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Finite Kripke Models of HA are Locally PA

D. van DALEN, H. MULDER, E. C. W. KRABBE, and A. VISSER

Introduction In a Kripke model of Heyting's Arithmetic, **HA**, the nodes, when viewed as classical structures, are models of classical arithmetic with (at least) Δ_1^0-induction. In general, it is an open problem which form of induction holds in the classical structures at the nodes of Kripke models. However, in the case of finite Kripke models (i.e., those containing a finite number of nodes) one can show that all these structures satisfy full induction, and consequently are models of full Peano Arithmetic, **PA.** It can also be shown that any Kripke model with an underlying model structure of type ω must contain an infinite number of such Peano models. These results were established in a workshop in Utrecht (1983).

1 Preliminaries Let L be a first-order language with logical constants: $\bot$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, $=$. Let $\neg\phi$ be short for $\phi \rightarrow \bot$ and let $\phi \leftrightarrow \psi$ be short for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. An extension L_D of L is obtained by adding an individual constant $\bar{c}$ for each element c of D. In practice, D shall always be the local domain D_α of some node α in a Kripke model, and we shall write L_α instead of L_{D_α}.

A *Kripke model* $\mathbf{K} = \langle K, \leq, D, I\rangle$ consists of a nonempty set K of nodes, partially ordered by $\leq$, a function D that assigns a nonempty local domain of individuals to each $\alpha \in K$, and a function I that assigns an interpretation function I_α to each $\alpha \in K$. Each I_α assigns values to the individual constants, the function symbols, and the predicate symbols of L_α, so as to provide for a *local model* $\mathbf{M}_\alpha = \langle D_\alpha, I_\alpha\rangle$. The different I_α agree on the values assigned to individual constants that belong to L. Moreover, D and I are to be cumulative in the following sense: if $\alpha \leq \beta$ then $D_\alpha \subseteq D_\beta$, and, for each function symbol or predicate symbol X, $I_\alpha(X) \subseteq I_\beta(X)$. $\mathbf{K}$ is called finite if K is finite.

Since we are interested in a theory with decidable equality it is no restriction to assume that '$=$' is interpreted by the actual identity in each node (cf. [1], p. 184).

Semantic evaluations proceed as usual. We write $\alpha \models \phi$ if ϕ is *true* in the (classical) model $\mathbf{M}_\alpha$, and $\alpha \Vdash \phi$ if α *forces* ϕ. Further, we write $\alpha \Vdash \Gamma$ if for each $\phi \in \Gamma$, $\alpha \Vdash \phi$. The symbol '$\vdash$' shall denote derivability on the strength of intuitionistic logic.

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It is well-known that $\Vdash$ is cumulative, i.e., if $\alpha \Vdash \phi$ then, for all β such that $\alpha \leq \beta$, $\beta \Vdash \phi$.

Deletion of some (but not all) nodes from a Kripke model $\mathbf{K}$ again yields a Kripke model. It suffices to restrict $\leq$, D, and I to the remaining set of nodes. If $\alpha \in K$ then the model obtained by deleting all β such that not $\alpha \leq \beta$ will be denoted as $\mathbf{K}^\alpha$ $(= \langle K^\alpha, \leq^\alpha, D^\alpha, I^\alpha \rangle)$, its relation of forcing as $\Vdash^\alpha$. Obviously, for all $\beta \in K^\alpha$ and for all $\phi \in L_\beta$: $\beta \Vdash^\alpha \phi$ iff $\beta \Vdash \phi$.

A *classical node* in a Kripke model K is to be a node α of K that forces all sentences $\forall x_1 \ldots \forall x_n(\phi \vee \neg\phi) \in L_\alpha$. We note the following properties of classical nodes:

(1) The following conditions are equivalent:
 (i) α is a classical node
 (ii) α forces all sentences $\forall x_1 \ldots \forall x_n(\phi \vee \neg\phi) \in L$
 (iii) For all $\phi \in L_\alpha$ $\alpha \models \phi$ iff $\alpha \Vdash \phi$.

