

On Purely Relevant Logics

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1 Introduction The system RMI_{\supset} (which consists of the implication-negation axioms of RM) was investigated in [3] and shown there to be an optimal relevance logic in its language. We note there, however, that one cannot add to it an R -style extensional conjunction \wedge , with $A \wedge B \rightarrow A$, $A \wedge B \rightarrow B$ as axioms and the adjunction rule of inference ($A, B \vdash A \wedge B$), without losing its relevance character (see [1], 29.5, and [3], III.8).

This state of affairs is not altogether surprising. Anderson and Belnap faced a similar problem when they came to add to R_{\supset} (or E_{\supset}) extensional connectives. In R_{\supset} , e.g., the meaning of \rightarrow is given by the "relevant deduction theorem", according to which a sentence of the form $A_1 \rightarrow (A_2 \rightarrow \dots \rightarrow (A_n \rightarrow B) \dots)$ is provable in R_{\supset} iff there is a proof in R_{\supset} of B from the assumptions A_1, \dots, A_n which *uses* all the A_i 's. (Here the meaning of "proof" is the usual one, while the meaning of "use" is to be understood according to the relevantist's analysis of this term (see [1], Chapter 1).) Accordingly, if one wishes to add to R_{\supset} an extensional conjunction such that $A \wedge B \vdash A$, $A \wedge B \vdash B$ and $A, B \vdash A \wedge B$ are all valid modes of inference, then he must recognize $A \wedge B \rightarrow A$, $A \wedge B \rightarrow B$ and $A \rightarrow (B \rightarrow A \wedge B)$ as valid sentences. However, it is well known that by adding these schemes to R_{\supset} we get classical logic.

Anderson and Belnap's first step in order to solve this difficulty was to give up $A \rightarrow (B \rightarrow A \wedge B)$ as a valid sentence and to introduce instead adjunction as a new, primitive rule of inference (besides M.P. for \rightarrow). A second, unavoidable step was to propose some new concepts of "proof" relative to which some version of the deduction theorem does hold. (In [1] and [5] three competing definitions can be found of what a "proof" in R or E is.¹ This is an obvious evidence that the relevantists have no clear intuition at this point.) These concepts of proofs all seem ad hoc and entail many absurdities. Consider an example: $A \wedge (B \rightarrow B)$ can be inferred, according to them, if we assume both A and $B \rightarrow B$ but not if we assume A alone, although $B \rightarrow B$ is a logical truth of the system and so it would be ridiculous to pretend assuming it.

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Our opinion is that the real source of these difficulties is the relevantist's unfortunate attempt to express pure *extensional* inferences (like that of A from $A \wedge B$) using *relevant* implication. Their failure to do so in the case of adjunction and the total impossibility of this idea² in the case of RMI_{\supset} are only indications of something more fundamental: when \rightarrow is relevant $A \wedge B \rightarrow A$ (in contrast to $A \wedge B \vdash A$) should be taken as valid only if the conjunction connective \wedge occurring in it has a relevant character too. In particular $A \wedge B \rightarrow B$ is to be valid only if a necessary condition for the truth of $A \wedge B$ is that A and B are relevant to each other.

In this paper we follow this line of thought by adding to R_{\supset} and RMI_{\supset} a *relevant* conjunction, instead of the meant-to-be extensional one of R and RM . Nevertheless, we shall try to keep as close as possible to the original systems and ideas of Anderson and Belnap (leaving a more radical approach for some other time). The most straightforward way to achieve this is to adjoin to R_{\supset} and RMI_{\supset} the rules and axioms of R_{fde} (the "first-degree-entailments" fragment of R and E^3) without any further modifications. In this way we obtain two systems, PR and PRM respectively, which might be called "purely relevant" since they possess the variable-sharing property with respect to both \rightarrow and \wedge . We show that R and RM can be obtained from PR and PRM by adding to them *axioms*, the intuitive meaning of which is that any two sentences are relevant to each other. We show further that the adjunction rule and the disjunction syllogism have quite a parallel role in the context of PR , PRM , R , and RM .⁴

For reasons which are discussed in [3], RMI_{\supset} seems to us preferable to R_{\supset} in the context of pure relevance logic. (For example, it has an appropriate idempotent relevant disjunction, which R_{\supset} has not.) Therefore, we devote our attention to PRM in the rest of this paper. We show, among other things, that this system has an infinite characteristic matrix, which resembles Sugihara matrix but has two "zeroes" instead of the single one of Sugihara. This limitation to precisely two zeroes seems nonintuitive to us, and indeed PRM contains some unpleasant theorems like $R(A, B) \vee R(A, C) \vee R(B, C)$, where $R(A, B)$ is a sentence of the language which intuitively means that A and B are relevant to each other. By weakening a little bit the distribution axiom of PRM (which was inherited from R), we get another system, PRM^* , which has a more satisfactory semantics and in which sentences of the above sort are not provable. However, PRM^* still contains nonintuitive theorems like $R(A, B) \vee R(A \wedge B, C)$ and so still leaves something to be desired.⁵ In order to get a really appropriate pure relevance logic we need a more radical approach to distribution and so a further departure from Anderson and Belnap systems is called for. This will be the subject of another paper.⁶

