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Strong Normalization for Typed Terms with Surjective Pairing

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1 Introduction In this note we describe a simple method for reducing strong normalization (henceforth to be abbreviated as SN) for PRIM($\mathfrak{J}$), the primitive recursive functionals with surjective pairing, to SN for PRIM($\mathfrak{J}_f$), the primitive recursive functionals with functional types only. The set of contraction rules includes the "surjectivity contraction", that is to say the contraction of a redex of the form $\boldsymbol{p}(\boldsymbol{p}_1 t)(\boldsymbol{p}_2 t)$ to t, for all t of the appropriate type, where $\boldsymbol{p}$ is a pairing operator with inverses $\boldsymbol{p}_1$, $\boldsymbol{p}_2$.

SN for PRIM($\mathfrak{J}$) almost automatically entails SN for certain related systems of typed terms and conversion rules, via suitable homomorphic embeddings into PRIM($\mathfrak{J}$) which preserve nontrivial reductions (t reduces trivially to t' iff t' and t are identical modulo renaming bound variables). Examples of such systems are: the typed $\lambda\eta$-calculus with surjective pairing; (terms describing) natural deductions in intuitionistic predicate logic, with respect to proper conversions (detour conversions) but without permutation conversions (cf. [10], 4.1.3); the terms in the first-order fragment $\mathbf{ML}_0$ of Martin-Löf's theory of types ([6]; $\mathbf{ML}_0$ is described for example in [2]). In the case of the first two examples our method for reducing SN for arbitrary terms to SN for the terms without $\times$ in their types can also be applied directly.

In the literature there are several proofs of SN for either PRIM($\mathfrak{J}$) itself [3] or for natural deduction systems containing at least the rules for $\rightarrow$, $\wedge$, $\forall$ and induction, which technically are very close to PRIM($\mathfrak{J}$), so that these proofs can be adapted to PRIM($\mathfrak{J}$) (e.g., [8], [4], [5]).

None of these published proofs consider the (analogue of) surjective pairing contraction SP. At least for Gandy's proof it is easy to show that SP is covered as well.

A straightforward extension of the proof in [10], 2.2, based on Tait's notion [9] of computability, is also possible, as was shown by R. de Vrijer (1982, unpublished). The result and method of Bercovici [1] are similar and can presumably be extended to PRIM($\mathfrak{J}$). De Vrijer adapted the definition of strong computability SC in [10], 2.2.13, as follows:

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$$\mathrm{SC}_0(t) := \mathrm{SN}(t),\ \mathrm{SC}_{\sigma\to\tau}(t) := \mathrm{SN}(t) \wedge \forall t'' \in \mathrm{SC}_\sigma \forall t' \succeq t(t't'' \in \mathrm{SC}_\tau),$$
$$\mathrm{SC}_{\sigma\times\tau}(t) := \mathrm{SN}(t) \wedge \forall t't''(t \succeq \boldsymbol{p}t't'' \to \mathrm{SC}_\sigma t' \wedge \mathrm{SC}_\tau t'').$$

However, we think that the interest of our result in Section 3 lies not so much in providing a new proof of SN for PRIM($\mathfrak{T}$) which includes SP, but rather in showing that SN, for the case where product types are present, is in fact a simple consequence of SN for terms without product types.

In the remainder of the paper most of the notation is standard. For notations not explained here, and other background information, see [10], 1.6.2-1.6.8, 1.6.16-1.6.17, 1.7, 1.8.1-1.8.4, and 2.2.

2 *Preliminaries*

Let $\mathfrak{T}$ be the set of symbols for finite types generated by the clauses $0 \in \mathfrak{T}$; if σ, $\tau \in \mathfrak{T}$ then also $(\sigma)\tau \in \mathfrak{T}$ and $\sigma \times \tau \in \mathfrak{T}$, and let $\mathfrak{T}_f$ be the subset of $\mathfrak{T}$ consisting of type symbols constructed without $\times$. $\mathfrak{T}_f$ is the set of *functional* or *implicational* types; types of the form $\sigma \times \tau$ are called *product types*.

If $\vec{\sigma} \equiv (\sigma_1, \ldots, \sigma_n)$, we sometimes write $(\vec{\sigma})\tau$ for $(\sigma_1)\ldots(\sigma_n)\tau$.

Let us write $t \succ_1 t'$ if t' is obtained from t by contracting a single redex in t; $\succ$ is the transitive closure of $\succ_1$, $\succeq$ the transitive *and* reflexive closure of $\succ_1$. The weak Church-Rosser property (WCR) and the Church-Rosser property (CR) can be stated as

WCR $\forall tt't''(t \succ_1 t' \wedge t \succ_1 t'' \to \exists t'''(t' \succeq t''' \wedge t'' \succeq t'''))$,
CR $\quad\forall tt't''(t \succeq t' \wedge t \succeq t'' \to \exists t'''(t' \succeq t''' \wedge t'' \succeq ''')).$

In the presence of SN, CR readily follows from WCR (see e.g., [10], 2.2.26), and since WCR is easily proved for the typed $\lambda\eta$-calculus with surjective pairing, our main result yields another proof of Pottinger's result [7] that CR holds for this calculus.

