the filter generated by the set $f_* fy \subseteq \mathfrak{B}$ and let C be the corresponding closed set in the Stone topology. It is readily seen that $C = \bigcap \{p/p \in F\} = \{x/f_*x = f_*y\}$. By Lemma 8 in C there exists a semimaximal element z; hence $f_*z = f_*y$ and $f_*z \mathcal{R}_{\mathfrak{A}} f_*y$. By Lemma 6 there exists an ultrafilter containing the set $F \cup \tau_{\mathfrak{B}}^{-1}\{z\}$; thus $C \cap \mathfrak{R}_{\mathfrak{B}}\{z\}$ is nonempty. Let $w \in C \cap \mathcal{R}_{\mathfrak{B}}\{z\}$; from $w \in \mathcal{R}_{\mathfrak{B}}\{z\}$ it follows $z \mathcal{R}_{\mathfrak{B}} w$ and from the semimaximality of z in C it follows that $w \mathcal{R}_{\mathfrak{B}} z$; hence $z \mathcal{R}_{\mathfrak{B}} z$.

6 Topological duality

First of all, our aim is to settle the question raised in the preceding section and to find a purely topological characterization of the dual function of a homomorphism between two *DA*'s.

Lemma 9 *Let $\mathfrak{A}$ be a DA and let $(X; T, \bar{T})$ be its bi-topological space. For every $x, y \in X$ the following conditions are equivalent:*
(α) $x \mathcal{R} y$
(β) *if G is an open neighborhood of x in $\bar{T}$ then $y \in \overline{G - \{x\}}$ (where $\overline{G - \{x\}}$ is the closure of $G - \{x\}$ in the Stone topology).*

Proof: $(\alpha) \Rightarrow (\beta)$ Suppose $x \mathcal{R} y$; if $x \neq y$, then every open set G of $\bar{T}$ containing x must contain y too. Hence $y \in G - \{x\} \subseteq \overline{G - \{x\}}$. If $x = y$, by Theorem 8 $y \in \overline{G - \{x\}}$ for every open set G of $\bar{T}$ containing x.

$(\beta) \Rightarrow (\alpha)$ Suppose $x \not\mathcal{R} y$. Let $x \neq y$; the subset $G = \mathcal{R}\{x\} \cup \{x\}$ is open in $\bar{T}$ and closed in T. The element y does not belong to $G \supseteq \overline{G - \{x\}}$. If $x = y$, by Theorem 8 there exists an open set G of $\bar{T}$ containing x such that $x = y \notin \overline{G - \{x\}}$.

Corollary 3 *If f: $\mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then the dual function f_*: $Y \to X$ satisfies the following conditions*:
(1) *f_* is continuous with respect to the Stone topologies $T_{\mathfrak{B}}$ and $T_{\mathfrak{A}}$*
(2) *f_* is open with respect to the topologies $\bar{T}_{\mathfrak{B}}$ and $\bar{T}_{\mathfrak{A}}$*
(3′) *if for every open neighborhood G of x in $\bar{T}_{\mathfrak{B}}$ the element y belongs to $\overline{G - \{x\}}$, then for every open neighborhood H of f_*x in $\bar{T}_{\mathfrak{A}}$ the element f_*y must belong to $\overline{H - \{f_*x\}}$.*

Conversely, if h: $Y \to X$ is a function which satisfies (1), (2), (3′), *then the dual function h_*: $\mathfrak{A} \to \mathfrak{B}$ is a homomorphism.*

Proof: By Theorem 9 and Lemma 9.

Let us briefly examine the dual space of a *subalgebra* of a *DA*. If $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, then there exists a one-one homomorphism ι from $\mathfrak{A}$ into $\mathfrak{B}$; by the previous corollary the dual function ι_* satisfies Conditions (1), (2), (3′) and is surjective; and conversely.

We have already seen a topological translation of the algebraic concept of quotient algebra (Corollary 2). In fact the set of all congruences of a *DA* $\mathfrak{A}$, the set of all its τ-filters (see Definition 2 in [10]), and the set of all quotient algebras of $\mathfrak{A}$ (up to isomorphisms) can be easily proved to be bijective. In any case, we can directly characterize the *τ-filters* (and hence the congruences) of a *DA* in topological terms.

