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On Proving Functional Incompleteness in Symbolic Logic Classes

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1 Introduction: Functional completeness A set of truth-functional connectives is said to be *functionally complete*[1] if every truth function can be represented by some formula which uses connectives only from that set. In the first semester of a sequence of introductory symbolic logic courses, one normally remarks that the usual connectives $\{\sim, \vee, \rightarrow, \wedge, \leftrightarrow\}$ form a functionally complete set. Typically, one does not rigorously prove this since such proof requires use of mathematical induction—a concept usually reserved for the second semester. However, a method of constructing disjunctive (and conjunctive) normal forms is often given, and the claim is made that every formula of the propositional logic can be treated by this method. The method (for disjunctive normal form) is this: given an arbitrary formula A with n distinct sentence letters in it, represent A's truth table in the usual way. For example consider (the three displayed T rows are the only T rows):

P_1	P_2	P_3	....	P_n	A
T	T	T	...	T	F
T	T	T	...	F	T
⋮					⋮
F	F	F	...	T	T
F	F	F	...	F	T

Now look at each row where A is assigned T and construct a formula which "describes" that row. For the above example, with the three T rows, we "describe" each T row by looking at the truth values (in that row) of the sentence letters. If the value in that row is T we employ the (unnegated) sentence letter; if its value is F we employ the negated sentence letter. We conjoin these employed formulas to make a formula which describes that row. For our three

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rows which are T in the above example we would obtain three formulas, each of which "describes" one of the T rows of A:

$$(P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge \sim P_n)$$
$$(\sim P_1 \wedge \sim P_2 \wedge \sim P_3 \wedge \ldots \wedge P_n)$$
$$(\sim P_1 \wedge \sim P_2 \wedge \sim P_3 \wedge \ldots \wedge \sim P_n).$$

It is now pointed out to the students that each of these three conjunctive formulas is true in exactly one row of its truth table—namely the one it "describes". We now disjoin all these "descriptive" formulas:

$$(P_1 \wedge P_2 \wedge P_3 \wedge \ldots \wedge \sim P_n) \vee (\sim P_1 \wedge \sim P_2 \wedge \sim P_3 \wedge \ldots \wedge P_n) \vee (\sim P_1 \wedge \sim P_2 \wedge \sim P_3 \wedge \ldots \wedge \sim P_n).$$

Since each of the disjuncts is true in exactly one row (namely a row in which A is T), and since there is a disjunct for every T row of A, and since the entire disjunction is true just in case at least one of its disjuncts is true, it follows that the entire disjunction is true precisely when A is. Thus the student becomes convinced that *any* propositional formula can be described using only $\sim, \vee, \wedge$.[2] That is, the student is convinced that $\{\sim, \vee, \wedge\}$ is functionally complete.

At this stage, the instructor reminds the class of certain equivalences such as:

$$(A \vee B) \leftrightarrow \sim(\sim A \wedge \sim B)$$
$$(A \wedge B) \leftrightarrow \sim(\sim A \vee \sim B).$$

By the use of these equivalences, one could replace all the $\vee$'s in the disjunctive normal form formula by $\sim$'s and $\wedge$'s. Therefore $\{\sim, \wedge\}$ is functionally complete. Or, all the $\wedge$'s could be replaced by $\sim$'s and $\vee$'s; so $\{\sim, \vee\}$ is functionally complete. To prove that a proposed set of connectives *is* functionally complete, all one needs to do is show that the connectives of an already-established-as-functionally-complete set can be "defined" by those in the proposed set. For example, to show that $\{\sim, \rightarrow\}$ is functionally complete, we appeal to the functional completeness of $\{\sim, \vee\}$ and the equivalence

$$(A \vee B) \leftrightarrow (\sim A \rightarrow B).$$

Possibly students are introduced to other truth functions such as the Scheffer strokes (NAND [$\uparrow$] and NOR [$\downarrow$]). The truth table for a $\downarrow$ is:

A	B	$(A \downarrow B)$
T	T	F
T	F	F
F	T	F
F	F	T

and the student is asked to show $\{\downarrow\}$ to be functionally complete in itself. After some trials, the student might come up with:

$$\sim A \leftrightarrow (A \downarrow A)$$
$$(A \vee B) \leftrightarrow ((A \downarrow B) \downarrow (A \downarrow B))$$

thereby showing that a functionally complete set of connectives $\{\sim, \vee\}$ can be "defined" by only $\downarrow$; so $\{\downarrow\}$ is functionally complete.