2 Preliminaries: Logical systems

The system R

Axioms

- R1** $A \rightarrow A$
R2 $(A \rightarrow B) \rightarrow. (B \rightarrow C) \rightarrow (A \rightarrow C)$
R3 $A \rightarrow (B \rightarrow C) \rightarrow. B \rightarrow (A \rightarrow C)$
R4 $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

- R5** $A \wedge B \rightarrow A$
R6 $A \wedge B \rightarrow B$
R7 $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow . A \rightarrow (B \wedge C)$
R8 $A \rightarrow A \vee B$
R9 $B \rightarrow A \vee B$
R10 $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow . (A \vee B) \rightarrow C$
R11 $(A \vee B) \wedge (A \vee C) \rightarrow . A \vee (B \wedge C)$
R12 $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$
R13 $\sim \sim A \rightarrow A.$

Rules of inference $\frac{A \quad A \rightarrow B}{B}$ (M.P.) $\frac{A \quad B}{A \wedge B}$ (Adj.)

R_{\rightarrow} and R_{\sim} are the $\{\rightarrow\}$ and the $\{\sim, \rightarrow\}$ fragments, respectively, of R .
 RM is the system obtained from R by replacing R1 by RM1: $A \rightarrow (A \rightarrow A)$.

RM_{\sim} is the system in the $\{\sim, \rightarrow\}$ language having RM1, R2-R6 as axioms, and M.P. as the rule of inference. RM_{\rightarrow} is the implicational fragment of RM_{\sim} .

A thorough investigation of both systems, including a characteristic matrix and Gentzen-type calculi, is given in [3].

RM_{\sim} is the implication-negation fragment of RM . It is a proper extension of RM_{\sim} , and is usually called Soboćinski three-valued logic (see [8], [7], and [1], pp. 148-149). RM_{\sim} (as well as RM_{\rightarrow}) has only the classical logic as a proper extension (see [3]).

RM_{\rightarrow} is the implicational fragment of RM (see [6] and [3] for two different formulations).

Other connectives

$$A + B =_{df} (\sim A) \rightarrow B$$

$$A \circ B = \sim(\sim A + \sim B).$$

The importance of \circ is due to the fact that $A \rightarrow (B \rightarrow C)$ and $(A \circ B) \rightarrow C$ are equivalent in R_{\sim} .

The system R_{fde} (first-degree fragment of R and E)

Axioms

- (1) $A \wedge B \rightarrow A$
- (2) $A \wedge B \rightarrow B$
- (3) $A \rightarrow A \vee B$
- (4) $B \rightarrow A \vee B$
- (5) $A \rightarrow \sim \sim A$
- (6) $\sim \sim A \rightarrow A$
- (7) $(A \vee B) \wedge (A \vee C) \rightarrow A \vee (B \wedge C).$

In (1)-(7) A, B, C do not contain \rightarrow .

Rules of inference:

$$(I1) \frac{A \rightarrow B, B \rightarrow C}{A \rightarrow C}$$

$$(I2) \frac{A \rightarrow B}{\sim B \rightarrow \sim A}$$

$$(I3) \frac{A \rightarrow B \quad A \rightarrow C}{A \rightarrow . B \wedge C}$$

$$(I4) \frac{A \rightarrow C \quad B \rightarrow C}{(A \vee B) \rightarrow C}.$$

All theorems of R_{fde} are first-degree-entailments, i.e., sentences of the form $A \rightarrow B$ where A and B do not contain \rightarrow .

An ordinary Sugihara matrix is a structure $\langle S, \leq, \sim, \rightarrow \rangle$ in which $\langle S, \leq \rangle$ is a linearly ordered set. \sim is a unary operation satisfying De-Morgan conditions ($\sim \sim a = a$, $a \leq b \Rightarrow \sim b \leq \sim a$), \rightarrow is a binary operation defined by:

$$a \rightarrow b: \begin{cases} \sim a \vee b & a \leq b \\ \sim a \wedge b & \text{otherwise,} \end{cases}$$

we call $a \in S$ designated iff $\sim a \leq a$.⁷ An ordinary Sugihara matrix S is called normal if $a \neq \sim a$ for $a \in S$. By *NS*, the normal Sugihara matrix, we shall mean here the integers without 0 (with the usual \leq and \sim), and by the abnormal Sugihara matrix we similarly mean the integers. We note the Meyer has shown that the normal Sugihara matrix *NS* as well as the abnormal one are characteristic for *RM* (see [1]).

I. Definition 1

(a) *PR* is the system obtained from *R* by replacing (adj) by the following relevant adjunction rule:

$$\frac{A \rightarrow B \quad A \rightarrow C}{A \rightarrow . B \wedge C} \quad (\text{re. adj})$$

(b) *PRM* is the system obtained from *RM* by the same replacement.