Theorem 11 *Let $\mathfrak{A}$ be a DA and let $(X; T, \bar{T})$ be the corresponding bi-topological space. There exists a natural bijection between the set of all τ-filters of $\mathfrak{A}$ and the set of all subsets of X which are simultaneously open in $\bar{T}$ and closed in the Stone topology T.* (Similar results—but with no reference to the topology $\bar{T}$—are stated in [1] and in [12].)

Proof: Let F be a τ-filter of $\mathfrak{A}$. We know that the subset $H = \cap F$ is closed in T; let us prove that H is open in $\bar{T}$. Suppose to the contrary that $x \in H$ and $x \mathcal{R} y$, but $y \notin H$. There exists a clopen set p of T such that $p \in F$ and $p \notin y$. Hence $\nu p \in y$; since $x \mathcal{R} y$, we have $x \notin \tau p$. Therefore $\tau p \notin F$, which is absurd.

Conversely, let H be closed in T and open in $\bar{T}$. The subset $F = \{p \in \mathfrak{A} / H \subseteq p\}$ is a filter of $\mathfrak{A}$; let us show that F is a τ-filter. Suppose that there exists $p \in F$ such that $\tau p \notin F$. Thus we can find an element x of H which does not belong to τp. Let y be an element of νp such that $x \mathcal{R} y$; since H is open in $\bar{T}$ it follows that $y \in H \subseteq p$, which is absurd.

Now, let us consider the *direct product* $\langle \mathfrak{C}; \tau_{\mathfrak{C}} \rangle$ of two *DA*'s $\langle \mathfrak{A}; \tau_{\mathfrak{A}} \rangle$ and $\langle \mathfrak{B}; \tau_{\mathfrak{B}} \rangle$. Let $(X; T_{\mathfrak{A}}, \bar{T}_{\mathfrak{A}})$, $(Y; T_{\mathfrak{B}}, \bar{T}_{\mathfrak{B}})$ and $(Z; T, \bar{T})$ be the bi-topological spaces corresponding to $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, respectively. Then the set Z is the disjoint union of X and Y. Moreover, both X and Y are open in $\bar{T}$ because no element of X is associated with an element of Y, and conversely. Therefore, with regard to both the topologies T and $\bar{T}$, the open subsets of Z are exactly the unions of the corresponding open subsets of X and Y. In other words, the subspaces X and Y are a disconnection of the space Z with respect to both the topologies T and $\bar{T}$. This property characterizes the bi-topological space of a direct product of *DA*'s in topological terms.

Theorem 12 *Let $(Z; T, \bar{T})$ be a relatively scattered bi-topological space. If there exist two subspaces X and Y of Z which are the components of a disconnection in both the topologies T and $\bar{T}$, then the dual DA $\mathfrak{C}$ of $(Z; T, \bar{T})$ is isomorphic to the direct product of the dual DA's of $(X; T_{|X}, \bar{T}_{|X})$ and $(Y; T_{|Y}, \bar{T}_{|Y})$.*

Proof: It is immediate that the spaces $(X; T_{|X}, \bar{T}_{|X})$ and $(Y; T_{|Y}, \bar{T}_{|Y})$ are relatively scattered; let $\langle \mathfrak{A}; \sigma_{\mathfrak{A}} \rangle$ and $\langle \mathfrak{B}; \sigma_{\mathfrak{B}} \rangle$ be the corresponding *DA*'s. Consider the function $f: \mathfrak{C} \to \mathfrak{A} \times \mathfrak{B}$ where $fp = (p \cap X, p \cap Y)$ for every $p \in \mathfrak{C}$. It is just an easy verification that f is a Boolean isomorphism. To prove that f is a homomorphism between *DA*'s it is enough to verify that $\sigma_{\mathfrak{C}} p \cap X = \sigma_{\mathfrak{A}}(p \cap X)$ and $\sigma_{\mathfrak{C}} p \cap Y = \sigma_{\mathfrak{B}}(p \cap Y)$. The operations $\sigma_{\mathfrak{C}}, \sigma_{\mathfrak{A}}, \sigma_{\mathfrak{B}}$ can be considered as the derived sets operators in the spaces $(Z; \bar{T})$, $(X; \bar{T}_{|X})$, $(Y; \bar{T}_{|Y})$, respectively. Since both X and Y are clopen sets also with respect to the topology $\bar{T}$, the previous equalities hold.

