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Modal Trees: Correction to a Decision Procedure for $S5$ (and T)

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Every now and then, this *Journal* has published extensions of the tree method given by Jeffrey in [5] to yield decision procedures for propositional calculi other than classical. This has been done, for instance, in [3] for modal logics T and $S5$, and in [1] for Lemmon's minimal tense logic K_t. In this paper, our main interest is focused on modal trees for $S5$; the attention paid by us here to modal trees for T is only of secondary importance. The existence of effective decision methods for both systems is well known from long ago (e.g., see [4]). The tree method of [5] being the simplest and most elegant decision procedure for classical propositional calculus, extending it in order to cover these propositional modal logics with a similar degree of simplicity is therefore desirable.

Unfortunately, the well conceived attempt made in [3] by Davidson, Jackson, and Pargetter fails to yield a successful decision procedure for $S5$, as we aim to show here. Davidson seems not to be conscious of the fault, since in [2] (a more recent paper) she repeats the same failure with regard to a testing procedure for modal trees for modal predicate logics. It seems that Jackson and Pargetter have not seen the point either, as is revealed in a footnote on p. 56 of [2], where Davidson declares her indebtness to them for their remarks and suggestions.

However, a quite successful decision procedure can be easily obtained by a small (although decisive) correction to the above procedures, as we are also going to show later. In doing this, we will be applying the strategy of Copeland in [1] for the analogous treatment of tense trees. In [3], the procedure is designed for T as well as for $S5$; applied to T it is in fact an effective decision procedure, in contrast to the case of $S5$. Our correction is designed in turn in such a way that it can also be applied to T, though in this case only a more elegant decision procedure, in the sense of yielding shorter trees in some cases, is to be obtained.

To begin, we summarize the modal rules of the tree method of [3] as follows (where ϕ is a wff, and ϕ^i is the result of indexing all its propositional letters with the same superscript i):

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$$(\sim L): \begin{array}{c} \sim L\phi^i \\ \vdots \\ | \\ M \sim \phi^i \end{array} \qquad (\sim M): \begin{array}{c} \sim M\phi^i \\ \vdots \\ | \\ L \sim \phi^i \end{array}$$

Systems T and $S5$ have these two rules in common, but they differ in the (M) and (L) rules:

$$(MT): \begin{array}{c} M\phi^i \\ \vdots \\ | \\ \phi^j \end{array}$$

where, if ϕ^i occurs as a full line in the path above ϕ^j, $j = i$; and if not, $j > i$ and j does not occur previously in the path.

$$(LT): \begin{array}{c} L\phi^i \\ \vdots \\ | \\ \phi^j \end{array}$$

where $j = i$ or j is the index of some wff above ϕ^j in the path obtained by an application of (MT) to a wff indexed with i.

$$(MS5): \begin{array}{c} M\phi^i \\ \vdots \\ | \\ \phi^j \end{array}$$

where, if ϕ^k occurs as a full line in the path above ϕ^j for some k, $j = k$; and if not, $j > i$ and j does not occur previously in the path.

$$(LS5): \begin{array}{c} L\phi^i \\ \vdots \\ | \\ \phi^j \end{array}$$

for any j occurring in the path.

In addition, of course, rules for truth-functional connectives are as usual (see [5]).

Further, the decision procedure is also described in [3] (and in [2] as a testing one) as usual for trees of consistency (see [5] again). Nevertheless, it is prescribed:

(i) the initial wffs to test for consistency are taken to be all indexed with 0;
(ii) if ϕ^i occurs as a full line in a tree, apply the relevant rule by writing the list(s) of conclusions of that rule *at the bottom of every path in which ϕ^i occurs*;
(iii) if the rule applied to a line was (L), write the index assigned to the conclusion next to that line and check that index (not the full line);

(iv) a line with a checked index may not have (L) applied to it to yield a wff with that index.

Now, our first remark is that the italicized part of (ii) is contrary to (L) and (M) rules in both systems T and $S5$. Sure enough, if we commit ourselves to follow the procedure as stated above, some applications of these rules will be not in accordance with the rules themselves; notice that if we list the conclusions of one of these rules applied to a wff at the bottom of every path passing through that wff, it can be the case that an index which is not provided in a given path by the nature of the rule would be introduced in that very same path. As an example of a simple form, consider the following tree which tests $\{p \vee q, Mq\}$ for consistency:

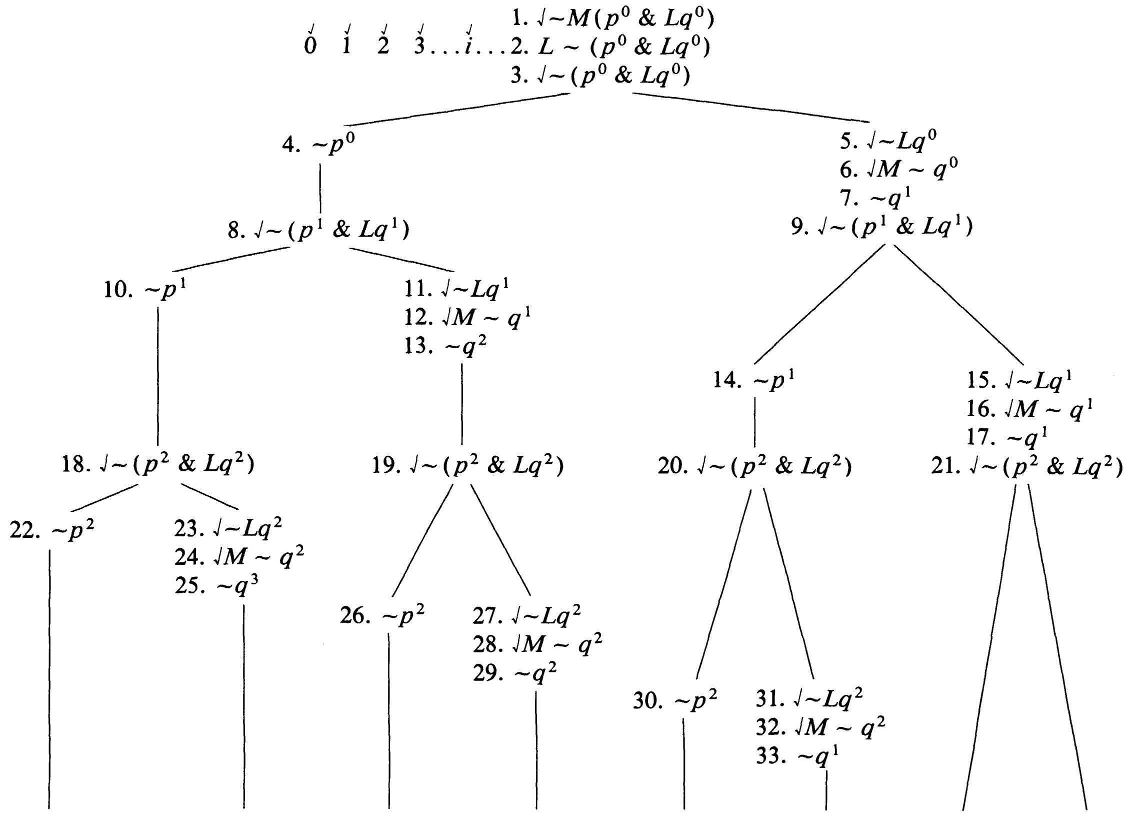
1. $\surd p^0 \vee q^0$		$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ initial wffs.
2. $\surd Mq^0$		
3. p^0	4. q^0	from 1 by (v).
5. q^1	6. q^1	from 2 by (MT) or $(MS5)$.

This tree follows the specified procedure, but in line 6 the index 1 should not have been introduced; according to (MT) or $(MS5)$ the wff of line 6 should have been indexed with 0. To state it briefly, the statements of (L) and (M) rules allow that a particular ϕ^j might be a conclusion of $L\phi^i$ or $M\phi^i$ relative to one path but not relative to another. Yet the procedure fails to take account of this.

All the same, this fact does *not* render a decision procedure necessarily inadequate by itself (this is our second remark) though it *does* when it is taken together with $(LS5)$ rule (this will be our third claim) so that the procedure is inadequate for $S5$. For the moment, let us see for instance that the procedure, in spite of its not taking account of that feature of the modal rules, is an effective decision procedure for T, as it will be manifested by considering the very restrictive nature of (MT) and (LT) as regards the introduction of new indices in a tree, in such a way that there can only be a finite stock of available indices; and this means that there will always be a finished tree.

Now, the case is not the same in $S5$. The strict application of the supposed decision procedure, when coming into play with the more liberal $(LS5)$ rule, has a very undesirable result: some trees for $S5$ run infinitely, and so we have no decision procedure at our disposal. The point can be illustrated with another example, as given on the next two pages (which tests the obvious consistency in T and in $S5$ of a very simple wff, following step by step the instructions of the specified procedure).

It should be clear that with this procedure a finished tree for T is reached at line 17. Instead, in $S5$ the index 2 occurring in line 13 allows the introduction of line 19, and the tree runs further on. However, following strictly the rules for T and $S5$, lines 8 and 10–13 (and subsequently all the paths below them) should not have been entered in the tree. The introduction of these lines is not in accordance with the (LT) or $(LS5)$ rule, as the case may be, since no wff of index 1 (which moreover—as a special requirement of T—has been obtained



initial wff.
from 1 by $(\sim M)$
from 2 by (LT) or $(LS5)$.

from 3 by $(\sim\&)$.
from 5 by $(\sim L)$.
from 6 by (MT) or $(MS5)$.
from 2 by (LT) or $(LS5)$.

from 8 by $(\sim\&)$.
from 11 by $(\sim L)$.
from 12 by (MT) or $(MS5)$.
from 9 by $(\sim\&)$.
from 15 by $(\sim L)$.
from 16 by (MT) or $(MS5)$.
from 2 by $(LS5)$.

from 18 by $(\sim\&)$
from 23 by $(\sim L)$.
from 24 by $(MS5)$.
from 19 by $(\sim\&)$.
from 27 by $(\sim L)$.
from 28 by $(MS5)$.
from 20 by $(\sim\&)$.
from 31 by $(\sim L)$.
from 32 by $(MS5)$.

