Finite Kripke Models of HA are Locally PA

D. van DALEN, H. MULDER, E. C. W. KRABBE, and A. VISSE
It is well-known that $\|\cdot\|$ is cumulative, i.e., if $\alpha \vdash \phi$ then, for all $\beta$ such that $\alpha \leq \beta$, $\beta \vdash \phi$.

Deletion of some (but not all) nodes from a Kripke model $K$ again yields a Kripke model. It suffices to restrict $\leq$, $D$, and $I$ to the remaining set of nodes. If $\alpha \in K$ then the model obtained by deleting all $\beta$ such that $\alpha \leq \beta$ will be denoted as $K^\alpha (= \langle K^\alpha, \leq^\alpha, D^\alpha, I^\alpha \rangle)$, its relation of forcing as $\|_{\alpha}$. Obviously, for all $\beta \in K^\alpha$ and for all $\phi \in L_\beta$: $\beta \|_{\alpha} \phi$ iff $\beta \| \phi$.

A **classical node** in a Kripke model $K$ is to be a node $\alpha$ of $AT$ that forces all sentences $\forall x_1 \ldots \forall x_n (\phi \lor \neg \phi) \in L_\alpha$. We note the following properties of classical nodes:

1. The following conditions are equivalent:
   (i) $\alpha$ is a classical node
   (ii) $\alpha$ forces all sentences $\forall x_1 \ldots \forall x_n (\phi \lor \neg \phi) \in L$
   (iii) For all $\phi \in L_\alpha$ $\alpha \vdash \phi$ iff $\alpha \| \phi$.

2. All final nodes (i.e., nodes such that for no $\beta < \alpha$: $\alpha \leq \beta$) are classical.

3. If $\alpha$ is classical, so are all $\beta$ such that $\alpha \leq \beta$.

4. Let $L$ be the language of arithmetic. If $\alpha$ is classical and $\alpha \| \text{HA}$ then $\alpha \| \text{PA}$. Moreover $\text{M}_\alpha$ will be a Peano model.

Let $\rho$ be any sentence of $L$. For each formula $\phi$ of $L$ we can construct another formula, $\phi^\rho$, by substituting $\phi^0 \lor \rho$ for each atomic component $\phi^0$ of $\phi$. The result, $\phi^\rho$, is called the Friedman translation of $\phi$ by $\rho$ in $L$. We write $\Gamma^\rho$ for $\{ \phi^\rho | \phi \in \Gamma \}$. We shall exploit the following facts about Friedman translations (cf. [2]):

(A) $\rho \vdash \phi^\rho$.

(B) If $\Gamma \vdash \phi$ then $\Gamma^\rho \vdash \phi^\rho$.

(C) Let $L$ be the language of arithmetic: if $\text{HA} \vdash \phi$ then $\text{HA} \vdash \phi^\rho$.

(D) Let $L$ be the language of arithmetic, $\phi \in \Sigma_1^0$, then $\text{HA} \vdash \phi^\rho \iff (\phi \lor \rho)$.

### 2 Pruning

**Definition 1** Let $K$ be a Kripke model, $\rho$ a sentence such that, for at least one node $\alpha \in K$, $\rho \in L_\alpha$ and $\alpha \| \rho$. Then the model obtained by **pruning** $\rho$-nodes from $K$ shall be the model obtained from $K$ by deleting all nodes that force $\rho$. This model will be denoted as $K^\rho (= \langle K^\rho, \leq^\rho, D^\rho, I^\rho \rangle)$, its forcing relation by $\|_{\rho}$.

**First Pruning Lemma** If $\beta \in K^\rho$ and $\phi, \rho \in L_\beta$ then: $\beta \| \phi^\rho$ iff $\beta \| \rho \phi$. 

**Proof:** This is proved by induction on $\phi$, the two relatively complex cases being '$\rightarrow$' and '∨'.

**Case** $\phi = \phi_1 \rightarrow \phi_2$. ($\Rightarrow$) Assume $\beta \|_{\rho} \phi_1 \rightarrow \phi_2$. Then, for some $\beta'$ such that $\beta \leq_{\rho} \beta'$, $\beta' \|_{\rho} \phi_1$ and $\beta' \|_{\rho} \phi_2$. Obviously, $\beta \leq \beta'$ and $L_{\beta'} \subseteq L_\beta$, so $\phi_1$, $\phi_2$, $\rho \in L_{\beta'}$. By the induction hypothesis $\beta' \|_{\rho} \phi_1^\rho$ and $\beta' \|_{\rho} \phi_2^\rho$, whence it follows that $\beta' \|_{\rho} \phi_1^\rho \rightarrow \phi_2^\rho$, i.e., $\beta' \|_{\rho} (\phi_1 \rightarrow \phi_2)^\rho$.