7 Applications First we study the product $\triangleleft_a$ of *DA*'s defined by Sonobe in [15] from a topological point of view. Let us recall that if $\langle \mathfrak{A}; \tau_{\mathfrak{A}} \rangle$ and $\langle \mathfrak{B}; \tau_{\mathfrak{B}} \rangle$ are two *DA*'s then by $\mathfrak{A} \triangleleft_a \mathfrak{B}$ (where $a \in \mathfrak{A}$ and $a \leq \tau_{\mathfrak{A}} 0$) we mean the Boolean Algebra $\mathfrak{A} \times \mathfrak{B}$ with the operation τ defined as follows: $\tau(p, 1) = (\tau_{\mathfrak{A}} p, 1)$ and

$\tau(p,q) = (a, \tau_{\mathfrak{B}} q)$ if $q \neq 1$. (The symbol $\mathfrak{A} \lhd \mathfrak{B}$ denotes $\mathfrak{A} \lhd_0 \mathfrak{B}$.) In [15] it is shown that $\mathfrak{A} \lhd_a \mathfrak{B}$ is a *DA*.

From a Boolean point of view, the $\lhd$ product coincides with the direct product; hence the set of all ultrafilters of $\mathfrak{A} \lhd_a \mathfrak{B}$ is the disjoint union $X \dot{\cup} Y$ of the sets X and Y of all ultrafilters of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Let $\mathcal{R}_{\mathfrak{A}}$, $\mathcal{R}_{\mathfrak{B}}$, $\bar{T}_{\mathfrak{A}}$ and $\bar{T}_{\mathfrak{B}}$ have the usual meaning.

It is readily verified that the relation $\mathcal{R}$ induced by τ on $X \dot{\cup} Y$ is the following:

if $x, z \in X$ then $x \mathcal{R} z$ iff $x \mathcal{R}_{\mathfrak{A}} z$
if $y, t \in Y$ then $y \mathcal{R} t$ iff $y \mathcal{R}_{\mathfrak{B}} t$
if $x \in X$ and $y \in Y$ then $y \not\mathcal{R} x$, while $x \mathcal{R} y$ iff $x \notin a$.

We notice that if $a = 0$ then $x \mathcal{R} y$ for every $x \in X$ and $y \in Y$.

Let $\bar{T}^{\lhd a}$ be the corresponding topology on the set $X \dot{\cup} Y$. A subset is open in $\bar{T}^{\lhd a}$ iff it is either the union of the space Y with an open set in $\bar{T}_{\mathfrak{A}}$, or the union of an open set in $\bar{T}_{\mathfrak{B}}$ with a subset of a (necessarily open in $\bar{T}_{\mathfrak{A}}$). Clearly $\bar{T}^{\lhd a}$ (if $a \neq 1$) is strictly less fine than the topology $\bar{T}_{\mathfrak{A}\times\mathfrak{B}}$ that is defined on the same set $X \dot{\cup} Y$ when we consider the usual direct product of *DA*'s. Intuitively, in $\bar{T}_{\mathfrak{A}\times\mathfrak{B}}$ the subsets X and Y are placed "side by side", while in $\bar{T}^{\lhd a}$ (using the same convention as in Figures 1 and 2) all the elements of X which do not belong to a are placed above Y, and the elements of a are at the bottom level in $X \dot{\cup} Y$. The following theorem characterizes the bi-topological spaces of *DA*'s which are of the form $\mathfrak{A} \lhd \mathfrak{B}$ and $\mathfrak{A} \lhd_a \mathfrak{B}$.