It is also straightforward to establish the functional *in*completeness of many sets of connectives without explicit excursion into mathematical induction. Thus, for example, the functional incompleteness of $\sim$ is easily established by pointing out that it is impossible to describe any binary truth table (except for the ones that are "really" just negation) with a unary operator. And it is easy to convince students that $\{\vee, \rightarrow, \wedge, \leftrightarrow\}$ is functionally incomplete since each connective in this set takes true subformulas onto true formulas so that no truth function having an F in the first row can be described by any formula using just these connectives alone.

Drawing connectives only from the standard set of five, $\{\vee, \wedge, \leftrightarrow, \rightarrow, \sim\}$, it follows (from what has already been established about incomplete sets of connectives) that any functionally complete set must

(i) contain $\sim$
(ii) contain at least one of the other four connectives.

From what has been shown about complete sets we know it suffices for one of these other connectives to be $\wedge$, $\vee$, or $\rightarrow$. To provide necessary and sufficient conditions for the functional completeness of any set drawn from the five standard connectives, only one question remains: Is $\{\sim, \leftrightarrow\}$ functionally complete?

Proof of the functional *in*completeness of this latter set is, for good reason, best handled only after techniques of mathematical induction have been introduced. For reasons that will be apparent in the next section, it is our view that this problem is not handled well in the standard texts.

2 A problem in functional incompleteness Proof of the incompleteness of $\{\sim, \leftrightarrow\}$ is assigned as a homework problem in various books, for instance [1], p. 212, #8; [3], p. 211, #10; [4], p. 177, #10; [7], p. 28, #1.34. A similar problem (the proof of which is, in all important respects, identical) is to prove that the set $\{\leftrightarrow, \not\leftrightarrow\}$ is functionally incomplete ([2], p. 135, #24.6).[3] And of course, rather than using the simple arguments we gave above, one could try to prove that the set $\{\leftrightarrow\}$ is functionally incomplete by induction (along the lines of the $\{\leftrightarrow, \sim\}$ problem). See here [9], p. 121, #1 and [5], p. 52, #6. Hunter ([6], pp. 89–90) uses the $\{\sim, \leftrightarrow\}$ problem as an example of how to use mathematical induction, and Bergmann, et al. ([1]) present a solution to their problem in their answer book (p. 132).

Despite the problem's popularity as an exercise for intermediate students, it is not a transparently easy problem. Intermediate logic-teacher folklore has it that the key to the solution is to prove that no truth table of any formula with two atomic sentences using only $\sim$ and $\leftrightarrow$ as connectives can have an odd number of F's (or, equivalently, an odd number of T's). Therefore, no truth table with (say) three F's in it can be expressed. This is the strategy employed by Hunter and by Bergmann, et al. But even given this hint the problem has some pitfalls. Indeed, the proofs by Bergmann, et al. and by Hunter are incorrect. Let us see how the Bergmann, et al. proof is started (the Hunter proof proceeds similarly). Where **P** is a formula containing two atomic components and only $\sim, \leftrightarrow$ as connectives, they argue:

> Our induction will proceed on the number of occurrences of connectives in **P**. However, the first case, that considered in the basis clause, is the case where **P** contains *one* occurrence of a connective. This is because every sentence that contains zero occurrences of connectives is an atomic sentence and thus cannot contain more than one atomic component.
>
> *Basis Clause:* The thesis holds for every sentence of [the logic] with exactly two atomic components and one occurrence of (one of) the connectives $\sim$ and $\leftrightarrow$.
>
> In this case **P** cannot be of the form $\sim Q$, for if the initial $\sim$ is the only connective in **P**, then Q is atomic, and hence **P** does not contain two atomic components. So **P** is of the form $Q \leftrightarrow R$, where Q and R are atomic sentences. $Q \leftrightarrow R$ will have to be true on the two partial assignments [i.e., rows of the truth table] to Q and R that assign the same truth-values to Q and R and false on the other two partial assignments to Q and R. Hence the thesis holds in this case. (p. 132)

That was the basis step. We see that the "smallest" biconditional formula under consideration has two *distinct* sentence letters in it, and that the "smallest" negated formula has an embedded biconditional. There *must* be two distinct sentence letters in it or else this wouldn't be a correct basis case. If there were only one sentence letter in it, then they could not rule out $\sim Q$ from the basis case. Furthermore, the two sentence letters must be distinct, or else they could not conclude that $Q \leftrightarrow R$ will have two true rows and two false rows. (After all, $Q \leftrightarrow Q$ does not have this feature.)