Theorem 1

(i) *PR* is equivalent to $R_{\supset} \cup R_{fde}$ (i.e., the system in the language of *R* which has as axioms and rules of inference exactly those of $R_{\supset} \cup R_{fde}$, all extended to the language of *R*).

(ii) *PRM* is equivalent to $RMI_{\supset} \cup R_{fde}$.

Proof: (i) It is not hard to see that $R_{\supset} \cup R_{fde} \subseteq PR$. We show here, as an example, that *PR* is closed under rule I4 of R_{fde} . We need first a lemma:

Lemma 1

(a) If $\vdash_{PR} A \rightarrow B$ and $\vdash_{PR} A \rightarrow C$ then $\vdash_{PR} (A \rightarrow B) \wedge (A \rightarrow C)$

(b) If $\vdash_{PR} C \rightarrow A$ and $\vdash_{PR} B \rightarrow A$ then $\vdash_{PR} (C \rightarrow A) \wedge (B \rightarrow A)$.

That *PR* is closed under I4 is an immediate consequence of part (b) of the lemma and Axiom R11.

Proof: (a) If $\vdash_{PR(M)} A \rightarrow B$ and $\vdash_{PR(M)} A \rightarrow C$, then $\vdash_{PR(M)} A \rightarrow (B \wedge C)$ by re. adj. Since $\vdash_{PR(M)} B \wedge C \rightarrow B$, also $A \rightarrow (B \wedge C) \rightarrow . A \rightarrow B$ is provable, and similarly $A \rightarrow (B \wedge C) \rightarrow . A \rightarrow C$ is. Using re. adj. once more, followed by M.P., we see that $(A \rightarrow B) \wedge (A \rightarrow C)$ is provable.

(b) Suppose $\frac{}{PR(M)} C \rightarrow A$ and $\frac{}{PR} B \rightarrow A$. Then $\frac{}{PR(M)} \sim A \rightarrow \sim C$ and $\frac{}{PR(M)} \sim A \rightarrow \sim B$. By part (a) $\frac{}{PR(M)} (\sim A \rightarrow \sim B) \wedge (\sim A \rightarrow \sim C)$. Since $\frac{}{R_{\supset}} (\sim A \rightarrow \sim B) \rightarrow (B \rightarrow A)$, we get easily that $\frac{}{PR(M)} (B \rightarrow A) \wedge (C \rightarrow A)$.

Returning now to the proof of Theorem 1, we note first that all the systems we deal with in this paper are easily shown to be closed under substitutions of equivalents. As a consequence we can replace everywhere $A \rightarrow (B \rightarrow C)$ by $A \circ B \rightarrow C$, $A \vee B$ by $\sim(\sim A \wedge \sim B)$, $A \rightarrow C$ by $\sim C \rightarrow \sim A$ and vice versa. Now, in order to show that $PR \subseteq R_{\supset} \cup R_{fde}$ we only have to show that Axioms R7 and R10 are deducible in $R_{\supset} \cup R_{fde}$, and by the above remarks it is clear that it suffices to deduce R7 (say). Since $\frac{}{R_{\supset} \cup R_{fde}} (A \rightarrow B) \wedge (A \rightarrow C) \rightarrow A \rightarrow B$ we have that $\frac{}{R_{\supset} \cup R_{fde}} [((A \rightarrow B) \wedge (A \rightarrow C)) \circ A] \rightarrow B$. Similarly $\frac{}{R_{\supset} \cup R_{fde}} [((A \rightarrow B) \wedge (A \rightarrow C)) \circ A] \rightarrow C$. From these theorems R7 follows easily using re. adj. and the equivalence of $A \rightarrow (B \rightarrow C)$ and $A \circ B \rightarrow C$.

The proof of (ii) is similar.

The next definition introduces a matrix that will be shown later to be characteristic for PRM :

Definition 2 AS , the *augmented Sugihara* matrix, is the matrix differing from the ordinary Sugihara matrix S by having two “zeroes”, 0_1 and 0_2 , for which $0_1 \vee 0_2 = 1$, $0_1 \wedge 0_2 = -1$, $\sim 0_1 = 0_1$, $\sim 0_2 = 0_2$. (As a consequence $0_1 \rightarrow 0_1 = 0_1$, $0_2 \rightarrow 0_2 = 0_2$, $0_1 \rightarrow 0_2 = 0_2 \rightarrow 0_1 = -1$.)

Note: In any other respect AS behaves exactly like S (e.g., $a \rightarrow b = \sim a \vee b$ if $a \leq b$, $a \rightarrow b = \sim a \wedge b$ otherwise). Hence, in order to show that a particular formula, known to be valid in S (i.e., a theorem of RM), is also valid in AS it is enough to check assignments involving both 0_1 and 0_2 . Using this fact the proof of the following is straightforward:

Theorem 2

Every theorem of PRM is valid in AS .

As an immediate corollary we have:

Theorem 3 PR and PRM have the *sharing-of-variable property* for both \rightarrow and \wedge , i.e., $A \rightarrow B$ and $A \wedge B$ are provable only if A and B share a variable.