Theorem 13 *Let $(Z; T, \bar{T})$ be a relatively scattered bi-topological space; call $\mathfrak{C}$ the corresponding DA.*

(α) *$\mathfrak{C}$ is of the form $\mathfrak{A} \lhd \mathfrak{B}$ iff there exists a nonempty subset p of Z such that*: (i) *p is clopen in T and open in $\bar{T}$;* (ii) *for every open set H of $\bar{T}$ either $p \subseteq H$ or $H \subseteq p$.*

(β) *$\mathfrak{C}$ is of the form $\mathfrak{A} \lhd_a \mathfrak{B}$ iff there exists a nonempty subset p of Z such that:* (i) *p is clopen in T and open in $\bar{T}$;* (ii) *for every open set H of $\bar{T}$, either $H \cap dZ \subseteq p \cap dZ$ or $H \supseteq p$.*

Proof: For the sake of brevity we only prove (β).

The necessity is trivial. Conversely, define a to be $\nu p \cap \nu dZ$; notice that a is open in $\bar{T}$ (as it contains only isolated points) and clopen in T (since dZ is a clopen set). We claim that $\mathfrak{C} = \mathfrak{A} \lhd_a \mathfrak{B}$ where $\mathfrak{A}$ and $\mathfrak{B}$ are the *DA*'s which correspond to the bi-topological relatively scattered spaces $(\nu p, T_{|\nu p}, \bar{T}_{|\nu p})$ and $(p, T_{|p}, \bar{T}_{|p})$. It is enough to show that $\bar{T} = \bar{T}^{\lhd a}$ where the last symbol denotes the topology which induces σ in the *DA* $\mathfrak{A} \lhd_a \mathfrak{B}$. Let H be an open set of $\bar{T}$. If $H \cap \nu p \cap dZ \neq \emptyset$, then, by our hypothesis, $H \supseteq p$. Hence $H = p \cup (H \cap \nu p)$, where $H \cap \nu p$ is an open set of $\bar{T}_{|\nu p}$. If, on the other hand, $H \cap \nu p \cap dZ = \emptyset$, we have $H = (H \cap p) \cup (H \cap \nu p \cap \nu dZ)$, where $H \cap p$ is open in $\bar{T}_{|p}$ and $H \cap \nu p \cap \nu dZ$ is open in $\bar{T}_{|\nu p}$ and it is contained in a. Therefore, in both cases H is open in $\bar{T}^{\lhd a}$.

Let K be an open set of $\bar{T}^{\lhd a}$. If K is the union of an open set of $\bar{T}_{|\nu p}$ and p, then K may be expressed as $p \cup H$ where H is open in $\bar{T}$. Hence K is open in $\bar{T}$. Similarly, any union $L \cup M$, where $L \subseteq \nu p \cap \nu dZ$ and M is open in $\bar{T}_{|p}$, can be proved to be open in $\bar{T}$.

In [15] subdirectly irreducible (shortly, *s.d.i.*) *DA*'s (see for instance [7]) are characterized by means of the product $\lhd$.

Proposition 3 *A DA $\mathfrak{A}$ is s.d.i. iff there exists a DA $\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to the DA $2 \lhd \mathfrak{B}$, where 2 is the DA with only two elements.*

Now we can find a topological characterization of s.d.i. *DA*'s. First of all, we recall that a minimal element with respect to the relation $\mathfrak{R}$ (in particular a minimum element) must be irreflexive.

Theorem 14 *Let $\mathfrak{A}$ be a DA and let $(X; T, \bar{T})$ be the corresponding bi-topological space. The following conditions are equivalent:*
(1) *$\mathfrak{A}$ is s.d.i.*
(2) *there exists a nonempty proper subset H of X which is open in $\bar{T}$, closed in T, and such that if $K \subset X$ is open in $\bar{T}$ and closed in T, then $K \subseteq H$*
(3) *in X there exists a minimum element y with respect to the relation $\mathfrak{R}$.*

Proof: (1) $\Rightarrow$ (2) By Proposition 3.