Now we come to the induction step. Here it is argued that if A and B both have an even number of T's and F's in their truth tables, then any formula joining them with $\leftrightarrow$ or $\sim$ also does. The negation case is simple: if A has an even number of T's and F's then $\sim A$ will also. When the connective is a $\leftrightarrow$, there are a variety of subcases, depending on just what even number of F's it is that A and B have (and when they each have 2 F's [and hence 2 T's], how they are "aligned" to one another). While tedious, this part of the proof is conceptually simple. Now since A has an even number of F's, and since there are two sentence letters in it, it must have 0, 2, or 4 F's. Now we just go through the possibilities. If A has 0 F's, B can have 0, 2, or 4. If they are both 0, then $(A \leftrightarrow B)$ is always T and therefore has 0 F's (an even number). If B has 2 F's, then $(A \leftrightarrow B)$ is F in those cases and T in the other 2 (and hence has an even number of F's). If B has 4 F's then $(A \leftrightarrow B)$ is always F (and hence has an even number of F's). Postponing the case where A has 2 F's, we move on to the case where A has 4 F's. Again B might have 0, 2, or 4 F's. If it is 0, then $(A \leftrightarrow B)$ is always F (4, an even number). If it is 2, then $(A \leftrightarrow B)$ is T in exactly those 2 cases (an even number). If it is 4, then $(A \leftrightarrow B)$ is always T (0 F's, an even number). The last case is where A has 2 F's. Here, B can have 0, 2, or 4 F's. If it is 0, then $(A \leftrightarrow B)$ is F exactly when A is F (2, an even number). If it is 4, then $(A \leftrightarrow B)$ is F exactly when A is T (2, an even number). The final possibility is that both A and B have 2 F's. Here, unlike the other cases, we need to consider how the two formulas have their T's and F's "aligned". Recall that there are supposed to be exactly two distinct sentence letters (call them p and q) in $(A \leftrightarrow B)$ and that they both occur in each of A and B. Thus we have these possibilities:

p	q	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
T	T	T	T	T	T	T	T	T	F	T	F	T	F	T	F	T	T	T	T	T	F	T	F
T	F	T	T	T	F	T	F	T	T	T	T	T	F	F	F	F	F	F	T	F	T
F	T	F	F	F	T	F	F	F	T	F	F	F	T	T	T	T	F	T	T	T	F
F	F	F	F	F	F	F	T	F	F	F	T	F	T	F	F	F	T	F	F	F	T

A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
T	F	T	T	T	F	T	F	T	F	F	F	F	F	F	F	F	F	F	F	F	F
F	F	F	F	F	T	F	T	F	F	T	T	T	T	T	F	T	T	T	F	F	F
T	T	F	F	F	T	F	F	F	T	T	T	T	F	T	T	F	F	F	T	T	T
F	T	T	T	T	F	T	T	T	F	F	F	F	T	F	T	T	T	T	T	T	T

(Since $(A \leftrightarrow B)$ has the same truth table as $(B \leftrightarrow A)$, all of the symmetric possibilities have been omitted. For instance, we did not consider the possibility:

A	B
T	T
F	T
T	F
F	F

because by symmetry it is handled by the second possibility, viz.,

A	B
T	T
T	F
F	T
F	F

This cuts the number of possibilities to be looked at from 36 to 21.) It will be noted that every possibility has either 0 F's (possibilities 1, 7, 12, 16, 19, 21) or 2 F's (possibilities 2, 3, 4, 5, 8, 9, 11, 14, 15, 17, 18, 20) or 4 F's (possibilities 6, 10, 13).

Thus the theorem has been proved—or has it? What does the theorem say about these formulas?

1. $p \leftrightarrow (p \leftrightarrow p)$
2. $\sim p \leftrightarrow (p \leftrightarrow q)$
3. $\sim (p \leftrightarrow p) \leftrightarrow (p \leftrightarrow \sim q)$
4. $p \leftrightarrow p$
5. $(p \leftrightarrow p) \leftrightarrow \sim (q \leftrightarrow q)$
6. $((p \leftrightarrow p) \leftrightarrow q) \leftrightarrow (\sim p \leftrightarrow (q \leftrightarrow p))$.

That is, formulas in which some arbitrarily deeply embedded biconditional does *not* have distinct sentence letters on each side, or in which (as in case 2) one side of such a biconditional does have further embedded two-variable biconditionals and the other does not. The "proof" just cited has in essence defined an "OK biconditional using p and q" as follows:

(a) If p and q are distinct sentence letters, then $(p \leftrightarrow q)$ is an OK biconditional.
(b) If A and B are both OK biconditionals, then $(A \leftrightarrow B)$ is an OK biconditional.
(c) If A is an OK biconditional, then so is $\sim A$.

The proof given shows that every OK biconditional will have 0, 2, or 4 F's in its truth table, and therefore not every truth table can be described by OK biconditionals. But as formulas 1–6 show, not every biconditional (using p and q) is an OK biconditional (using p and q), so the original problem has not been solved.