Proof: Suppose A and B do not share a variable. Let v be a valuation such that $v(P) = 0_1$ for every P occurring in A , $v(P) = 0_2$ for every P occurring in B . Then $v(A) = 0_1$, $v(B) = 0_2$, $v(A \wedge B) = v(A \rightarrow B) = -1$. Hence, by Theorem 2 $\frac{}{PRM} A \rightarrow B$, $\frac{}{PRM} A \wedge B$.

We introduce now two kinds of sentences, both expressing formally that A and B are relevant to each other:

Definition 3

- (a) $R^{\wedge}(A, B) \equiv_{df} (A \rightarrow A) \wedge (B \rightarrow B)$
 (b) $R^{+}(A, B) \equiv_{df} (A \rightarrow A) + (B \rightarrow B)$.

The next theorem contains the most important proof-theoretical properties of PRM :

Theorem 4(a) $A, B, R^\wedge(A, B) \vdash_{\overline{PR}} A \wedge B$ (b) $\vdash_{\overline{PRM}} (A \wedge B) \rightarrow (A + B)$ (c) $R^+(A, B), (A \rightarrow B) + (B \rightarrow A)$ and $(A \rightarrow B) \vee (B \rightarrow A)$ are equivalent in \overline{PRM} (d) For any particular A and B , $R^+(A, B)$ and $R^\wedge(A, B)$ are interdeducible in \overline{PRM} :

$$R^+(A, B) \vdash_{\overline{PRM}} R^\wedge(A, B) \quad R^\wedge(A, B) \vdash_{\overline{PRM}} R^+(A, B)$$

(e) $A \rightarrow B \vdash_{\overline{PRM}} R^\wedge(A, B)$ (f) (i) $\vdash_{\overline{PRM}} R^\wedge(A, A)$; (ii) $\vdash_{\overline{PRM}} R^\wedge(A, B) \rightarrow R^\wedge(B, A)$ (g) $R^\wedge(A, B) \vdash_{\overline{PRM}} R^\wedge(A, \sim B)$ (h) (i) $R^\wedge(A, B) \vdash_{\overline{PRM}} R^\wedge(A, B \rightarrow C)$; (ii) $R^\wedge(A, B) \vdash_{\overline{PRM}} R^\wedge(A, C \rightarrow B)$ (i) (i) $R^\wedge(A, B) \vdash_{\overline{PRM}} R^\wedge(A \wedge C, B \wedge C)$; (ii) $R^\wedge(A, B) \vdash_{\overline{PRM}} R^\wedge(A \vee C, B \vee C)$ (j) If $T, A \vdash_{\overline{PRM}} C$ and $T, B \vdash_{\overline{PRM}} C$ then $T, A \vee B \vdash_{\overline{PRM}} C$ (k) $B \vdash_{\overline{PRM}} A \vee (A \rightarrow B)$.

Proof: (a) Since $\vdash_{\overline{R^-}} A \rightarrow ((A \rightarrow A) \rightarrow A)$, we have that $A \vdash_{\overline{PRM}} (A \rightarrow A) \rightarrow A$. Hence, by R^\wedge definition, $A \vdash_{\overline{PR}} R^\wedge(A, B) \rightarrow A$. Similarly, $B \vdash_{\overline{PR}} R^\wedge(A, B) \rightarrow B$. By re. adj. $A, B \vdash_{\overline{PR}} R^\wedge(A, B) \rightarrow A \wedge B$ and (a) follows.

(b) $\vdash_{\overline{RMI^-}} (C \rightarrow A) \rightarrow ((C \rightarrow B) \rightarrow .C \rightarrow .A + B)$.⁸ Substituting $A \wedge B$ for C we obtain (b).

(c) $\vdash_{\overline{RMI^-}} R^+(A, B) \Leftrightarrow ((A \rightarrow B) + (B \rightarrow A))$. Also, since $\vdash_{\overline{R^-}} (A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A + B) \rightarrow C)$ we have, by taking C to be $A \vee B$, $\vdash_{\overline{PR}} A + B \rightarrow A \vee B$. Hence $\vdash_{\overline{PR}} ((A \rightarrow B) + (B \rightarrow A)) \rightarrow (A \rightarrow B) \vee (B \rightarrow A)$. Finally, since $\vdash_{\overline{RMI^-}} (A \rightarrow B) \rightarrow R^+(A, B)$, $\vdash_{\overline{RMI^-}} (B \rightarrow A) \rightarrow R^+(A, B)$, also $\vdash_{\overline{PRM}} (A \rightarrow B) \vee (B \rightarrow A) \rightarrow R^+(A, B)$.

