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Frege's Permutation Argument

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At the beginning of section 10 of his *Grundgesetze*, Frege complains that the stipulation embodied in his abstraction principle[1] "by no means fixes completely the denotation of a name like '$\grave{\epsilon}\Phi(\epsilon)$' " ([1], p. 46), and he proceeds to give a technical argument–which, following Dummett, we shall call the permutation argument–in support of this claim. He writes:

> If we assume that $X(\xi)$ is a function that never takes on the same value for different arguments, then for objects whose names are of the form '$X(\grave{\epsilon}\Phi(\epsilon))$' just the same distinguishing mark for recognition holds, as for objects signs for which are of the form '$\grave{\epsilon}\Phi(\epsilon)$'. To wit, '$X(\grave{\epsilon}\Phi(\epsilon)) = X(\grave{\alpha}\Psi(\alpha))$' then also has the same denotation as '$-\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$'. From this it follows that by identifying the denotation of '$\grave{\epsilon}\Phi(\epsilon) = \grave{\alpha}\Psi(\alpha)$' with that of '$-\!\overset{\mathfrak{a}}{\smile}\!-\Phi(\mathfrak{a}) = \Psi(\mathfrak{a})$', we have by no means fully determined the denotation of a name like '$\grave{\epsilon}\Phi(\epsilon)$'—at least if there does exist such a function $X(\xi)$ whose value for a value-range as argument is not always the same as the value-range itself.[2]

Later in section 10 Frege, appealing to a variant of the permutation argument, argues that "it is always possible to stipulate that an arbitrary value-range is to be the True and another the False" (p. 48). In a challenging article full of interesting observations [3], Peter Schroeder-Heister claims this argument is fallacious.[3] According to Schroeder-Heister, the identifiability thesis (the thesis that it is always possible to stipulate that an arbitrary value-range is to be the True and another the False) cannot be established in the way attempted by Frege and is, in any case, false. We hope to show that Schroeder-Heister's model-theoretic reconstruction of Frege's argument misrepresents the argument and (in particular) its conclusion, the identifiability thesis. We believe that, correctly construed, the argument constitutes a perfectly sound demonstration of the identifiability thesis.[4]

In fact, Schroeder-Heister considers several ways in which Frege's argument can be translated into a model-theoretic framework. In each case he shows that

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the identifiability thesis is not forthcoming. Since, in our view, the fault lies in the particular model-theoretic framework with which Schroeder-Heister is operating, we need not consider all the variations he discusses but, instead, will focus attention on what we regard as the most faithful of the reconstructions offered and this will indicate where the fault lies in all of them.

Central to the model-theoretic framework with which Schroeder-Heister operates is his notion of an interpretation. An interpretation assigns to each closed abstract of Frege's formal language (that is, to each expression of the form '$\grave{\epsilon}\Phi(\epsilon)$' which has no free variables) some object from a given domain. Consider now the following reconstruction of Frege's argument. Let the interpretation I assign object ω_1 ($\neq$ the True) to the closed abstract A and another object ω_2 ($\neq$ the False) to the closed abstract B. We now define the permutation X as follows:

$X(x)$ = the True if $x = \omega_1$
$X(x)$ = the False if $x = \omega_2$
$X(x) = \omega_1$ if x = the True
$X(x) = \omega_2$ if x = the False
$X(x) = x$ for all other objects x.

I' is now defined as that interpretation which assigns $X(I(C))$ to each closed abstract C. We now conclude: I' is a model of the abstraction principle if I is. If this conclusion is correct then the identifiability thesis is immediately forthcoming.

But, as Schroeder-Heister demonstrates, this conclusion is mistaken. If I is a model of the abstraction principle, then since $\ulcorner \smile^{\mathfrak{a}} (\mathfrak{a} = -A) = (\mathfrak{a} = \perp)\urcorner$[5] is true under I so is $\ulcorner \grave{\epsilon}(\epsilon = -A) = \grave{\epsilon}(\epsilon = \perp)\urcorner$. I', however, is not then a model of the abstraction principle because $\ulcorner \grave{\epsilon}(\epsilon = -A) = \grave{\epsilon}(\epsilon = \perp)\urcorner$ remains true under I' (since I assigns the same object to both $\ulcorner \grave{\epsilon}(\epsilon = -A)\urcorner$ and to '$\grave{\epsilon}(\epsilon = \perp)$', and hence $\ulcorner X(\grave{\epsilon}(\epsilon = -A))\urcorner$ denotes the same object as '$X(\grave{\epsilon}(\epsilon = \perp))$') whereas $\ulcorner \smile^{\mathfrak{a}} (\mathfrak{a} = -A) = (\mathfrak{a} = \perp)\urcorner$ is false under I' (because under I', A denotes the True).

In a similar fashion, Schroeder-Heister is able to show that the identifiability thesis as he is led to construe it is false. He proceeds essentially as follows: Let A_1 and B_1 be the closed abstracts '$\grave{\epsilon}(\epsilon = \top)$' and $\ulcorner \grave{\epsilon}(\epsilon = A_1)\urcorner$ respectively. There is an interpretation which satisfies the abstraction principle and which assigns different objects to A_1 and B_1. But any interpretation which satisfies the abstraction principle and which assigns the True to A_1 must also assign the True to B_1 and so cannot assign the False to B_1. A_1 and B_1 thus constitute a counterexample to what Schroeder-Heister takes to be the identifiability thesis: that if two arbitrary abstracts are assigned different objects by some interpretation which satisfies the abstraction principle then we can provide a new interpretation which satisfies the abstraction principle and which assigns the True to one and the False to the other.