$34.\ \sim p^2$

$35.\ \surd\sim Lq^2$ from 21 by $(\sim\&)$.

$36.\ \surd M \sim q^2$ from 35 by $(\sim L)$.

$37.\ \sim q^1$ from 36 by $(MS5)$.

$38.\ \surd\sim(p^3\ \&\ Lq^3)$ $39.\ \surd[=38]$ $40.\ \surd[=38]$ $41.\ \surd[=38]$ $42.\ \surd[=38]$ $43.\ \surd[=38]$ $44.\ \surd[=38]$ $45.\ \surd[=38]$ from 2 by $(LS5)$.

$46.\ \sim p^3$ | $47.\ \surd\sim Lq^3$ from 38 by $(\sim\&)$.

$48.\ \surd M \sim q^3$ from 47 by $(\sim L)$.

$49.\ \sim q^4$ from 48 by $(MS5)$.

. . . .

. . . .

$n.\ \surd\sim(p^i\ \&\ Lq^i)$ from 2 by $(LS5)$.

$m.\ \sim p^i$ | $m+1.\ \surd\sim Lq^i$ from n by $(\sim\&)$.

$m+2.\ \surd M \sim q^i$ from $m+1$ by $(\sim L)$.

$m+3.\ \sim q^{i+1}$ from $m+2$ by $(MS5)$.

. . . .

. . . .

by an application of (MT) to a wff of index 0) occurs in the leftmost path above 8. According to the rules and the other instructions of the procedure the path had to be finished at line 4. Again, similar remarks can be made with regard to lines 18, 20, and 21. So, in contrast to T, the tree runs infinitely in $S5$, as can be seen regarding its leftmost path, which includes an endless reiterative cycle which always provides a new index to be checked in a new application of $(LS5)$ to line 2 for yielding a conclusion with that index; furthermore this conclusion should be written in turn at the bottom of all paths of the tree according to (ii), and so on.

As far as we can see, these problems may be solved with a very simple correction in the testing procedure. It merely consists of the addition of an exceptive clause in (ii) in the fashion of Copeland in [1]; the new aspect of (ii) would be as follows:

(ii′) if ϕ^i occurs as a full line in a tree, apply the relevant rule of inference by writing the list(s) of conclusions of that rule at the bottom of every path in which ϕ^i occurs, *except* in the case of (L) where the conclusion may be written *only* in such of these paths which already contain the index occurring in the conclusion.

The tree method for propositional classical calculus guarantees that we always have a finished tree for any wff to test for consistency. In its modal extension for T and $S5$, all the rules except (L) are applied to a given line just once, and each application of (L) to a given line must yield a wff of different index. Nevertheless, in the case of $S5$ without the exception of (ii′) it may happen that the application of $(LS5)$ to a given line could be done an infinite number of times, by allowing infinite new indices to be checked in its application. Really our correction only guarantees that there can only be a finite stock of available indices taking into account the nature of $(MS5)$ and the fact that initial wffs to test are finite in number and length, and therefore that there will always be a finished tree for $S5$.

It is worth noticing, however, that our correction does not remove completely the discordance between the procedure and the modal rules. In fact, the procedure remains contrary, in the explained sense, to $(MS5)$ rule (and of course to (MT) since the correction is designed equally for both systems); but there is no problem here, since *if* the procedure is corrected according to (ii′) the application of this rule is so restrictive as regards the introduction of new indices that there is no fear about the infiniteness of the trees. $(LS5)$ is the only rule that can give rise to infinite trees if we do not impose the correction in the procedure, because of its special nature and the fact that, in contrast to $(MS5)$, it can be applied more than once to a given line of a tree.

Nevertheless, a more elegant procedure, yielding still shorter trees in some cases, could be obtained by a stronger correction on (ii):

(ii″) if ϕ^i occurs as a full line in a tree, apply the relevant rule of inference by writing the list(s) of conclusions of that rule at the bottom of every path in which ϕ^i occurs, *except* in the case of (M) or (L) rules; if the applied rule was (M), write at the bottom of every path passing through ϕ^i the appropriate conclusion according to the specifications of (M) *for each path*; if the applied rule was (L), the conclusion may be written *only* in such of these paths which already contain the index occurring in the conclusion.

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