($\Rightarrow$) Assume $\beta \|_{\rho} (\phi_1 \rightarrow \phi_2)^\rho$, i.e., $\beta \|_{\rho} \phi_1^\rho \rightarrow \phi_2^\rho$. Then, for some $\beta'$ such that $\beta \leq \beta'$, $\beta' \|_{\rho} \phi_1^\rho$ and $\beta' \|_{\rho} \phi_2^\rho$. Since $\rho \vdash \phi_2^\rho$ (fact A, Section 1), it follows
that $\beta' \Vert \rho$. Hence $\beta' \in K^\rho$ and $\beta \leq^\rho \beta'$. Obviously $\phi_1, \phi_2, \rho \in L_{\beta'}$, so we can apply the induction hypothesis to obtain $\beta' \Vert^\rho \phi_1$ and $\beta' \Vert^\rho \phi_2$, whence it follows that $\beta' \Vert^\rho \phi_1 \to \phi_2$.

**Case $\phi = \forall x \phi_1$.** ($\Rightarrow$) Assume $\beta \Vert^\rho \forall x \phi_1(x)$ (writing '$\phi_1(x)$' for '$\phi_1$'). Then, for some $\beta'$ such that $\beta \leq^\rho \beta'$, and for some $c \in D_{\beta'}$, $\beta' \Vert^\rho \phi_1(c)$. Obviously, $\beta \leq \beta'$ and $\beta \in L_{\beta'}$, so $\forall x \phi_1, \rho \in L_{\beta'}$. Moreover $D_{\beta'} = D_{\beta'}$, so $c \in D_{\beta'}$ and $\phi_1(c) \in L_{\beta'}$. By the induction hypothesis $\beta' \Vert^\rho (\phi_1(c))^\rho$. Since $\rho$ is a sentence, $(\phi_1(c))^\rho = (\phi_1)^{\{c/\}}$. It follows that $\beta \Vert^\rho \forall x(\phi_1)^{\{c/\}}$, i.e., $\beta \Vert^\rho (\forall x \phi_1)^\rho$.

($\Leftarrow$) Assume $\beta \Vert (\forall x \phi_1)^\rho$, i.e., $\beta \Vert^\rho \forall x(\phi_1)^{\{c/\}}$. Then, for some $\beta'$ such that $\beta \leq \beta'$, and for some $c \in D_{\beta'}$, $\beta' \Vert (\phi_1)^{\{c/\}}$, i.e., $\beta' \Vert (\phi_1(c))^\rho$. Since $\rho \vdash (\phi_1(c))^\rho$ (fact A), it follows that $\beta' \Vert \rho$. Hence $\beta' \in K^\rho$ and $\beta \leq^\rho \beta'$. Obviously, $(\phi_1(c)), \rho \in L_{\beta'}$, so we can apply the induction hypothesis to obtain $\beta' \Vert^\rho \phi_1(c)$. Since $c \in D_{\beta'} (=D_{\beta'})$, it follows that $\beta \Vert^\rho \forall x \phi_1$.

**Second Pruning Lemma** Let $L$ be the language of arithmetic. If $\beta \in K^\rho$ and $\rho \in L_{\beta}$ and $\beta \vdash HA$ then $\beta \vdash^\rho HA$.

**Proof:** Assume $\beta \in K^\rho$, $\rho \in L_{\beta}$, $\beta \vdash HA$. Let $\phi$ be any theorem of HA. Since $HA \vdash \phi$ (fact C), it follows that $\beta \vdash \phi$. According to the first pruning lemma and $\phi \in L_{\beta}$, $\beta \vdash^\rho \phi$. Hence $\beta \vdash^\rho HA$.

**3 Spotting Peano models** From now on we shall assume that $L$ is (any suitable variant or extension of) the language of arithmetic.