(2) All *final nodes* (i.e., nodes such that for no $\beta \neq \alpha$: $\alpha \leq \beta$) are classical.

(3) If α is classical, so are all β such that $\alpha \leq \beta$.

(4) Let L be the language of arithmetic. If α is classical and $\alpha \Vdash \mathbf{HA}$ then $\alpha \Vdash \mathbf{PA}$. Moreover $\mathbf{M}_\alpha$ will be a Peano model.

Let ρ be any sentence of L. For each formula ϕ of L we can construct another formula, ϕ^ρ, by substituting $\phi_0 \vee \rho$ for each atomic component ϕ_0 of ϕ. The result, ϕ^ρ, is called the *Friedman translation of ϕ by ρ in L*. We write Γ^ρ for $\{\phi^\rho | \phi \in \Gamma\}$. We shall exploit the following facts about Friedman translations (cf. [2]):

(A) $\rho \vdash \phi^\rho$.

(B) If $\Gamma \vdash \phi$ then $\Gamma^\rho \vdash \phi^\rho$.

(C) Let L be the language of arithmetic: if $\mathbf{HA} \vdash \phi$ then $\mathbf{HA} \vdash \phi^\rho$.

(D) Let L be the language of arithmetic, $\phi \in \Sigma_1^0$, then $\mathbf{HA} \vdash \phi^\rho \leftrightarrow (\phi \vee \rho)$.

2 *Pruning*

Definition 1 Let $\mathbf{K}$ be a Kripke model, ρ a sentence such that, for at least one node $\alpha \in K$, $\rho \in L_\alpha$ and $\alpha \nVdash \rho$. Then the model obtained by *pruning ρ-nodes* from $\mathbf{K}$ shall be the model obtained from $\mathbf{K}$ by deleting all nodes that force ρ. This model will be denoted as $\mathbf{K}^\rho (= \langle K^\rho, \leq^\rho, D^\rho, I^\rho \rangle)$, its forcing relation by $\Vdash^\rho$.

First Pruning Lemma *If $\beta \in K^\rho$ and ϕ, $\rho \in L_\beta$ then: $\beta \Vdash \phi^\rho$ iff $\beta \Vdash^\rho \phi$.*

Proof: This is proved by induction on ϕ, the two relatively complex cases being '$\rightarrow$' and '$\forall$'.

Case $\phi = \phi_1 \rightarrow \phi_2$. ($\Rightarrow$:) Assume $\beta \nVdash^\rho \phi_1 \rightarrow \phi_2$. Then, for some β' such that $\beta \leq^\rho \beta'$, $\beta' \Vdash^\rho \phi_1$ and $\beta' \nVdash^\rho \phi_2$. Obviously, $\beta \leq \beta'$ and $L_\beta \subseteq L_{\beta'}$, so ϕ_1, ϕ_2, $\rho \in L_{\beta'}$. By the induction hypothesis $\beta' \Vdash \phi_1^\rho$ and $\beta' \nVdash \phi_2^\rho$, whence it follows that $\beta' \nVdash \phi_1^\rho \rightarrow \phi_2^\rho$, i.e., $\beta' \nVdash (\phi_1 \rightarrow \phi_2)^\rho$.

($\Leftarrow$:) Assume $\beta \nVdash (\phi_1 \rightarrow \phi_2)^\rho$, i.e., $\beta \nVdash \phi_1^\rho \rightarrow \phi_2^\rho$. Then, for some β' such that $\beta \leq \beta'$, $\beta' \Vdash \phi_1^\rho$ and $\beta' \nVdash \phi_2^\rho$. Since $\rho \vdash \phi_2^\rho$ (fact A, Section 1), it follows that $\beta' \nVdash \rho$. Hence $\beta' \in K^\rho$ and $\beta \leq^\rho \beta'$. Obviously $\phi_1, \phi_2, \rho \in L_{\beta'}$, so we can apply the induction hypothesis to obtain $\beta' \Vdash^\rho \phi_1$ and $\beta' \nVdash^\rho \phi_2$, whence it follows that $\beta' \nVdash^\rho \phi_1 \to \phi_2$.