For sequences of terms $\vec{t} \equiv (t_1, \ldots, t_n)$, $\vec{s} \equiv (s_1, \ldots, s_n)$ we write $\vec{t} \succ_1 \vec{s}$ iff for some i $t_i \succ_1 s_i$ and $t_j \equiv s_j$ for all $j \neq i$; $\vec{t} \succeq \vec{s}$, $\vec{t} \succ \vec{s}$ are defined accordingly.

The terms of PRIM($\mathfrak{T}$) are built by application and λ-abstraction from variables ($x, y, z, \ldots$) of all finite types, constants 0 (zero), S (successor), pairing and unpairing operators $\boldsymbol{p}$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ for all appropriate types, as well as sequences $\vec{\boldsymbol{r}} \equiv (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)$ of constants for simultaneous recursion for all appropriate types. (We have dropped all explicit references to types, but of course the recursion constants and $\boldsymbol{p}$, $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ must be regarded as syntactically distinct for different types.) Terms which differ only in the naming of their bound variables are regarded as identical. In substitutions we tacitly assume bound variables to be renamed, if necessary, so as to avoid clashes of variables.

Let t, t', t'' be arbitrary terms, $\vec{t} \equiv (t_1, \ldots, t_n)$, $\vec{t}' \equiv (t_1', \ldots, t_n')$ arbitrary sequences of terms. The *contractions* for PRIM($\mathfrak{T}$) are now given by $(\lambda x.t)(t')$ contr $t[x/t']$, $\lambda x.tx$ contr t if x does not occur free in t, $\boldsymbol{p}_1(\boldsymbol{p}tt')$ contr t, $\boldsymbol{p}_2(\boldsymbol{p}tt')$ contr t', $\boldsymbol{p}(\boldsymbol{p}_1 t)(\boldsymbol{p}_2 t)$ contr t (surjectivity of pairing); and if $\vec{\boldsymbol{r}} \equiv (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)$ is a sequence of simultaneous recursion constants, then $\boldsymbol{r}_i\vec{t}\vec{t}'0$ contr t_i, $\boldsymbol{r}_i\vec{t}\vec{t}'(St'')$ contr $t_i'(\vec{\boldsymbol{r}}\vec{t}\vec{t}'t'')t''$.

3 *The main result*

The reduction of SN for PRIM($\mathfrak{T}$) to SN for PRIM($\mathfrak{T}_f$) is obtained via a mapping * on types of $\mathfrak{T}$ and terms of PRIM($\mathfrak{T}$) transforming any $\sigma \in \mathfrak{T}$ into a finite sequence $\sigma^* \equiv (\sigma_1, \ldots, \sigma_n)$ (n depending on σ) of types

$\sigma_i \in \mathfrak{T}$, and each term $t \in \sigma$ of PRIM($\mathfrak{T}$) into a sequence of terms $t^* \equiv (t_1, \ldots, t_n)$ with $t_i \in \sigma_i$ $(1 \le i \le n)$. For the types * is defined by

(i) $0^* := 0$;
if $\sigma^* \equiv (\sigma_1, \ldots, \sigma_n)$, $\tau^* \equiv (\tau_1, \ldots, \tau_m)$ then
(ii) $(\sigma \times \tau)^* := (\sigma_1, \ldots, \sigma_n,\ \tau_1, \ldots, \tau_m)$, for which we simply write σ^*, τ^*; and
(iii) $(\sigma \to \tau)^* := ((\sigma^*)\tau_1, \ldots, (\sigma^*)\tau_m)$.

Thinking of the formation of function types as implication and of $\times$ as conjunction, we see that * on the types corresponds to a well-known procedure for eliminating $\wedge$ from intuitionistic propositional $(\to, \wedge)$-logic, by which each $(\to, \wedge)$-formula A is replaced by a finite sequence $A_1, \ldots, A_n$ of $(\to)$-formulas such that A is deducible iff the A_i are all deducible.

* is also defined for terms of PRIM($\mathfrak{T}$) as follows.