What is needed to fix up this proof? Obviously there is a problem in the basis case for we need to be able to consider such formulas as p and $(p \leftrightarrow p)$ and the like, so that we can apply the induction step to such formulas as $(p \leftrightarrow (p \leftrightarrow q))$ or $(p \leftrightarrow p) \leftrightarrow (p \leftrightarrow q)$. The problem with this is that the induction hypothesis—that the formula has 0, 2, or 4 F's (or T's) in its truth table—is *false* for such formulas as p or $(p \leftrightarrow p)$, since there are but two rows in p's truth table and only one is F.

In the sections following, we present two ways to prove the problem correctly. The first way follows the "proof" just given but makes allowances for the non-OK biconditionals by altering the basis step. The second way makes use of some properties of formulas which are composed only of negations and biconditionals. Both methods make heavy use of mathematical induction, which is, after all, frequently the point of the problem to begin with—to allow the student to demonstrate his or her ability with this technique. At the end we will mention still another way it might be proved. This way is more general and gives more information about the property of functional completeness/incompleteness, but does not easily allow the student to practice mathematical induction.

3 Solution I: The straightforward solution Let us recall what we are trying to do. We wish to show that no formula made up exclusively of $\sim$ and $\leftrightarrow$, and using only the sentence letters p and q, will have exactly three F's in its truth table. Since there is such a truth table, we conclude that $\{\sim, \leftrightarrow\}$ is functionally incomplete.

The problem with the previously given "proof" is that some of the formulas to be considered use only *one* of p and q, and hence do not have a four-row truth table suitable for use in the argument (and were therefore incorrectly omitted from the discussion). This is the case that Bergmann, et al. explicitly excluded from their basis case, and is therefore the reason that the induction step does not consider all the formulas which might be of interest in determining whether some formula might have (say) three T's and one F. The correct proof, following the general method employed by Bergmann, et al., would alter the basis step. Again, we will show that no formula using only the sentence letters p and/or q, and the connectives $\sim$ and/or $\leftrightarrow$, can have anything except 0, 2, or 4 F's in its truth table. However, obviously, a caveat must be made here: when the formula under consideration employs only *one* sentence letter we will want to look to the *four*-rowed truth table which uses *both* p and q. For example, if the formula under consideration is $(\sim(p \leftrightarrow p) \leftrightarrow p)$, its normal two-rowed truth table would be:

p	$(\sim(p \leftrightarrow p) \leftrightarrow p)$
T	F
F	T

But we will want to look instead at this four-rowed truth table:

p	q	$(\sim(p \leftrightarrow p) \leftrightarrow p)$
T	T	F
T	F	F
F	T	T
F	F	T

(Obviously, this four-rowed truth table is to be constructed from the two-rowed truth table in such a way as to agree in its assignment of T/F to the formula on the basis of the value of the component sentence letter. Whenever a particular value of the sentence letter in the two-rowed truth table assigns a value to the formula, the four-rowed truth table will assign that value to the formula for the same values of the sentence letter. This guarantees that there is a unique four-rowed truth table for each particular two-rowed truth table.)

We are now in a position to prove that $\{\sim, \leftrightarrow\}$ is an incomplete set of connectives. Our proof will be somewhat smoother than the Bergmann, et al. "proof", since the inclusion of all formulas into the discussion makes the proof of the basis case simpler. Nonetheless, the induction step remains messy, and we will just cite the induction step given earlier (which now works correctly for the new basis case). Throughout this proof the term 'truth table' refers to the four-rowed truth table constructed as needed in the manner indicated in the last paragraph; we call this the "expanded truth table " when the formula has only one sentence letter.

Theorem *Every formula A constructed from p, q, $\sim$, $\leftrightarrow$ has* 0, 2, *or* 4 *F's in its (expanded) truth table.*

Proof: Basis step. A has 0 connectives. Here A is a single sentence letter, either p or q.

The two-rowed truth table for A would look like (when the sentence letter is p):

p	$A(=p)$
T	T
F	F

The only way to construct a four-row truth table (incorporating q) in such a way that it assigns to A whatever X did for that value of p is this:

p	q	$A(=p)$
T	T	T
T	F	T
F	T	F
F	F	F

which, as we see, has two F's in it, so the basis case is proved.

Induction step. We assume that every formula A with fewer than k connectives ($k \geq 1$) has either 0, 2, or 4 F's in its (expanded) truth table, and will prove that therefore any formula with k connectives has 0, 2, or 4 F's in its (expanded) truth table. Here we appeal to the induction carried out earlier. That proof, it will be recalled, considered every combination of subformulas having 0, 2, or 4 F's in their truth table. Here we consider the four-rowed truth tables, and again every combination of subformulas having 0, 2, or 4 F's. If the formula A is $\sim B$, then (since B's four-rowed truth table obeys the induction hypothesis), B will have either 0, 2 or 4 F's in it and hence A will have either 4, 2, or 0 F's in its truth table. If the formula A is $(B \leftrightarrow C)$ then the induction hypothesis holds for each of B and C. That is, the four-rowed truth tables of B and C each have 0, 2, or 4 F's. Here we consider all the possibilities mentioned earlier: (a) B has 0 or 4 F's, (b) B has 2 F's while C has 0 or 4 F's, and (c) both B and C have 2 F's. (In this last case we need to consider the way these F's are "aligned".) In any of these cases we find that the theorem holds.