Case $\phi = \forall x \phi_1$. ($\Rightarrow$:) Assume $\beta \nVdash^\rho \forall x \phi_1(x)$ (writing '$\phi_1(x)$' for 'ϕ_1'). Then, for some β' such that $\beta \leq^\rho \beta'$, and for some $c \in D^\rho_{\beta'}$, $\beta' \nVdash^\rho \phi_1(\bar{c})$. Obviously, $\beta \leq \beta'$ and $L_\beta \subseteq L_{\beta'}$, so $\forall x \phi_1, \rho \in L_{\beta'}$. Moreover $D^\rho_{\beta'} = D_{\beta'}$, so $c \in D_{\beta'}$ and $\phi_1(\bar{c}) \in L_{\beta'}$. By the induction hypothesis $\beta' \nVdash (\phi_1(\bar{c}))^\rho$. Since ρ is a sentence, $(\phi_1(\bar{c}))^\rho = (\phi_1^\rho)\,[^{\bar{c}}/x]$. It follows that $\beta \nVdash \forall x(\phi_1^\rho)$, i.e., $\beta \nVdash (\forall x \phi_1)^\rho$.

($\Leftarrow$:) Assume $\beta \nVdash (\forall x \phi_1)^\rho$, i.e., $\beta \nVdash \forall x(\phi_1^\rho)$. Then, for some β' such that $\beta \leq \beta'$, and for some $c \in D_{\beta'}$, $\beta' \nVdash (\phi_1^\rho)[^{\bar{c}}/x]$, i.e., $\beta' \nVdash (\phi_1(\bar{c}))^\rho$. Since $\rho \vdash (\phi_1(c))^\rho$ (fact A), it follows that $\beta' \nVdash \rho$. Hence $\beta' \in K^\rho$ and $\beta \leq^\rho \beta'$. Obviously, $(\phi_1(\bar{c}))$, $\rho \in L_{\beta'}$, so we can apply the induction hypothesis to obtain $\beta' \nVdash^\rho \phi_1(\bar{c})$. Since $c \in D_{\beta'}(=D^\rho_{\beta'})$, it follows that $\beta \nVdash^\rho \forall x \phi_1$.

Second Pruning Lemma *Let L be the language of arithmetic. If $\beta \in K^\rho$ and $\rho \in L_\beta$ and $\beta \Vdash$ **HA** then $\beta \Vdash^\rho$ **HA**.*

Proof: Assume $\beta \in K^\rho$, $\rho \in L_\beta$, $\beta \Vdash$ **HA**. Let ϕ be any theorem of **HA**. Since **HA** $\vdash \phi^\rho$ (fact C), it follows that $\beta \Vdash \phi^\rho$. According to the first pruning lemma and $\phi \in L \subseteq L_\beta$, $\beta \Vdash^\rho \phi$. Hence $\beta \Vdash^\rho$ **HA**.

3 Spotting Peano models From now on we shall assume that L is (any suitable variant or extension of) the language of arithmetic.

Theorem 1 *The local models in finite Kripke models of Heyting arithmetic are Peano models.*

Proof: Let **K** be a finite Kripke model, $\alpha \in K$, $\alpha \Vdash$ **HA**. Avoiding α, we shall apply several prunings to **K**. Construct a sequence of models $\mathbf{K}^{(0)}, \ldots, \mathbf{K}^{(n)}$ as follows. Let $\mathbf{K}^{(0)}$ be **K**. Let $K^{(i)}$ be given and assume $\alpha \in \mathbf{K}^{(i)}$. If there is a sentence $\rho \in L^{(i)}_\alpha$ such that $\alpha \nVdash^{(i)} \rho$ whereas some $\beta \in K^{(i)}$ can be found such that $\beta \Vdash^{(i)} \rho$, take any such ρ and let $\mathbf{K}^{(i+1)}$ be the model obtained by pruning ρ-nodes from $\mathbf{K}^{(i)}$. Otherwise, if there is no such ρ, the construction will halt. Let n be the stage where the process halts.