(2) $\Rightarrow$ (3) Suppose that X has no minimum element. Since H is closed in T there exists a clopen set p of T such that $H \cap p = \emptyset$. Let z be a maximal element of p with respect to $\mathfrak{R}$. By our assumption there exists an element x different from z such that $x \,\mathfrak{R}\, z$. The subset $K = H \cup \{z\} \cup \mathfrak{R}\{z\}$ is strictly contained in X (because $x \notin K$), it is open in $\bar{T}$ and closed in T, but $H \subset K$ because $z \notin H$.

(3) $\Rightarrow$ (1) Let y be the minimum element of X. We claim that the subset $X - \{y\}$ is closed, and as a consequence clopen in T. Indeed, assuming the contrary, it can be shown that y satisfies the hypothesis of Theorem 7 and hence it is reflexive, which is absurd. It follows that $X - \{y\}$ is a relatively scattered space; let $\mathfrak{B}$ be its dual *DA*. It is immediate that $\mathfrak{A}$ is isomorphic to $2 \lhd \mathfrak{B}$; by Proposition 3 we can conclude that $\mathfrak{A}$ is s.d.i.

It is of interest to discuss connections between topological and algebraic properties. We recall that a topological space X is said to be *irreducible* if any two nonempty open sets intersect; and that a point y of a topological space X is called *generic* if $c\{y\} = X$, where c is the closure operator.

Definition 5 (essentially from [15]) A *DA* $\mathfrak{A}$ is said to be *co-s.d.i.* if the lattice of all congruences of $\mathfrak{A}$ has one and only one co-atom θ which contains every proper congruence.

Theorem 15 *Let $\mathfrak{A}$ be a DA and let $(X; T, \bar{T})$ be its bi-topological space. The following conditions are equivalent:*
(1) *$\mathfrak{A}$ is co-s.d.i.*
(2) *there is a maximum τ-filter in $\mathfrak{A}$*
(3) *in X there exists a nonempty subset H which is open in $\bar{T}$, closed in T, such that if $K \neq \emptyset$ is open in $\bar{T}$ and closed in T, then $H \subseteq K$*
(4) *there exists a DA $\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to $\mathfrak{B} \lhd 2$*
(5) *$\tau 0$ is an atom of $\mathfrak{A}$*
(6) *the space $(X; \bar{T})$ contains a generic point*
(7) *the space $(X; \bar{T})$ is irreducible.*

Proof: We only sketch the proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Theorem 11. By the definition of $\triangleleft$ product, Conditions (3), (4), (5) are equivalent. If $\tau 0$ is an atom of $\mathfrak{A}$, then $\tau 0$, regarded as an ultrafilter, is a generic point: hence X is irreducible. If $\tau 0$ is not an atom of $\mathfrak{A}$, then there exist two different elements x, y of X such that $\tau 0 \in x$ and $\tau 0 \in y$. The subsets $\{x\}$ and $\{y\}$ are disjoint open sets of $\bar{T}$; therefore X is not irreducible. This concludes the proof.

In a similar way, other algebraic properties can be discussed in topological terms, and conversely. As an example, we state without proof the following

Theorem 16 *Consider the properties:*
(α) *the DA $\mathfrak{A}$ is s.d.i.*
(β) *X has finitely many minimal elements, and for every nonminimal $x \in X$, there exists a minimal element $y \in X$ such that $y \; \mathcal{R} \; x$*
(γ) *$(Z; \bar{T}_{|Z})$ is compact (where Z is the set of all irreflexive elements of X)*
(δ) *the DA $\mathfrak{A}$ is not ω-consistent (see [9]).*
Exactly the following implications hold:

$$(\alpha) \Rightarrow (\beta) \Leftrightarrow (\gamma) \Rightarrow (\delta).$$

We conclude examining a possible application of another topological concept. Recall that a *sheaf* on a topological space X can be regarded as a local homeomorphism of a topological space Y onto X. In the following the symbols **H**,**S**,**P** have the usual algebraic meaning (see [7]).