**Theorem 1** The local models in finite Kripke models of Heyting arithmetic are Peano models.

**Proof:** Let $K$ be a finite Kripke model, $\alpha \in K$, $\alpha \vdash HA$. Avoiding $\alpha$, we shall apply several prunings to $K$. Construct a sequence of models $K^{(0)}, \ldots, K^{(n)}$ as follows. Let $K^{(0)}$ be $K$. Let $K^{(i)}$ be given and assume $\alpha \in K^{(i)}$. If there is a sentence $\rho \in L_{\beta}^\rho$ such that $\alpha \vdash^\rho \rho$ whereas some $\beta \in K^{(i)}$ can be found such that $\beta \vdash^\rho \rho$, take any such $\beta$ and let $K^{(i+1)}$ be the model obtained by pruning $\rho$-nodes from $K^{(i)}$. Otherwise, if there is no such $\rho$, the construction will halt. Let $n$ be the stage where the process halts.

**Claim** $\alpha$ is a classical node in $K^{(n)}$. For, let $\rho$ be any sentence $\forall x_1 \ldots \forall x_n (\phi \lor \neg \phi) \in L_{\beta}^\rho$. Let $\beta$ be some final node such that $\alpha \leq \beta$. $\beta$ is classical (fact 2, Section 1) and $L_{\beta}^{(n)} \subseteq L_{\beta}^\rho$. Hence $\beta \vdash^\rho \rho$, and by definition of $n \alpha \vdash^\rho \rho$. Further, it follows from $\alpha \vdash HA$, by the second pruning lemma, that $\alpha \vdash^\rho HA$ (for all $1 \leq i \leq n$). Hence $M_{\alpha}^{(n)}$ will be a Peano model (fact 4). But $M_{\alpha}^{(n)} = M_\alpha$.

**Corollary** Let $\alpha$ be a node in a Kripke model $K$ such that $\alpha \vdash HA$. Let $K^\alpha$ be finite. Then $M_\alpha$ is a Peano model.

There seem to be no straightforward extensions of this result to infinite Kripke models. However, if the underlying structure is of type $\omega$, we have:

**Theorem 2** A Kripke model of HA over $\omega$ (with its natural order) contains infinitely many local Peano models.

**Proof:** Let $K = \langle \omega, \leq, D, I \rangle$ be a Kripke model of HA (i.e., for each $n \in \omega$, $n \vdash HA$), where $\leq$ is the natural ordering on $\omega$. 
Case 1. Let \( K \) contain a classical node \( n \). Then all \( m > n \) will be classical as well (fact 3, Section 1). For each such \( m \), since \( m \vDash \text{HA} \), \( M_n \) will be a Peano model (fact 4).

Case 2. Let \( K \) contain no classical nodes. Consider the set \( A = \{ n \mid n \in \omega \text{ and for all } \phi \in L_n : n + 1 \vDash \phi \text{ then } n \vDash \phi \} \). We shall first show that

(i) \( \omega \sim A \) is infinite.

Suppose \( \omega \sim A \) were finite. Let \( n \) be such that, for all \( m \geq n, m \in A \). Since \( n \) is not classical, there is a sentence \( \forall x_1 \ldots \forall x_r (\phi \vee \neg \phi) \in L_n \) such that \( n \not\vDash \forall x_1 \ldots \forall x_r (\phi \vee \neg \phi). \) Hence, for some \( m \geq n \) and for certain \( c_1, \ldots, c_r \in D_m, m \vDash \phi(c_1 \ldots c_r) \vee \neg \phi(c_1 \ldots c_r). \) Then \( m \vDash \phi' \) and \( m \not\vDash \neg \phi' \). Hence, for some \( k > m, k \vDash \phi' \). Let \( k^* \) be minimal with the property: \( k^* > m, k^* \vDash \phi' \). Then \( k^* - 1 \not\vDash \phi' \) and \( k^* - 1 \geq m \geq n \). Since \( \phi' \in L_{k^* - 1} \), it follows that \( k^* - 1 \in \omega \sim A \), contradicting the choice of \( n \). Therefore (i) holds.