Claim α is a classical node in $\mathbf{K}^{(n)}$. For, let ρ be any sentence $\forall x_1 \ldots \forall x_n(\phi \vee \neg\phi) \in L^{(n)}_\alpha$. Let β be some final node such that $\alpha \leq \beta$. β is classical (fact 2, Section 1) and $L^{(n)}_\alpha \subseteq L^{(n)}_\beta$. Hence $\beta \Vdash^{(n)} \rho$, and by definition of n $\alpha \Vdash^{(n)} \rho$. Further, it follows from $\alpha \Vdash$ **HA**, by the second pruning lemma, that $\alpha \Vdash^{(i)}$ **HA** (for all $1 \leq i \leq n$). Hence $\mathbf{M}^{(n)}_\alpha$ will be a Peano model (fact 4). But $\mathbf{M}^{(n)}_\alpha = \mathbf{M}_\alpha$.

Corollary *Let α be a node in a Kripke model* **K** *such that $\alpha \Vdash$* **HA**. Let K^α *be finite. Then $\mathbf{M}_\alpha$ is a Peano model.*

There seem to be no straightforward extensions of this result to infinite Kripke models. However, if the underlying structure is of type ω, we have:

Theorem 2 *A Kripke model of* **HA** *over ω (with its natural order) contains infinitely many local Peano models.*

Proof: Let $\mathbf{K} = \langle \omega, \leq, D, I \rangle$ be a Kripke model of **HA** (i.e., for each $n \in \omega$, $n \Vdash$ **HA**), where $\leq$ is the natural ordering on ω.

Case 1. Let **K** contain a classical node n. Then all $m \geq n$ will be classical as well (fact 3, Section 1). For each such m, since $m \Vdash \mathbf{HA}$, $\mathbf{M}_n$ will be a Peano model (fact 4).

Case 2. Let **K** contain no classical nodes. Consider the set $A = \{n | n \in \omega$ and for all $\phi \in L_n$: if $n + 1 \Vdash \phi$ then $n \Vdash \phi\}$. We shall first show that

(i) $\omega \sim A$ is infinite.

Suppose $\omega \sim A$ were finite. Let n be such that, for all $m \geq n$, $m \in A$. Since n is not classical, there is a sentence $\forall x_1 \dots \forall x_r(\phi \vee \neg\phi) \in L_n$ such that $n \nVdash \forall x_1 \dots \forall x_r(\phi \vee \neg\phi)$. Hence, for some $m \geq n$ and for certain $c_1, \dots, c_r \in D_m$, $m \nVdash \phi(\bar{c}_1 \dots \bar{c}_r) \vee \neg\phi(\bar{c}_1 \dots \bar{c}_r)$. Let $\phi' = \phi(\bar{c}_1, \dots, \bar{c}_r)$. Then $m \nVdash \phi' \vee \neg\phi'$, so $m \nVdash \phi'$ and $m \nVdash \neg\phi'$. Hence, for some $k > m$, $k \Vdash \phi'$. Let k^* be minimal with the property: $k^* > m$, $k^* \Vdash \phi'$. Then $k^* - 1 \nVdash \phi'$ and $k^* - 1 \geq m \geq n$. Since $\phi' \in L_{k^*-1}$, it follows that $k^* - 1 \in \omega \sim A$, contradicting the choice of n. Therefore (i) holds.

Let $\mathbf{K}^-(= \langle K^-, \leq^-, D^-, I^- \rangle)$ be the model obtained from **K** by deleting all nodes in A. Forcing in $\mathbf{K}^-$ will be denoted by $\Vdash^-$. It can be shown, by a simultaneous induction on ϕ for all $n \in K^-$, that the following holds:

(ii) For all $n \in K^-$, $\phi \in L_n$, $n \Vdash \phi$ iff $n \Vdash^- \phi$.

We consider the case of the implication.

$\phi = \phi_1 \rightarrow \phi_2$. ('$\Rightarrow$':) Assume $n \nVdash^- \phi_1 \rightarrow \phi_2$. Then, for some m such that $n \leq^- m$, $m \Vdash \phi_1$ and $m \nVdash^- \phi_2$. Obviously $n \leq m$ and $\phi_1, \phi_2 \in L_m$. According to the induction hypothesis $m \Vdash \phi_1$ and $m \nVdash \phi_2$. Hence $n \nVdash \phi_1 \rightarrow \phi_2$.