(d) That $R^\wedge(A, B) \vdash_{\overline{PR}} R^+(A, B)$ follows immediately from (b). For the other direction we have $\vdash_{\overline{RMI^-}} \sim(A \rightarrow A) \rightarrow (A \rightarrow A)$ and $R^+(A, B) \vdash_{\overline{RMI}} \sim(A \rightarrow A) \rightarrow (B \rightarrow B)$. Hence, by re. adj. we deduce that $R^+(A, B) \vdash_{\overline{PRM}} \sim(A \rightarrow A) \rightarrow R^\wedge(A, B)$ and so $R^+(A, B) \vdash_{\overline{PRM}} \sim R^\wedge(A, B) \rightarrow (A \rightarrow A)$. Similarly $R^+(A, B) \vdash_{\overline{PRM}} \sim R^\wedge(A, B) \rightarrow (B \rightarrow B)$. Applying re. adj. we get: $R^+(A, B) \vdash_{\overline{PRM}} \sim R^\wedge(A, B) \rightarrow R^\wedge(A, B)$. But $\vdash_{\overline{R^-}} (\sim C \rightarrow C) \rightarrow C$, so $R^+(A, B) \vdash_{\overline{PRM}} R^\wedge(A, B)$.

(e) This is a direct consequence of (c) and (d).

(f) We leave the proof to the reader.

(g) Since $\vdash_{\overline{RMI^-}} R^+(A, B) \rightarrow R^+(A, \sim B)$, (g) is a consequence of (d).

(h) Similar to (g).

(i) We show, for example, that $R^\wedge(A, B) \vdash_{\overline{PRM}} R^\wedge(A \wedge C, B \wedge C)$. Now we have: $\vdash_{\overline{RMI^-}} R^+(A, B) \rightarrow (A \rightarrow (B \rightarrow (\sim B \rightarrow A)))$. Hence, by (d), $R^\wedge(A, B) \vdash_{\overline{PRM}} A \rightarrow (B \rightarrow (\sim B \rightarrow A))$ and so:

$$(1) R^\wedge(A, B) \vdash_{\overline{PRM}} (A \wedge C) \rightarrow ((B \wedge C) \rightarrow (\sim B \rightarrow A)).$$

Also, since $\vdash_{\overline{RMI^-}} (B + C) \rightarrow (C \rightarrow (\sim B \rightarrow C))$, we have that $\vdash_{\overline{PRM}} (B + C) \rightarrow (A \wedge C \rightarrow (\sim B \rightarrow C))$. Hence, by (b), $\vdash_{\overline{PRM}} (B \wedge C) \rightarrow ((A \wedge C) \rightarrow (\sim B \rightarrow C))$, or:

$$(2) \frac{}{\overline{PRM}}(A \wedge C) \rightarrow ((B \wedge C) \rightarrow (\sim B \rightarrow C)).$$

From (1) and (2) we infer $R^\wedge(A, B) \frac{}{\overline{PRM}}(A \wedge C) \rightarrow ((B \wedge C) \rightarrow (\sim B \rightarrow A \wedge C))$, so:

$$(3) R^\wedge(A, B) \frac{}{\overline{PRM}}(A \wedge C) \rightarrow ((B \wedge C) \rightarrow (\sim(A \wedge C) \rightarrow B)).$$

Using (b) and the fact that $\frac{}{\overline{RMI_\rightarrow}}(A + C) \rightarrow (C \rightarrow (\sim C \rightarrow A))$ we infer that:

$$(4) \frac{}{\overline{PRM}}(A \wedge C) \rightarrow ((B \wedge C) \rightarrow (\sim C \rightarrow A)).$$

Also, since $\frac{}{\overline{RMI_\rightarrow}}C \rightarrow (C \rightarrow (\sim C \rightarrow C))$, we have that

$$(5) \frac{}{\overline{PRM}}(A \wedge C) \rightarrow ((B \wedge C) \rightarrow (\sim C \rightarrow C)).$$

Applying re. adj. to (4) and (5) followed by a contraposition we get:

$$(6) \frac{}{\overline{PRM}}(A \wedge C) \rightarrow (B \wedge C) \rightarrow (\sim(A \wedge C) \rightarrow C).$$

(6) and (3) give, again by re. adj.: $R^\wedge(A, B) \frac{}{\overline{PRM}}(A \wedge C) \rightarrow (B \wedge C) \rightarrow (\sim(A \wedge C) \rightarrow (B \wedge C))$, i.e., $R^\wedge(A, B) \frac{}{\overline{PRM}}R^+(A \wedge C, B \wedge C)$ (since $\frac{}{\overline{RMI_\rightarrow}}R^+(A, B) \Leftrightarrow (A \rightarrow B \rightarrow (\sim A \rightarrow B))$). Now use (d).

(j) As a consequence of (a), (h) and (i) we have: $A \vee B, A \vee (B \rightarrow C) \frac{}{\overline{PRM}}(A \vee B) \wedge (A \vee (B \rightarrow C))$. Hence, using distribution,⁹ $A \vee B, A \vee (B \rightarrow C) \frac{}{\overline{PRM}}A \vee (B \wedge (B \rightarrow C))$. Similarly: $A \vee (B \rightarrow C), A \vee (B \rightarrow D) \frac{}{\overline{PRM}}A \vee ((B \rightarrow C) \wedge (B \rightarrow D))$.