How, then, does Schroeder-Heister's model-theoretic reconstruction of Frege's argument go awry? We can best make this plain by presenting our rival construal of the argument likewise in a model-theoretic framework. This will show that the argument provides a sound demonstration of the identifiability thesis. Consider then the following reconstruction of Frege's argument. Let f be a function which, when assigned to the abstraction operator,[6] yields a model of the abstraction principle; and let ω_1 and ω_2 be any two of its values. Now define the permutation X as before. Then the composition of X and f will also, when assigned to the abstraction operator, yield a model of the abstraction principle.[7] It follows that it is compatible with the stipulation embodied in the abstraction principle that any value of the abstraction function[8] should be the True and any other the False; and this, we propose, is just the identifiability thesis.

This reconstruction of Frege's argument differs from (each of) Schroeder-Heister's in that evaluations of closed abstracts are determined by the assignment of a particular function to the abstraction operator. By contrast, Schroeder-Heister considers evaluations which result from assigning objects to closed abstracts directly.[9] As a result he countenances evaluations which have no counterpart on our construal. It is this lenience which enables him to demonstrate that Frege's argument as he reconstructs it is fallacious. For recall the role of the interpretation I' in that demonstration. The evaluation determined by I' is one of those which have no counterpart on our construal; it does not correspond to any assignment to the abstraction operator. Again, it is this lenience that leads Schroeder-Heister to misconstrue the identifiability thesis in such a way that it admits of a counterexample. The counterexample, as we have seen, concerns the two abstracts $A_1 =$ '$\grave{\epsilon}(\epsilon = \top)$' and $B_1 = \ulcorner \grave{\epsilon}(\epsilon = A_1) \urcorner$. Schroeder-Heister shows, quite correctly, that it is not compatible with the stipulation embodied in the abstraction principle that A_1 should denote the True and B_1 the False. But on our reading of the identifiability thesis, Frege does not suggest that it is. Any evaluation which results from assigning the True to A_1 and the False to B_1 will also be one with no counterpart on our construal. If we do stipulate that A_1 is to denote the True, as Frege has shown that we are free to do, then we shall *ipso facto* have stipulated that B_1 is to denote the True as well and we must choose some *other* value of the abstraction function to be the False. But, as Frege rightly claims, any other value will do.

NOTES

1. Frege's abstraction principle (his notorious Axiom V) is expressed in his notation (omitting the assertion sign) as follows: '$((\grave{\epsilon}f(\epsilon) = \grave{\alpha}g(\alpha)) = -\!\overset{\mathfrak{a}}{\smile}\!\!- f(\mathfrak{a}) = g(\mathfrak{a})$' where '$f$' and '$g$' are assumed to be bound by initial universal quantifiers. The expressions '$\grave{\epsilon}\Phi(\epsilon)$' and '$\grave{\alpha}\Psi(\alpha)$' are employed by Frege to denote the value-ranges of the functions $\Phi(\xi)$ and $\Psi(\xi)$ respectively—'Φ' and 'Ψ' here being purely schematic. The second-level function name '$-\!\overset{\mathfrak{a}}{\smile}\!\!-\phi(\mathfrak{a})$' corresponds roughly to our universal quantifier.

2. [1], p. 46. We have taken the liberty of replacing Furth's 'course-of-values' with 'value-range'.

3. Note that what Schroeder-Heister terms the permutation argument is not the initial argument of section 10 (i.e., not what we call the permutation argument) but its later variant.

4. Of course, given the inconsistency of Frege's system, the identifiability thesis is in fact false but we are disregarding this inconsistency here and throughout the paper.

5. The first-level function $-\xi$ has the True as its value if its argument is the True and the False as its value otherwise. '$\bot$' is an abbreviation for '$\lnot\!\overset{\mathfrak{a}}{\smile}\!-\mathfrak{a} = \mathfrak{a}$' and '$\bot$' (which we shall shortly encounter) is an abbreviation for '$-\!\overset{\mathfrak{a}}{\smile}\!-\mathfrak{a} = \mathfrak{a}$'.

6. The abstraction operator, which in Frege's notation is '$\grave{\epsilon}\phi(\epsilon)$', is the second-level function name which is used to form abstracts—'ϕ' here indicates its argument-place. The assignments which we are envisaging do not affect the values of any other primitive symbols, nor the rules for evaluating complex formulas, which are all assumed to be fixed.

7. For a fuller exposition of Frege's argument along these lines, see [2].

8. The abstraction function is just the second-level function denoted by the abstraction operator, namely that function which maps first-level functions onto their value-ranges.

9. Since the abstraction operator is a primitive symbol, and closed abstracts are complex, his approach is less in keeping with standard model theory than ours, because in standard model theory primitive symbols are interpreted first and complex symbols are then evaluated accordingly. Furthermore, as Schroeder-Heister himself points out, on his approach it is possible to provide a semantics for Frege's formal language only by assuming that every object in the domain of quantification has a name in the language. On our approach this questionable assumption does not need to be made. Schroeder-Heister's justification for proceeding in the way he does is that if "one wanted to take into account the internal structure of abstracts, one would already have to presuppose that closed abstracts are used to denote sets" ([3], p. 70). But this is correct only if one presupposes which second-level function the abstraction operator stands for and, on our way of construing Frege's argument, no such presupposition is being made.

REFERENCES

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