This proves that every formula A made up only of p and q, with only the connectives $\sim$ and $\leftrightarrow$ will have the following property: if A contains both p and q, then A's truth table will contain either 0, 2, or 4 F's in it; and if A contains only one of p or q, then its expanded truth table will contain either 0, 2, or 4 F's in it. From this it follows that $\{\sim, \leftrightarrow\}$ is functionally incomplete because there is no formula constructible from p, q and these connectives that will have (say) three F's in its truth table.

4 Solution II: Some tricks with biconditionals The previous solution was pretty messy. Expanding truth tables and going through all the cases in various of the theorems was extremely tedious. And it all rested on the insight that $\sim, \leftrightarrow$ could not describe a truth table with 3 F's in it. Another insight we might have into the problem has to do with the relationship between the truth of a formula composed exclusively of $\leftrightarrow$'s and the number of occurrences of the sentence letters in it.

Lemma 1 *Let A be a formula composed exclusively of p, q, and $\leftrightarrow$. Let $V(x)$ be a truth value assignment function which represents "True" by $+1$ and "False" by -1. And finally, let n_x^A be the number of x's in A. Then:*

$$V(A) = V(p)^{n_p^A} \times V(q)^{n_q^A}.$$

(That is, given a particular row of a truth table, the truth value—$+1$ or -1—of A can be computed by counting the numbers of times p and q occur in A, raising the values of each of p and q—as given by that row—to those numbers, and multiplying these results. Note that $+1$ to any non-negative power is $+1$, and that -1 to any non-negative power is either $+1$ or -1.)

Proof: By induction on the number of connectives in A.

Basis step. A is either p or q. (Say p for definiteness). Then trivially:

$$V(A) = V(p) = V(p)^1 \times 1 = V(p)^1 \times V(q)^0 = V(p)^{n_p^A} \times V(q)^{n_q^A}.$$

Induction step. Assume the result holds for any formula with fewer than k $\leftrightarrow$'s. A is a formula with k $\leftrightarrow$'s, call it $(B \leftrightarrow C)$. By the induction hypothesis:

$$V(B) = V(p)^{n_p^B} \times V(q)^{n_q^B}$$
$$V(C) = V(p)^{n_p^C} \times V(q)^{n_q^C}.$$

Note that (from the $\leftrightarrow$ truth table) $V(B \leftrightarrow C) = V(B) \times V(C)$. Substituting the previous formulas into this last formula, and using the arithmetic fact that $a^b \times a^c = a^{b+c}$, we have:

$$V(A) = V(B \leftrightarrow C) = V(p)^{n_p^B + n_p^C} \times V(q)^{n_q^B + n_q^C}.$$

But $n_p^A = n_p^B + n_p^C$, and $n_q^A = n_q^B + n_q^C$, whence we have:

$$V(A) = V(p)^{n_p^A} \times V(q)^{n_q^A}$$

which was the required result.

Although we will not prove it here, there is a simple generalization of Lemma 1 to the case of an arbitrary number of sentence letters.

Lemma 2 *Let $p_1, p_2, \ldots p_n$ be all the sentence letters in A. Then*

$$V(A) = V(p_1)^{n_{p_1}^A} \times V(p_2)^{n_{p_2}^A} \times \ldots \times V(p_n)^{n_{p_n}^A}.$$

A simple consequence of Lemma 1 is this:

Lemma 3 *A formula A containing only $p, q, \leftrightarrow$ is valid iff both of p and q occur an even number of times.*

Proof: (a) Assume that both p and q occur an even number of times. We show that $V(A) = 1$. The result of Lemma 1 holds for any row of the truth table: One takes $V(p)$ and $V(q)$ in that row and raises them to the appropriate power. Since both p and q occur an even number of times, the result for each will be $+1$. Thus, the product will be $+1$.

(b) Assume that A is valid (i.e., $V(A)$ is always 1). We show that both p and q must occur an even number of times. If p, for example, could occur an odd number of times, consider the row of the truth table where p is F and q is T. Now apply Lemma 1 to this row:

$$V(A) = V(p)^{n_p^A} \times V(q)^{n_q^A}.$$

Since $V(q) = 1$, it follows that $V(q)^{n_q^A} = 1$. So $V(A) = V(p)^{n_p^A}$. But $V(p) = -1$ and n_p^A is odd. Therefore $V(A) = -1$ in this row, contrary to the assumption that A is valid. By a similar argument, q must also occur an even number of times.

