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More on Trees and Finite Satisfiability: The Taming of Terms

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As Boolos showed in [1], the new rule proposed by Burgess provides (together with the usual tableaux rules) a simple and elegant method for testing finite satisfiability of first-order sentences. We present an extension of the method to the case of languages which contain function symbols. For these languages the usual rule "from $\forall x \phi(x)$ infer $\phi(t)$ for any term t" produces in some cases only infinite models. For instance, when applied to $\forall x Rf(x)$ it gives us the infinite one-branch tree:

$$\begin{array}{c} \forall x Rf(x) \\ | \\ Rf(a) \\ | \\ Rf(f(a)) \\ | \\ \vdots \end{array}$$

Where language contains equality we can in principle get rid of function symbols in standard way, replacing them by new predicate symbols (which are to serve as their "graphs"), but it is more natural to try to put up with terms.

In order to avoid ambiguities we first list some necessary definitions.

A *tableau* for a sentence ϕ is a tree built up by placing ϕ at the top node and then applying the following reduction rules.

($\neg\neg$) If $\neg\neg\phi$ lies on a branch B we extend it to (B, ϕ).

($\wedge$) If $\phi \wedge \psi$ lies on B, we extend it to (B, ϕ, ψ) (similarly for $\neg\vee$, $\neg\rightarrow$).

($\vee$) If $\phi \vee \psi$ lies on B, then B splits into two extensions (B, ϕ), (B, ψ) (similarly for $\neg\wedge$ and $\rightarrow$).

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(new ∃). If $\exists x\phi(x)$ lies on B and $a_1,\ldots,a_k$ are all constants occurring on B, then B splits into $k+1$ extensions $(B,\phi(a_1)),\ldots,(B,\phi(a_k))$, $(B,\phi(a))$ where a is new for B; i.e., does not occur on B (similarly for $\neg\forall$).

(∀) If $\forall x\phi(x)$ lies on B, extend it to $(B,\phi(a))$ where a is a constant on B to which (∀) has not been applied yet (on that branch) and if there is no constant on B, then a is new for B (similarly for $\neg\exists$).

(F) 1° If for some constants $b_1,\ldots,b_n$ and some term $fb_1\ldots b_n$ $\phi(fb_1\ldots b_n)$ occurs on a branch B and if (F) has been applied to no occurrence of $fb_1\ldots b_n$ on B yet, then B splits into $(B,\phi(a_1)),\ldots,(B,\phi(a_k))$, $(B,\phi(a))$ where $a_1,\ldots,a_k$ are all constants occurring on B and a is new for B. We say that $fb_1\ldots b_n$ is *associated* (by (F)) with $a_1,\ldots,a_k$, a on $(B,\phi(a_1)),\ldots,(B,\phi(a_k))$, $(B,\phi(a))$ respectively.

2° If $fb_1\ldots b_n$ is associated with b on B, $\phi(fb_1\ldots b_n)$ lies on B and $\phi(b)$ does not, then extend B to $(B,\phi(b))$.

It is easy to see that this variant of the method is still sound and complete for unsatisfiability; i.e., we can prove:

Hintikka's Lemma *If a branch B is finished and open then there is a model M satisfying all θ on B.*

The proof goes more or less as usual: define $|M| = \{a \mid a$ is a constant occurring on $B\}$ and take $M\models Ra_1\ldots a_\ell$ iff $Ra_1\ldots a_\ell$ occurs on B and $f^M(b_1,\ldots,b_n)=b$ if $fb_1\ldots b_n$ is associated on B with b (arbitrary otherwise). Then use induction on the complexity of a formula defined as the number of all occurrences of logical and functional symbols (constants excluded). For instance, if $\phi(b)$ appears on B as a result of an application of (F) to $\phi(fb_1\ldots b_n)$ and if $M\models\phi(b)$, then obviously $M\models\phi(fb_1\ldots b_n)$ (by the definition of f^M). It follows that M is finite if B is.

In order to prove that the method is sound for finite satisfiability we use the notions of good model and good branch from [1] and check that an application of a rule to a formula on a good branch produces at least one good extension of that branch (see Lemma in [1]). Again, the only interesting case is that of (F) and the argument is partly the same as for (new ∃). For if there are $k+1$ extensions of B (as described above) and N is good and $N\models B$ then there are two possibilities. If $f^N(b_1,\ldots,b_n)=a_i$ for some $i=1,\ldots,k$ then $(B,\phi(a_i))$ is the extension we need; otherwise choose $e\in N$ such that $f^N(b_1,\ldots,b_n)=e$; then $N^a_e\models\phi(a)$ and for all $i=1,\ldots,k$ $e\neq a_i$ so N^a_e is good and $(B,\phi(a))$ is a good extension. If there is only one extension and (F) was applied before, then $f^N(b_1,\ldots,b_n)=b$; hence $N\models\phi(b)$, i.e., $(B,\phi(b))$ is a good extension.

Given a finite model M of ϕ, choose the branch B of a (finished) tableau for ϕ such that the leftmost good (with respect to M) extension of any initial segment of B is also contained in B. We claim that B is finite. To prove that, first note that on B there can be at most m $(= card|M|)$ new constants $a_1,\ldots,a_m$. For otherwise at some step a_{m+1} would be introduced as a result of an application of either (new ∃) to $\exists x\theta(x)$ or (F) to $\theta(fc_1\ldots c_n)$ and in both cases we get extensions $\theta(b_1),\ldots,\theta(b_\ell)$, $\theta(a_{m+1})(\ell\geq m)$. The last formula, however, could not lie on B, for then for some good N and all $i=1,\ldots,\ell$ $N\not\models$

$\theta(b_i)$, which contradicts the assumption that $M \models \exists x\theta(x)$ (or $M \models \theta(fc_1 \ldots c_n)$), since among b_i's are the names of all elements of M and B is good.

The existence of the bound on the number of constants on B implies that the rule ($\forall$) (and ($\neg\exists$)) will be applied on B only finitely many times; i.e., B is finished after some step.

We note an alternative treatment of equality. The following rules ensure that $=$ is always interpreted as a congruence:

(i) Extend B by $a = a$ where a is the first constant (on B) to which this was not applied before.

(ii) If $a_1 = b_1, \ldots, a_k = b_k$ appear on B together with $\phi(a_1, \ldots, a_k)$, where ϕ is atomic or a negation of an atomic sentence, then extend B by $\phi\ (b_1, \ldots, b_k)$.

(iii) Suppose that the following conditions hold for some $a_1, \ldots, a_k, b_1, \ldots, b_k$ and f (on B):

(a) $a_1 = b_1, \ldots, a_k = b_k$ all appear on B

(b) $fa_1 \ldots a_k$ is associated with a (by (F))

(c) $fb_1 \ldots b_k$ is associated with b (by (F)).

Then extend B by $a = b$.

REFERENCES

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