Let \( K^- (= (K^-, \leq^-, D^-, I^-)) \) be the model obtained from \( K \) by deleting all nodes in \( A \). Forcing in \( K^- \) will be denoted by \( \vDash^- \). It can be shown, by a simultaneous induction on \( \phi \) for all \( n \in K^- \), that the following holds:

(ii) For all \( n \in K^-, \phi \in L_n, n \vDash \phi \) iff \( n \not\vDash \phi \).

We consider the case of the implication.

\( \phi = \phi_1 \rightarrow \phi_2. \) (\( \rightarrow \)) Assume \( n \not\vDash \phi_1 \rightarrow \phi_2 \). Then, for some \( m \) such that \( n \leq^-, m, m \vDash \phi_1 \) and \( m \not\vDash \phi_2 \). Obviously \( n \leq m \) and \( \phi_1, \phi_2 \in L_m \). According to the induction hypothesis \( m \vDash \phi_1 \) and \( m \not\vDash \phi_2 \). Hence \( n \not\vDash \phi_1 \rightarrow \phi_2 \).

(\( \rightarrow \)) Assume \( n \not\vDash \phi_1 \rightarrow \phi_2 \). Then, for some \( m \), such that \( n \leq m, m \not\vDash \phi_1 \) and \( m \vDash \phi_2 \). Suppose first that \( m \in K^- \). Since \( \phi_1, \phi_2 \in L_m \), it follows by the induction hypothesis that \( m \vDash \phi_1 \) and \( m \not\vDash \phi_2 \). Obviously \( n \leq m \), so \( n \not\vDash \phi_1 \rightarrow \phi_2 \). Now suppose that \( m \notin K^- \). Since (i) holds there is a \( k > m \) such that \( k \in K^- \). Let \( k^* \) be minimal with that property: \( k^* > m, k^* \in K^- \). Then, for all \( k \) such that \( m \leq k < k^* \), \( k \in A \) and also \( \phi_2 \in L_k \). By definition of \( A \) the following holds: if \( k \not\vDash \phi_2 \) then \( k + 1 \not\vDash \phi_2 \). Hence, since \( m \not\vDash \phi_2 \), \( k^* \not\vDash \phi_2 \). On the other hand \( k^* \vDash \phi_1 \) (cumulation). Since \( \phi_1, \phi_2 \in L_{k^*} \), it follows by the induction hypothesis that \( k^* \vDash \phi_1 \) and \( k^* \not\vDash \phi_2 \). Since obviously \( n \leq k^* \) we may conclude that \( n \not\vDash \phi_1 \rightarrow \phi_2 \). The case \( \phi = \forall x \phi_1 \) can be treated similarly, whereas the other cases are even simpler. So (ii) holds.

An immediate consequence of (ii) is that for each node \( n \in K^- \), \( n \not\vDash \text{HA} \). We shall now show that \( M_n \) is a Peano model. Since \( n \notin A \), there is a sentence \( \rho \in L_n^-(=L_n) \) such that \( n \not\vDash \rho \) and \( n + 1 \vDash \rho \). According to (ii) \( n \not\vDash \rho \), hence the model \( K^- \rho \) exists and contains \( n \). By the second pruning lemma it follows that \( n \not\vDash-\rho \text{HA} \). Moreover \( n \) is a final node of \( K^- \rho \). For if \( n <^-, m \) it follows that \( n + 1 \leq m \), therefore \( m \vDash \rho \) (cumulation) and by (ii) \( m \not\vDash-\rho \). Hence \( m \) will be pruned away. Since \( n \) is final it is classical in \( K^- \rho \) (fact 2, Section 1) and so \( M_{n^-} \) is a Peano model (fact 4). But \( M_n = M_{n^-} \rho \), hence each of the infinitely many \( M_n \) such that \( n \in K^- \) is a Peano model.

4 Other applications of pruning Friedman's proof of Markov's rule (MR) (cf. Friedman, [2]) has a model theoretic version.
MR Let $\phi \in \Sigma^0_1$. Then $\text{HA} \vdash \forall x_1 \ldots \forall x_n \neg \neg \phi \rightarrow \text{HA} \vdash \forall x_1 \ldots \forall x_n \phi$.