Again this can easily be generalized, although we shall not prove it.

Lemma 4 *A formula whose only connective is $\leftrightarrow$ is valid just in case every sentence letter in it occurs an even number of times.*

Some trivial consequences of Lemma 2 are:

Lemma 5 *The bracketing and order of occurrence of sentence letters in a formula with $\leftrightarrow$ as its only connective are irrelevant to its truth value (so long as it is well-formed).*

Proof: This follows from the fact that Lemma 2 only counts numbers of occurrences of sentence letters, not location.

Lemma 6 *Any formula with $\leftrightarrow$ as its only connective is either valid or else truth functionally equivalent to one in which all sentence letters occurring an even number of times have been eliminated, and in which the number of occurrences of each sentence letter occurring an odd number of times has been reduced to one.*

Proof: This follows from the fact that, by Lemma 2, only those sentence letters occurring an odd number of times can affect the value of $V(A)$ and they affect it in the same way regardless of the (odd) number of their occurrences.

Another interesting consequence of Lemma 2 is the following.

Lemma 7 *Any formula whose only connective is $\leftrightarrow$ is either valid, or else has the same number of T's and F's in its truth table.*

Proof: By Lemma 6, every formula with only $\leftrightarrow$ as connectives is either valid or equivalent to a formula consisting of single occurrences of various sentence letters. But by Lemma 2, the truth value in a particular row of a formula of the latter type will be T or F depending only on whether the number of sentence letters receiving the value F is even or odd. But in any truth table, exactly half the rows have an odd number of F's. Hence any formula with only $\leftrightarrow$ as its connectives has either all T's or else an equal number of T's and F's in its truth table.

Now let's add negation into the picture. In light of the propositional theorems (a) $(A \leftrightarrow B) \leftrightarrow (B \leftrightarrow A)$ and (b) $(A \leftrightarrow \sim B) \leftrightarrow \sim(A \leftrightarrow B)$, the following is true (we will provide an inductive proof of it).

Lemma 8 *Any formula A whose only connectives are $\leftrightarrow$ and $\sim$ is truth functionally equivalent either to some formula whose only connective is $\leftrightarrow$, or to the negation of such a formula.*

Proof: By induction on the number of connectives in A.

Basis step. A has 0 connectives. Then A is a sentence letter, and thus is itself a formula whose only connective is $\leftrightarrow$.

Induction step. Suppose that any formula using $\leftrightarrow$ and $\sim$ with fewer than k connectives ($k \geq 1$) is equivalent to some formula using only $\leftrightarrow$ or to the negation of such a formula. We prove that any formula A with k connectives must be also.

Case i. A is $\sim B$. By the induction hypothesis B either is equivalent to X or to $\sim X$, where X contains only $\leftrightarrow$'s. Trivially, then, A is equivalent to $\sim X$ or to $\sim\sim X$ (i.e., to X).

Case ii. A is $(B \leftrightarrow C)$. By the induction hypothesis, B is equivalent either to X or to $\sim X$ and C is equivalent either to Y or to $\sim Y$, where X and Y contain only $\leftrightarrow$'s. There are thus four subcases:
Subcase α. B equivalent to X, C to Y. Then A is equivalent to $(X \leftrightarrow Y)$.
Subcase β. B equivalent to X, C to $\sim Y$. Then A is equivalent to $\sim(X \leftrightarrow Y)$.
Subcase γ. B equivalent to $\sim X$, C to Y. Then A is equivalent to $\sim(X \leftrightarrow Y)$.
Subcase δ. B equivalent to $\sim X$, C to $\sim Y$. Then A is equivalent to $(X \leftrightarrow Y)$.

Thus every formula containing only $\sim$ and $\leftrightarrow$ is equivalent to one containing only $\leftrightarrow$*'s or to the negation of such a formula. With Lemmas* 7 *and* 8 *at hand, we can easily prove the functional incompleteness of* $\sim, \leftrightarrow$.

Theorem *Every formula whose only connectives are* $\sim$ *and* $\leftrightarrow$ *has either all F's or all T's or else the same number of T's and F's in its truth table.*

Proof: From Lemma 8, every formula A using only $\sim$ and $\leftrightarrow$ is equivalent either to X or to $\sim X$, where X contains only $\leftrightarrow$'s. By Lemma 7, X is either always T or else contains the same number of T's and F's in its truth table. If A is equivalent to X, then A will have either all T's in its truth table or else the same number of T's and F's. If A is equivalent to $\sim X$, then A will have either all F's in its truth table or else the same number of T's and F's.