('$\Leftarrow$':) Assume $n \nVdash \phi_1 \rightarrow \phi_2$. Then, for some m, such that $n \leq m$, $m \Vdash \phi_1$ and $m \nVdash \phi_2$. Suppose first that $m \in K^-$. Since $\phi_1 \phi_2 \in L_m$, it follows by the induction hypothesis that $m \Vdash^- \phi_1$ and $m \nVdash^- \phi_2$. Obviously $n \leq^- m$, so $n \nVdash^- \phi_1 \rightarrow \phi_2$. Now suppose that $m \notin K^-$. Since (i) holds there is a $k > m$ such that $k \in K^-$. Let k^* be minimal with that property: $k^* > m$, $k^* \in K^-$. Then, for all k such that $m \leq k < k^*$, $k \in A$ and also $\phi_2 \in L_k$. By definition of A the following holds: if $k \nVdash \phi_2$ then $k + 1 \nVdash \phi_2$. Hence, since $m \nVdash \phi_2$, $k^* \nVdash \phi_2$. On the other hand $k^* \Vdash \phi_1$ (cumulation). Since $\phi_1, \phi_2 \in L_{k^*}$, it follows by the induction hypothesis that $k^* \Vdash^- \phi_1$ and $k^* \nVdash^- \phi_2$. Since obviously $n \leq^- k^*$ we may conclude that $n \nVdash^- \phi_1 \rightarrow \phi_2$. The case $\phi = \forall x \phi_1$ can be treated similarly, whereas the other cases are even simpler. So (ii) holds.

An immediate consequence of (ii) is that for each node $n \in K^-$ $n \Vdash^-$ **HA.** We shall now show that $\mathbf{M}_n$ is a Peano model. Since $n \notin A$, there is a sentence $\rho \in L_n^- (=L_n)$ such that $n \nVdash \rho$ and $n + 1 \Vdash \rho$. According to (ii) $n \nVdash \rho$, hence the model $\mathbf{K}^{-\rho}$ exists and contains n. By the second pruning lemma it follows that $n \Vdash^{-\rho}$ **HA.** Moreover n is a final node of $\mathbf{K}^{-\rho}$. For if $n <^- m$, it follows that $n + 1 \leq m$, therefore $m \Vdash \rho$ (cumulation) and by (ii) $m \Vdash^- \rho$. Hence m will be pruned away. Since n is final it is classical in $\mathbf{K}^{-\rho}$ (fact 2, Section 1) and so $\mathbf{M}_n^{-\rho}$ is a Peano model (fact 4). But $\mathbf{M}_n = \mathbf{M}_n^{-\rho}$, hence each of the infinitely many $\mathbf{M}_n$ such that $n \in K^-$ is a Peano model.

4 Other applications of pruning Friedman's proof of Markov's rule (**MR**) (cf. Friedman, [2]) has a model theoretic version.

MR *Let* $\phi \in \Sigma_1^0$. *Then* **HA** $\vdash \forall x_1 \ldots \forall x_n \neg\neg\phi \to$ **HA** $\vdash \forall x_1 \ldots \forall x_n \phi$.