Theorem 17 *Let $\mathfrak{A}, \mathfrak{B}$ be two DA's and let $(X; T_{\mathfrak{A}}, \bar{T}_{\mathfrak{A}})$, $(Y; T_{\mathfrak{B}}, \bar{T}_{\mathfrak{B}})$ be their bi-topological spaces. If $\mathfrak{A}$ is finite and $B \in \mathbf{HP}\mathfrak{A}$, then there exists a local homeomorphism h of $(Y; \bar{T}_{\mathfrak{B}})$ into $(X; \bar{T}_{\mathfrak{A}})$. (In other words, h is a sheaf on a subspace of X.) Moreover, h is continuous with respect to the topologies $T_{\mathfrak{B}}$ and $T_{\mathfrak{A}}$.*

We need two lemmas.

Lemma 10 *Let $\mathfrak{A}$ be a finite Boolean Algebra. Consider the Algebra $\mathfrak{A}^I$, where I is any set. Then the Stone space of $\mathfrak{A}^I$ is homeomorphic to the topological product between the Stone spaces of $\mathcal{P}(I)$ and of $\mathfrak{A}$. In other words, every ultrafilter x of $\mathfrak{A}^I$ can be identified with a pair (u, a) where u is an ultrafilter of $\mathcal{P}(I)$ and a is an atom of $\mathfrak{A}$.*

Proof: We only sketch the proof. If $p \in \mathfrak{A}$, we write $\bar{p}$ to denote the element of $\mathfrak{A}^I$ such that $(\bar{p})_i = p$ for all $i \in I$. Let x be an ultrafilter of $\mathfrak{A}^I$. If $a_0, \ldots, a_{n-1}$ are the atoms of $\mathfrak{A}$, then there exists one and only one $j \in n$ such that $\bar{a}_j \in x$. Consider the set $u = \{H/H \subseteq I$ and there exists a $q \in x$ such that $H = \{h/q_h \neq 0\}\}$; it is readily seen that u is an ultrafilter of $\mathcal{P}(I)$. Thus, the ultrafilter x of $\mathfrak{A}^I$ is associated with the pair (u, a_j). Conversely, let a be an atom of $\mathfrak{A}$ and let u be an ultrafilter of $\mathcal{P}(I)$. It is not difficult to verify that the set $x = \{q \in \mathfrak{A}^I / \{h/a \leq q_h\} \in u\}$ is an ultrafilter of $\mathfrak{A}^I$.

Lemma 11 *Let $\mathfrak{A}$ be a finite DA. Let $(X; \mathcal{R})$ and $(W; \mathcal{R}')$ be the dual spaces of $\mathfrak{A}$ and $\mathfrak{A}^I$, respectively. Then, $(u, a) \; \mathcal{R}' \; (v, b)$ iff $u = v$ and $a \; \mathcal{R} \; b$.*

Proof: (⇒) We prove first that $a \,\mathcal{R}\, b$. Let p be an element of $\mathfrak{A}$ which contains b; we must prove that $a \leq \sigma p$. Since the element $\bar{p}$ belongs to (v,b), by the hypothesis $\overline{\sigma p} \in (u,a)$. Therefore the subset $\{h/a \leq \overline{\sigma p}_h\} \in u$, that means $a \leq \sigma p$. Now let us show that $u = v$. If $K \in v$ then there exists a $q \in (v,b)$ such that $K = \{k/q_k \neq 0\} = \{k/b \leq q_k\}$. From $a \,\mathcal{R}\, b$ it follows $a \leq \sigma b \leq \sigma q_k$ for all $k \in K$, while $a \not\leq \sigma q_k = 0$ for all $k \notin K$. Therefore the subset $\{h/a \leq (\sigma q)_h\} = \{h/(\sigma q)_h \neq 0\}$ equals K. Since $\sigma q \in (u,a)$, we can conclude that $K \in u$.

Conversely, let $H \in u$ and assume that $H \notin v$. Then the element q of $\mathfrak{A}^I$, such that $q_i = 0$ iff $i \in H$ and $q_i = b$ iff $i \notin H$, belongs to (v,b). From $(u,a) \,\mathcal{R}\, (v,b)$ it follows that $\sigma q \in (u,a)$. As $\{k/\sigma q_k \neq 0\} \subseteq I - H$, the set $I - H$ would belong to u, which is absurd.