5 Solution III: Some deeper insights into functional incompleteness The highly motivated and deeply inquiring student probably wishes to know *why* $\{\sim, \rightarrow\}$ does form a functionally complete set but $\{\sim, \leftrightarrow\}$ does not. Is there anything that can be done to explain why? In this section we give an explanation of a very general nature (due to [8]). We will not prove the central theorem, but will content ourselves with explaining it and explaining how to use it to determine whether an arbitrary set of connectives is or is not functionally complete. (Although we know precisely which subsets of the standard five connectives are functionally complete, there are, of course, many other possible connectives to consider.)

We start by defining the notion of a *dummy variable* in a truth function. Intuitively it is a variable (or position) which never makes a difference in evaluating a formula. For example, suppose we make up the truth function $(p \circ q)$ like this:

p	q	$(p \circ q)$
T	T	F
T	F	T
F	T	F
F	F	T

Note that this function "really" is just the negation of a q—the value of p in no way ever makes a difference. More formally, if f is a truth function of n variables and

$$f(x_1, x_2, \ldots x_i, F, x_{i+2}, \ldots, x_n) = f(x_1, x_2, \ldots x_i, T, x_{i+1}, \ldots x_n)$$

for all the possible values of the other variables, then x_{i+1} is a dummy variable for f. That is, it never matters what the value of the $(i + 1)$st position is.

We next describe five classes of possible truth functions:

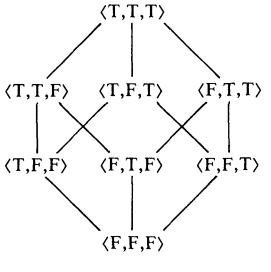
Type 1: Functions closed under T. For an arbitrary truth function f, f is closed under T iff $f(\mathrm{T},\mathrm{T},\ldots \mathrm{T}) = \mathrm{T}$.

Type 2: Functions closed under F. For an arbitrary truth function f, f is closed under F iff $f(\mathrm{F},\mathrm{F},\ldots \mathrm{F}) = \mathrm{F}$.

Type 3: Linear functions. A linear function is one in which every non-dummy variable always makes a difference. That is, given any row of a truth table, if you ignore values of dummy variables, a change in the value

of one variable (holding all others constant) will create a change in the value of the function. So: each variable either never makes a difference ("dummy variable") or else it *always* makes difference. This means that linearity can be easily tested in the following way. First, delete dummy variables. Then, a function is linear if one of the following two situations occurs: (a) In every row in which the value of the function is T, there are an even number of T's assigned to the arguments of the function; and in every row in which the function is F, there are an odd number of T's assigned to the arguments of the function; or (b) In every row in which the value of the function is T, there are an odd number of T's assigned to the arguments of the function; and in every row in which the function is F, there are an even number of T's assigned to the arguments of the function. That this is an adequate test can be seen by considering a simple example. We ignore any dummy variables. Now suppose $f(\mathrm{T},\mathrm{T},\mathrm{T}) = \mathrm{T}$; then since a change in an argument must result in a change of the function value, $f(\mathrm{T},\mathrm{T},\mathrm{F}) = \mathrm{F}$, and applying this fact again we get $f(\mathrm{F},\mathrm{T},\mathrm{F}) = \mathrm{T}$ and so on. Here everywhere the value of the function is T there are an odd number of T's in the arguments and everywhere the function is F there are an even number of T's. Had the value of $f(\mathrm{T},\mathrm{T},\mathrm{T}) = \mathrm{F}$, the reverse would have been the case. A function f is nonlinear iff, after deleting dummy variables, there is at least one n-tuple where $f(x_1,\ldots,\mathrm{T},\ldots x_n) = f(x_1,\ldots,\mathrm{F},\ldots x_n)$. Also note that since this i^{th} position of the nonlinear function is not a dummy position, there is also at least one sequence of truth values $\langle y_1,\ldots y_n\rangle$ such that $f(y_1,\ldots,\mathrm{T},\ldots y_n) \neq f(y_1,\ldots,\mathrm{F},\ldots y_n)$.

Type 4: Monotonic functions. A monotonic function is one in which the value of the function "follows" the values of the arguments. That is, if f is an n-adic monotonic function and $\langle x_1,\ldots x_n\rangle$ and $\langle y_1\ldots y_n\rangle$ are sequences of truth values, then: if $\langle x_1,\ldots x_n\rangle \leq \langle y_1\ldots y_n\rangle$ then $f(x_1,\ldots x_n) \leq f(y_1\ldots y_n)$. To make sense of this we note first that we consider $\mathrm{F} < \mathrm{T}$, and so what $f(x_1,\ldots x_n) \leq f(y_1\ldots y_n)$ rules out is that $f(x_1,\ldots x_n) = \mathrm{T}$ while $f(y_1\ldots y_n) = \mathrm{F}$. A sequence of truth values $\langle x_1,\ldots x_n\rangle$ is $\leq$ to another sequence $\langle y_1\ldots y_n\rangle$ just in case whenever $x_i = \mathrm{T}$ then so is $y_i = \mathrm{T}$. More pictorially we can represent this as a lattice with the sequence $\langle \mathrm{T},\mathrm{T},\ldots \mathrm{T}\rangle$ at the top and $\langle \mathrm{F},\mathrm{F},\ldots \mathrm{F}\rangle$ at the bottom. For example, the 3-tuple lattice is:



A sequence is $<$ another just in case the former is below the latter along lines in the lattice. A function f is monotonic iff applying it successively

to tuples downwards along the lines never results in having a F followed by a T. Or: is nonmonotonic iff there is at least one place where $f(x_1, \ldots x_n) = \mathrm{T}$ while $f(y_1 \ldots y_n) = \mathrm{F}$ and yet $\langle y_1 \ldots y_n \rangle$ is above $\langle x_1 \ldots x_n \rangle$ in the lattice.

Type 5: Self-dual functions. A truth function f is self-dual if its reading from top to bottom is the same as the complement of reading it from bottom to top (in the standard ordering of values for arguments). So, for example, if f yields the values FTFFTTFT (reading from top to bottom of its truth table) we can reverse the order to get TFTTFFTF and complement this to get FTFFTTFT. This is the same as what we started with, and so f is self-dual. The function whose truth table reads FFTFTTFT is *not* self-dual. That is, a self-dual function obeys the following condition: for every row of the truth table $\langle x_1, \ldots x_n \rangle$, $f(x_1, \ldots x_n) \neq f(-x_1, \ldots -x_n)$, where $-x_i$ is the opposite truth value from x_i. A function is *not* self-dual just in case there is a row of the truth table $\langle y_1 \ldots y_n \rangle$ such that $f(y_1 \ldots y_n) = f(-y_1 \ldots -y_n)$.

Having defined these classes of truth functions, we are in a position to state (but not to prove):

Theorem (Post's Functional Completeness Theorem) *A set X of truth functions (of two-valued logic) is functionally complete iff, for each of the five defined classes, there is a member of X which does not belong to that class.*[4]

One can state in class (and even illustrate the truth of) the following statements of what types the usual connectives are.

$\wedge, \vee$: Classes 1, 2, 4
$\rightarrow$: Class 1
$\sim$: Classes 3, 5
$\leftrightarrow$: Classes 1, 3
$\not\rightarrow$: Classes 2, 3
T_k: Classes 1, 3, 4
F_k: Classes 2, 3, 4
$=$: Classes 1, 2, 3, 4, 5

where T_k is any k-place truth-function whose value is always T; F_k is any k-place truth function whose value is always F; and $=$ is the (monadic) identity function: $=(p)$ is always the value of p.

Given Post's Functional Completeness Theorem and this classification of the usual connectives, we can easily prove that $\{\sim, \leftrightarrow\}$ is not functionally complete because both connectives belong to Class 3. On the other hand, $\{\sim, \wedge\}$ or $\{\sim, \vee\}$ or $\{\sim, \rightarrow\}$ or $\{\rightarrow, \mathrm{F}_k\}$ are each functionally complete. In class one might also demonstrate that the Sheffer stroke functions $\uparrow$ and $\downarrow$ do not belong to any of the five classes.

6 Conclusion The point of this note has been to discuss the notion of functional completeness and the proofs of the completeness/incompleteness of various sets of truth functions. We showed that the published proofs of the incompleteness of $\{\sim, \leftrightarrow\}$ are incorrect and stated how to repair them. We also

gave an alternate way to solve this problem relying on some idiosyncracies of $\leftrightarrow$. The point of such proofs was really to give students practice in mathematical induction, rather than to give them insight into the notion of functional completeness. So we concluded by trying to indicate the underlying features of truth functions that will yield a functionally complete set.

Although the proof of Post's Functional Completeness Theorem is probably beyond the scope of an intermediate symbolic logic class, the mere presentation of it can deepen the students' understanding of how truth-functions work. This is especially so if it is given after the students have been assigned a series of problems with the instructions "either prove to be functionally incomplete or state the definitions which will reduce the set of connectives to a set known to be complete". At such a stage the students will have tried to figure out exactly what properties of truth functions are relevant to having a complete set.

NOTES

1. Alternative terms for this are *adequate* and *expressively complete.*
2. For formulas without any T's, any contradiction using only $\sim,\vee,\wedge$ can be chosen to express them.
3. The symbol $\not\leftrightarrow$ stands for "exclusive disjunction".
4. We would like to thank Norman Martin (University of Texas) for information about Post's Functional Completeness Theorem.

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