Since it is easy to show that $\frac{}{\overline{PR}}B \wedge (B \rightarrow C) \rightarrow C$ and that $E \rightarrow F \frac{}{\overline{PRM}}(G \vee E) \rightarrow (G \vee F)$, we can conclude that

$$(1) A \vee B, A \vee (B \rightarrow C) \frac{}{\overline{PRM}}A \vee C$$

$$(2) A \vee (B \rightarrow C), A \vee (B \rightarrow D) \frac{}{\overline{PRM}}A \vee (B \rightarrow (C \wedge D)).$$

We can now use (1) and (2) to follow the proof in [1], pp. 301–302 and show that if $T, B \frac{}{\overline{PRM}}C$, then $T, A \vee B \frac{}{\overline{PRM}}A \vee C$ and, then, (j).

(k) Since $\frac{}{\overline{PRM}}A \rightarrow A \vee B$ we have, by (e), that $\frac{}{\overline{PRM}}R^\wedge(A, A \vee B)$ and so, by (g) and (f), $\frac{}{\overline{PRM}}R(\sim A, A \vee B)$. It follows now, by (a), that $\sim A, A \vee B \frac{}{\overline{PRM}}\sim A \wedge (A \vee B)$, and so, using distribution and $B \frac{}{\overline{PR}}A \vee B$, that $\sim A, B \frac{}{\overline{PRM}}(\sim A \wedge A) \vee (\sim A \wedge B)$. But $\frac{}{\overline{PR}}(\sim A \wedge A) \rightarrow A$ and $\frac{}{\overline{PRM}}(\sim A \wedge B) \rightarrow (A \rightarrow B)$ (by (b)), hence $\sim A, B \frac{}{\overline{PRM}}A \vee (A \rightarrow B)$. Obviously $A, B \frac{}{\overline{PR}}A \vee A \rightarrow B$, and so, by (j) $\sim A \vee A, B \frac{}{\overline{PRM}}A \vee (A \rightarrow B)$. But $\frac{}{\overline{PR}}A \vee \sim A$ (since $\frac{}{\overline{R_\rightarrow}}A + \sim A$ and $\frac{}{\overline{PR}}(A + \sim A) \rightarrow (A \vee \sim A)$). Hence (k).

We turn now to prove the converse of Theorem 2, i.e., completeness of *PRM* relative to *AS*. The proof is similar to that used by Dunn in [4] in order to show the completeness of *RM* relative to Sugihara matrix (first shown by Meyer). As a first step we generalize the semantics:

Definition 4 By an *augmented Sugihara matrix* we mean either an ordinary Sugihara matrix or any structure obtained from a *normal* Sugihara matrix M , having a minimal designated value (we denote by 1), by adjoining to M two “zeroes” 0_1 and 0_2 and defining: $0_1 \rightarrow 0_1 = \sim 0_1 = 0_1$, $0_2 \rightarrow 0_2 = \sim 0_2 = 0_2$, $0_1 \vee 0_2 = 1$, $0_1 \wedge 0_2 = -1$, $0_1 \rightarrow 0_2 = 0_2 \rightarrow 0_1 = -1$. The designated values of the augmented matrix are those of M , together with 0_1 and 0_2 .

Note: It is not difficult to see that, like in an ordinary Sugihara matrix, the designated values are those a 's satisfying $\sim a \leq a$ and that $a \rightarrow b = \sim a \vee b$ if $a \leq b$,

$\sim a \wedge b$ otherwise. (The order relation is determined, of course, by \wedge and \vee .) Further, any augmented Sugihara matrix is a distributive lattice in which \sim satisfies De-Morgan conditions ($\sim \sim a = a$, $a \leq b \Leftrightarrow \sim b \leq \sim a$) and the normal members of which form a chain.

Theorem 2* If $T \vdash_{PRM} A$, M is an augmented Sugihara matrix, and v is a valuation in M in which every member of T has a designated value, then $v(A)$ is designated too.

We leave to the reader the proof of the above theorem, as well as of the following lemma:

Lemma 2

(a) If M is an augmented Sugihara matrix, v a valuation in M , A a sentence all atomic variables of which are among $\{P_1, \dots, P_n\}$, then $v(A) \in \{0_1, 0_2, \pm 1, \pm v(P_1), \dots, \pm v(P_n)\}$.

(b) If, under the above conditions, $v(A)$ is not designated then A is not valid in AS (Definition 2).

Theorem 5 The completeness theorem

(a) If T is a PRM-theory, $T \not\vdash_{PRM} \phi$, then there is an augmented Sugihara matrix M and a valuation v in M such that $v(C)$ is designated for any $C \in T$, but $v(\phi)$ is not.

(b) $\vdash_{PRM} A$ iff A is valid in AS.