Proof: Assume $\phi_0 \in \Sigma^0_1$, $\text{HA} \vdash \forall x_1 \ldots \forall x_n \neg \neg \phi_0$, but $\text{HA} \nvdash \forall x_1 \ldots \forall x_n \phi_0$. By the completeness theorem there is a Kripke model $K$ of $\text{HA}$ with a node $\alpha$ such that $\alpha \not\Vdash \forall x_1 \ldots \forall x_n \phi_0$. Therefore, $K$ contains a node $\beta$ such that, for certain $c_1, \ldots, c_n \in D_\beta$, $\beta \Vdash \phi_0(c_1, \ldots, c_n)$. Put $\phi = \phi_0(c_1, \ldots, c_n)$, then $\phi \in L_\beta$ and $\beta \Vdash \phi$. Hence $K^\phi$ exists and $\beta \in K^\phi$. According to the second pruning lemma, $\beta \Vdash \phi$. Consequently $\beta \Vdash \neg \neg \phi$ and there is some $\gamma \in K^\phi$ such that $\gamma \Vdash \phi$. By the first pruning lemma $\gamma \Vdash \phi$. Since $\phi \in \Sigma^0_1$, $\phi$ is equivalent to $\phi \lor \phi$ in $\text{HA}$ (fact D, Section 1). Since $\gamma \Vdash \text{HA}$, $\gamma \Vdash \phi \lor \phi$. Therefore $\gamma \Vdash \phi$. This means that $\gamma$ must have been pruned away, contradicting $\gamma \in K^\phi$.

In the same way we can formulate a model-theoretic version of Visser’s proof of the following (cf. [4]):

VR Let $\phi \in \Sigma^0_1$. Then $\text{HA} \vdash \forall x_1 \ldots \forall x_n (\neg \neg \phi \rightarrow \phi)$ implies $\text{HA} \vdash \forall x_1 \ldots \forall x_n (\phi \lor \neg \phi)$.

Proof: Assume $\phi_0 \in \Sigma^0_1$, $\text{HA} \vdash \forall x_1 \ldots \forall x_n (\neg \neg \phi_0 \rightarrow \phi_0)$, but $\text{HA} \nvdash \forall x_1 \ldots \forall x_n (\phi_0 \lor \neg \phi_0)$. By the completeness theorem there is a Kripke model $K$ of $\text{HA}$ with a node $\alpha$ such that $\alpha \not\Vdash \forall x_1 \ldots \forall x_n (\phi_0 \lor \neg \phi_0)$. Therefore, $K$ contains a node $\beta$ such that for certain $c_1, \ldots, c_n \in D_\beta$, $\beta \Vdash \phi_0 \lor \neg \phi$, where $\phi = \phi_0(c_1, \ldots, c_n)$. Certainly, $\neg \phi \in L_\beta$ and $\beta \not\Vdash \neg \phi$, hence $K^{\neg \phi}$ exists and $\beta \in K^{\neg \phi}$. According to the second pruning lemma $\beta \Vdash \neg \neg \phi$, so $\beta \Vdash \neg \neg \phi \rightarrow \phi$. Consider any $\gamma \in K^{\neg \phi}$ such that $\beta \leq^{\neg \phi} \gamma$. For such $\gamma$: $\gamma \Vdash \neg \phi$, whereas $\neg \phi \in L_\gamma$, therefore there is some $\gamma'$ such that $\gamma \leq \gamma'$ and $\gamma' \Vdash \phi$. Since $\gamma' \Vdash \neg \phi$ it follows that $\gamma' \in K^{\neg \phi}$ and $\gamma \leq^{\neg \phi} \gamma'$. Obviously, $\gamma' \Vdash \phi \lor \neg \phi$. Since $\phi \in \Sigma^0_1$, $\phi \lor \neg \phi$ is equivalent to $\phi^{\neg \phi}$ in $\text{HA}$ (fact D). By the first pruning lemma $\gamma' \Vdash \neg \neg \phi$. Therefore $\gamma' \Vdash \neg \phi$. Since this holds for any $\gamma$ such that $\beta \leq^{\neg \phi} \gamma$ we can conclude that $\beta \Vdash \neg \neg \phi$ and therefore $\beta \Vdash \neg \phi$. Applying the first pruning lemma once more we get $\beta \Vdash \phi^{\neg \phi}$, and, again by fact D, $\beta \Vdash \phi \lor \neg \phi$, a contradiction.

REFERENCES