Proof: Assume $\phi_0 \in \Sigma_1^0$, **HA** $\vdash \forall x_1 \ldots \forall x_n \neg\neg\phi_0$, but **HA** $\nvdash \forall x_1 \ldots \forall x_n \phi_0$. By the completeness theorem there is a Kripke model **K** of **HA** with a node α such that $\alpha \not\Vdash \forall x_1 \ldots \forall x_n \phi_0$. Therefore, **K** contains a node β such that, for certain $c_1, \ldots, c_n \in D_\beta$, $\beta \not\Vdash \phi_0(\bar{c}_1, \ldots, \bar{c}_n)$. Put $\phi = \phi_0(\bar{c}_1, \ldots, \bar{c}_n)$, then $\phi \in L_\beta$ and $\beta \not\Vdash \phi$. Hence $\mathbf{K}^\phi$ exists and $\beta \in K^\phi$. According to the second pruning lemma, $\beta \Vdash^\phi$ **HA**, so $\beta \Vdash^\phi \neg\neg\phi$. Consequently $\beta \not\Vdash^\phi \neg\phi$ and there is some $\gamma \in K^\phi$ such that $\gamma \Vdash^\phi \phi$. By the first pruning lemma $\gamma \Vdash \phi^\phi$. Since $\phi \in \Sigma_1^0$, ϕ^ϕ is equivalent to $\phi \vee \phi$ in **HA** (fact D, Section 1). Since $\gamma \Vdash$ **HA**, $\gamma \Vdash \phi \vee \phi$. Therefore $\gamma \Vdash \phi$. This means that γ must have been pruned away, contradicting $\gamma \in K^\phi$.

In the same way we can formulate a model-theoretic version of Visser's proof of the following (cf. [4]):

VR *Let* $\phi \in \Sigma_1^0$. *Then* **HA** $\vdash \forall x_1 \ldots \forall x_n(\neg\neg\phi \to \phi)$ *implies* **HA** $\vdash \forall x_1 \ldots \forall x_n(\phi \vee \neg\phi)$.

Proof: Assume $\phi_0 \in \Sigma_1^0$, **HA** $\vdash \forall x_1 \ldots \forall x_n(\neg\neg\phi_0 \to \phi_0)$, but **HA** $\nvdash \forall x_1 \ldots \forall x_n(\phi_0 \vee \neg\phi_0)$. By the completeness theorem there is a Kripke model **K** of **HA** with a node α such that $\alpha \not\Vdash \forall x_1 \ldots \forall x_n(\phi_0 \vee \neg\phi_0)$. Therefore, **K** contains a node β such that for certain $c_1, \ldots, c_n \in D_\beta$, $\beta \not\Vdash \phi \vee \neg\phi$, where $\phi = \phi_0(\bar{c}_1, \ldots, \bar{c}_n)$. Certainly, $\neg\phi \in L_\beta$ and $\beta \not\Vdash \neg\phi$, hence $\mathbf{K}^{\neg\phi}$ exists and $\beta \in \mathbf{K}^{\neg\phi}$. According to the second pruning lemma $\beta \Vdash^{\neg\phi}$ **HA**, so $\beta \Vdash^{\neg\phi} \neg\neg\phi \to \phi$. Consider any $\gamma \in \mathbf{K}^{\neg\phi}$ such that $\beta \leq^{\neg\phi} \gamma$. For such γ: $\gamma \not\Vdash \neg\phi$, whereas $\neg\phi \in L_\gamma$, therefore there is some γ' such that $\gamma \leq \gamma'$ and $\gamma' \Vdash \phi$. Since $\gamma' \not\Vdash \neg\phi$ it follows that $\gamma' \in K^{\neg\phi}$ and $\gamma \leq^{\neg\phi} \gamma'$. Obviously, $\gamma' \Vdash \phi \vee \neg\phi$. Since $\phi \in \Sigma_1^0$, $\phi \vee \neg\phi$ is equivalent to $\phi^{\neg\phi}$ in **HA** (fact D). By the first pruning lemma $\gamma' \Vdash^{\neg\phi} \phi$. Therefore $\gamma \not\Vdash^{\neg\phi} \neg\phi$. Since this holds for any γ such that $\beta \leq^{\neg\phi} \gamma$ we can conclude that $\beta \Vdash^{\neg\phi} \neg\neg\phi$ and therefore $\beta \Vdash^{\neg\phi} \phi$. Applying the first pruning lemma once more we get $\beta \Vdash \phi^{\neg\phi}$, and, again by fact D, $\beta \Vdash \phi \vee \neg\phi$, a contradiction.

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Centrale Interfaculteit Filosofisch Instituut
Rijksuniversiteit Utrecht
Heidelberglaan 2
Postbus 80.103
3508 TC Utrecht
Netherlands