Proof: (b) follows immediately from (a), Theorem 2, and Lemma 2. To show (a), let T and ϕ be as above. Following [1] and [4], we call a theory “prime” if, whenever $T \vdash A \vee B$, then either $T \vdash A$ or $T \vdash B$. Using Theorem 4(j), we can extend T to a prime theory T_0 such that $T_0 \not\vdash_{PRM} \phi$. Let $M = \langle M, \leq, \sim, \vee, \wedge, \rightarrow, T_M \rangle$ be the Lindenbaum algebra of T , constructed in the usual manner,¹⁰ and denote by $[A]$ the equivalence class of a sentence A . It is easy to see that M is a distributive lattice in which \sim satisfies De-Morgan conditions. Also, since $T_0 \vdash_{PRM} A \vee \sim A$ and T_0 is prime, $a \in T_M$ or $\sim a \in T_M$ for any $a \in M$. Since $\vdash_{RMI_{\sim}} A \Leftrightarrow (\sim A \rightarrow A)$, $a \in T_M$ iff $\sim a \leq a$. Further, if $a \in T_M$ and $a \leq b$, then $b \in T_M$. As usual, if we define $v(A) = [A]$, we get a well-defined valuation for which exactly the theorems of T_0 are true. Hence all theorems of T are true for it and ϕ is not.

Call now $a \in M$ “normal” if $a \neq \sim a$. Since $\vdash_{RMI_{\sim}} A \rightarrow \sim A \rightarrow (A \Leftrightarrow \sim A)$, $\vdash_{RMI} (A \Leftrightarrow \sim A) \rightarrow A$, $\vdash_{RMI} (A \Leftrightarrow \sim A) \rightarrow \sim A$, a is abnormal iff both a and $\sim a$ are in T_M .

To end the proof we must show that M is (isomorphic to) an augmented Sugihara matrix. For this it suffices to show that: (a) The normal members of T_M form a chain under \leq . (b) If a and b are abnormal and different, then $a \not\leq b$, $b \not\leq a$. (c) If a is abnormal, b normal and designated, then $a \leq b$. (d) There are at most two abnormal members in M . (Note: if there are two, 0_1 and 0_2 , then (a)–(c) imply that $0_1 \vee 0_2$ is a first normal and designated member of M .)

$$(e) \quad a \rightarrow b = \begin{cases} \sim a \vee b & a \leq b \\ \sim a \wedge b & a \not\leq b. \end{cases}$$

A key result in the proof of (a) and (c) will be the following consequence of Theorem 4(k) and the primeness of T_0 :

(*) If $T_0 \not\vdash A$, $T_0 \vdash B$, then $T_0 \vdash A \rightarrow B$.

To prove (a) suppose $[A], [B] \in T_M$ and that both are normal. Then $T_0 \not\vdash \sim A$ and $T_0 \vdash B$. (*) implies, then, that $T_0 \vdash_{\overline{RM}} \sim A \rightarrow B$. But $\vdash_{\overline{RM}\supset} (\sim A \rightarrow B) \rightarrow R^+(A, B)$, so by Theorem 4(c) and the primeness of T_0 , either $[A] \leq [B]$ or $[B] \leq [A]$.

For (b), assume $[A] = [\sim A]$, $[B] = [\sim B]$ and $[A] \leq [B]$. Then $T_0 \vdash A \rightarrow B$ and so $T_0 \vdash \sim B \rightarrow \sim A$. But $[B] = [\sim B]$ and $[\sim A] = [A]$ and so $T_0 \vdash B \rightarrow A$ also. Hence $[A] = [B]$.

For (c) suppose $[A]$ is abnormal and $[B]$ is normal and designated. Then $T_0 \not\vdash \sim B$, $T_0 \vdash \sim A$. (*) entails, therefore, that $T_0 \vdash \sim B \rightarrow \sim A$, and so $T_0 \vdash A \rightarrow B$ and $[A] \leq [B]$.

(d) follows easily from (a)–(c) and the fact that $\langle M, \leq, \sim \rangle$ is a *distributive* De-Morgan lattice. For (e) we note first that since $\vdash_{\overline{PR}} (A + B) \rightarrow A \vee B$ and $\vdash_{\overline{PRM}} A \wedge B \rightarrow A + B$ (Theorem 4(b)), it is always the case that $\sim a \wedge b \leq a \rightarrow b \leq \sim a \vee b$ ($a, b \in M$). Suppose now that $[A] \leq [B]$, i.e., $T_0 \vdash A \rightarrow B$. Since $\vdash_{\overline{RM}\supset} (A \rightarrow B) \rightarrow (\sim A \rightarrow (A \rightarrow B))$ and $\vdash_{\overline{RM}\supset} (A \rightarrow B) \rightarrow (B \rightarrow (A \rightarrow B))$, the assumption $T_0 \vdash A \rightarrow B$ implies that $T_0 \vdash (\sim A \vee B) \rightarrow (A \rightarrow B)$ and so $[A] \rightarrow [B] = \sim[A] \vee [B]$ in this case.

Suppose finally that $[A] \not\leq [B]$. Then $T_0 \not\vdash A \rightarrow B$ and so $T_0 \vdash \sim(A \rightarrow B)$. But $\vdash_{\overline{RM}\supset} \sim(A \rightarrow B) \rightarrow ((A \rightarrow B) \rightarrow \sim A)$, $\vdash_{\overline{RM}\supset} \sim(A \rightarrow B) \rightarrow (A \rightarrow B) \rightarrow B$, so the present assumption implies that $T_0 \vdash (A \rightarrow B) \rightarrow (\sim A \wedge B)$ and so $[A \rightarrow B] = \sim[A] \wedge [B]$.