(iv) $0^* := 0$, $S^* := S$;
(v) to each variable $x \in \sigma$ we assign a sequence $x^* \equiv (x_1, \ldots, x_n)$ with $x^* \in \sigma^*$ (i.e., $x_i \in \sigma_i$) such that for $x \neq y$ x^* and y^* are disjoint, and the variables in each sequence are all distinct;
(vi) when t^* has been defined we put $(\lambda x.t)^* := \lambda x^*.t^*$, i.e., if $t^* \equiv (t_1, \ldots, t_m)$, $(\lambda x.t)^*$ is the sequence $(\lambda x^*.t_1, \ldots, \lambda x^*.t_m)$.
(vii) Let $\vec{\boldsymbol{r}} \equiv (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n)$ be a sequence of simultaneous recursion constants, with $\boldsymbol{r}_i \in (\sigma_1) \ldots (\sigma_n)(\tau_1) \ldots (\tau_n)(0)\sigma_i$, where $\tau_i \equiv (\sigma_1) \ldots (\sigma_n)(0)\sigma_i$. Suppose $\sigma_i^* = (\sigma_{i,1}, \ldots, \sigma_{i,p(i)})$ then we take

$$\boldsymbol{r}_i^* := \boldsymbol{r}'_{i,1}, \ldots, \boldsymbol{r}'_{i,p(i)}, \qquad \boldsymbol{r}^* := \boldsymbol{r}_1^*, \ldots, \boldsymbol{r}_n^*,$$

where $\boldsymbol{r}^*$ is now a sequence of simultaneous recursors with $\boldsymbol{r}'_{i,j} \in (\sigma_1^*) \ldots (\sigma_n^*)(\tau_1^*) \ldots (\tau_n^*)(0)\sigma_{i,j}$.
(viii) $\boldsymbol{p}^* := \lambda x^* y^*.(x^*, y^* \equiv \lambda x^* y^*.x^*, \lambda x^* y^*.y^*$
$\boldsymbol{p}_1^* := \lambda x^* y^*.x^*$, $\boldsymbol{p}_2^* := \lambda x^* y^*.y^*$,
$(ix) (tt')^*$ is $t^* t'^*$.

Lemma *If* $t \succ_1 s$ *in* PRIM($\mathfrak{T}$), *then* $t^* \succ s^*$ *in* PRIM($\mathfrak{T}_f$).

The proof of this lemma is completely straightforward, and immediately implies

Theorem SN *for* PRIM($\mathfrak{T}_f$) *implies* SN *for* PRIM($\mathfrak{T}$).

Remarks: (i) The mapping * can easily be extended to formulas of intuitionistic finite-type arithmetic with pairing ($\mathbf{N\text{-}HA}_p^\omega$ in [10], 1.8.2), leading to a proof that $\mathbf{N\text{-}HA}_p^\omega$ is an "almost-definitional" extension of $\mathbf{N\text{-}HA}^\omega$, and a fortiori conservative over $\mathbf{N\text{-}HA}^\omega$.

(ii) We have dealt with sequences of constants for simultaneous recursion directly; a slight simplification would result by formulating PRIM($\mathfrak{T}$) with single recursion constants only.

Usually one takes only single recursors as primitives, and the existence of sequences of definable closed terms for simultaneous recursion is deduced from this (cf. [10], 1.7). However, the proof that these defined constants satisfy the equations for simultaneous recursion requires (quantifier-free) induction, and so the corresponding conversion rules do not follow from those for single recursors unless we restrict attention to closed terms.

REFERENCES

[1] Bercovici, I., "Strong normalization for typed lambda calculus with surjective pairing—Tait's method," to appear.

[2] Diller, J. and A. S. Troelstra, "Realizability and intuitionistic logic," *Synthese*, vol. 60 (1984), pp. 253-282.

[3] Gandy, R. O., "Proofs of strong normalization," pp. 457-477 in *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, eds. J. P. Seldin and J. R. Hindley, Academic Press, London, 1980.

[4] Jervell, H. R., "A normal form in first order arithmetic," pp. 93-108 in *Proceedings of the Second Scandinavian Logic Symposium*, ed. J. E. Fenstad, North-Holland Publishing Co., Amsterdam, 1971.

[5] Leivant, D., "Strong normalization for arithmetic," pp. 182-197 in *Proof Theory Symposium Kiel 1974*, eds. J. Diller and G. H. Müller, Springer Verlag, Berlin, 1975.

[6] Martin-Löf, P., "Constructive mathematics and computer programming," pp. 153-175 in *Logic, Methodology and Philosophy of Science VI*, eds. J. Loś, H. Pfeiffer, and K.-P. Podewski, North-Holland Publishing Co., Amsterdam, and Polish Scientific Publishers, Warszawa, 1982.

[7] Pottinger, G., "The Church-Rosser theorem for the typed λ-calculus with surjective pairing," *Notre Dame Journal of Formal Logic*, vol. 22 (1981), pp. 264-268.

[8] Prawitz, D., "Ideas and results in proof theory," pp. 235-307 in *Proceedings of the Second Scandinavian Logic Symposium*, ed. J. E. Fenstad, North-Holland Publishing Co., Amsterdam, 1971.

[9] Tait, W. W., "Intensional interpretations of functionals of finite type I," *The Journal of Symbolic Logic*, vol. 32 (1967), pp. 198-212.

[10] Troelstra, A. S. (editor), *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, Springer-Verlag, Berlin, 1973.

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