(⇐) Let $q \in (v,b)$; hence $K = \{k/b \leq q_k\} \in v = u$. From $a \,\mathcal{R}\, b$ it follows that the set $H = \{h/a \leq (\sigma q)_h\}$ contains K; therefore $H \in u$. We can conclude $\sigma q \in (u,a)$. This completes the proof.

From an intuitive point of view Lemma 11 says that $(W, \mathcal{R}')$ is constituted by as many copies of $(X, \mathcal{R})$ as there are ultrafilters in $\mathcal{P}(I)$.

Proof of Theorem 17: We denote the set of ultrafilters of $\mathfrak{A}^I$ by W. Since there exists a homeomorphism f of $\mathfrak{A}^I$ onto $\mathfrak{B}$, then the dual function f_* of Y into W is one-one, that is, Y can be represented as a subset of W. Let π be the projection of W onto X (see Lemma 10). Of course, the function $h = \pi \circ f_*$ is continuous with respect to the Stone topologies. Now, for every $y \in Y$ consider the set $G = \mathcal{R}_{\mathfrak{B}}\{y\} \cup \{y\}$ which is open in $\bar{T}_{\mathfrak{B}}$. In view of Lemma 11, the subset G is contained in $X_u = \{(u,a)/a \in X\} \subseteq W$ for some ultrafilter u of $\mathcal{P}(I)$. We have that $h(u,b) = b$ for every $(u,b) \in G$; therefore $hG = \mathcal{R}_{\mathfrak{A}}\{hy\} \cup \{hy\}$. We can conclude that h is a homeomorphism of G onto hG.

Theorem 18 *Let $\mathfrak{A}, \mathfrak{B}$ be two DA's and let $(X; T_{\mathfrak{A}}, \bar{T}_{\mathfrak{A}})$, $(Y; T_{\mathfrak{B}}, \bar{T}_{\mathfrak{B}})$ be their bi-topological spaces. If $\mathfrak{A}$ is finite and if there exists a local homeomorphism of $(Y; \bar{T}_{\mathfrak{B}})$ into $(X; \bar{T}_{\mathfrak{A}})$, which is continuous with respect to the Stone topologies, then $\mathfrak{B} \in$ **SPH**$\mathfrak{A}$.*

Proof: For every $x \in Y$ consider the subset $H_y = \mathcal{R}_{\mathfrak{B}}\{y\} \cup \{y\}$. By our hypotheses and Theorem 11, each H_y represents a quotient DA $\mathfrak{A}_y$ of $\mathfrak{A}$. Consider the disjoint union of all H_y and call it W. Note that W can be identified with the set $\{(w,y)/w \in H_y\}$. In this sense, we have $W \subseteq Y \times Y$. Introduce in $Y \times Y$ the product of the Stone topologies, and the product between the topology $\bar{T}_{\mathfrak{B}}$ and the discrete topology. Then restrict these topologies to W: in this way W becomes a relatively scattered bi-topological space. Let $\mathfrak{C}$ be the corresponding DA. Since there are just finitely many different $\mathfrak{A}_y$ (up to isomorphism), we have $\mathfrak{C} \in$ **HP**$\mathfrak{A}$. Define a function f: $W \to Y$ as follows: $f(w,y) = w$. The function f satisfies the conditions of Corollary 3 and it is surjective. Hence there exists a one-one homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$.

Corollary 4
(a) *If $\mathfrak{A}$ is a finite DA, we have* **HP**$\mathfrak{A} \subseteq$ **SPH**$\mathfrak{A}$.
(b) **SPH**$\mathfrak{A} =$ **HSP**$\mathfrak{A}$.
(Similar results hold for finitely many finite DA's.)

Proof: (a) By Theorems 17 and 18.

(b) By part (a) and recalling that *DA*'s have the CEP (congruence extension property).

To obtain Corollary 4 in an algebraic way, one could apply Jónnson's Lemma (see [7]), in view of the CEP and of the distributivity of the congruence lattice of a *DA*.

NOTE

1. We use capital Gothic letters both for Boolean Algebras and Diagonalizable Algebras. So a *DA* can be denoted either by $\langle \mathfrak{A}; \tau \rangle$ or simply by $\mathfrak{A}$.

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