This completes the proof of the theorem.

As an immediate corollary we have:

Theorem 6 *PRM is decidable.*

II. (A) Adjunction and the disjunctive syllogism

Theorem 7

(a) Adding *adj* to *PR* we get *R*.

(b) Adding $R^\wedge(A, B)$ or $R^+(A, B)$ to *PRM* we get *RM*.

(c) Adding $R^+(A, B)$ to *PR* we get *RM*.

Proof: (a) Follows immediately from the definitions of *PR* and *R*.

(b) Follows from Theorem 4 (a) and (b).

(c) In [3] it is shown that, if we add $R^+(A, B)$ to R_\supset , we get RM_\supset , which is an extension of RM_\supset . Hence $PR \cup \{R^+(A, B)\}$ is equivalent to $PRM \cup \{R^+(A, B)\}$ and part (c) follows from part (b).

Remark: Theorem 7 shows that *R* and *RM* are really close relatives. Both are obtained from *PR* by adding schemes that are interdeducible in *PRM* and which intuitively mean that any two sentences are relevant to each other.

Theorem 8 *The adjunction rule (adj) and the disjunctive syllogism (γ) are equivalent in the context of PR: closing PR by either of these rules gives R. Similar relations hold between PRM and RM.*

Proof: That $R(M)$ results from adding *adj.* to $PR(M)$ is trivial. By Meyer-Dunn theorem ([1], Section 25) we know also that R and RM are closed under γ . Finally, in order to show that the system obtained from $PR(M)$ by adding (γ) is $R(M)$, it is enough, by Theorem 7, to show that $R^{\wedge}(A, B)$ is derivable in this system. But $\vdash_{PR} \sim(R^{\wedge}(A, B)) \Leftrightarrow (\sim(A \rightarrow A) \vee \sim(B \rightarrow B))$, hence $\vdash_{PR} \sim(A \rightarrow A) \vee (\sim(B \rightarrow B) \vee R^{\wedge}(A, B))$. Since $\vdash_{PR} \sim(C \rightarrow C)$, two applications of (γ) give $R^{\wedge}(A, B)$.

Theorem 8 shows great similarity in the role of *adj.* and γ with respect to relevance logic.¹¹ (This must not surprise us: *Adj.* can be used, in order to derive the “paradox” $A \rightarrow (B \rightarrow A)$ in a way which is parallel to that in which γ is used in order to derive $\sim A \rightarrow (A \rightarrow B)$.) This similarity is strengthened by the next theorem which shows that $\sim A \wedge (A \vee B) \rightarrow B$ can also be included in a system of “relevance” logic.

Theorem 9 *The system resulting from R by replacing R7 and R10 by $\sim A \wedge (A \vee B) \rightarrow B$ has the sharing-of-variable property for \rightarrow . This remains the case even if we add also the associative and commutative laws for \vee and \wedge , all forms of De-Morgan laws and all tautologies in the $\{\sim, \vee, \wedge\}$ language.*

Proof: We use a structure similar to the matrix M_0 of [1], pp. 252–253. We only change the definition of $x \wedge y$, $x \vee y$ as follows: \wedge behaves like classical conjunction on $\{-1, 1\}$ and on $\{-2, 2\}$. Also $x \wedge y = +0$ if x, y are both designated but the previous cases do not apply, $x \wedge y = -3$ otherwise. $x \vee y$ is defined to be $-(-x \wedge -y)$. It is now easy to check that all theorems of the system described in the formulation of the theorem get designated values under each valuation in this structure. Now if A and B share no variable, we can define $V(P)$ to be 1 if P is a propositional variable of A , $V(P) = 2$ otherwise. Then $V(A) \in \{1, -1\}$, $V(B) \in \{-2, 2\}$ and $V(A \rightarrow B) = -3$, which is not designated. Hence $A \rightarrow B$ is not a theorem in this case.

II. (B) A maximal pure relevant logic

Definition 5

- (a) A_2 is the submatrix of AS consisting of $\{-1, 1, 0_1, 0_2\}$.
- (b) PRM_2 is the set of sentences valid in A_2 .

Note: A_2 is a combination of two known four-valued matrices: the truth-table for \sim and \rightarrow and the designated values are like those in the matrix we call by the same name in [3], and which is isomorphic to the matrix introduced by Parks in [7] (see [1], p. 168). On the other hand, the truth tables for \sim, \vee and \wedge are like those in the Smiley matrix and introduced in [1], Section 13.3.¹² It is important that the order relation defined on the Smiley matrix according to the lattice operations \vee and \wedge corresponds exactly to the one induced on the Parks matrix by \rightarrow and by the choice of designated values (i.e., $a \leq b$ iff $a \rightarrow b$ is designated).

Theorem 10

- (a) PRM_2 has the variable-sharing property for both \rightarrow and \wedge .
- (b) Every logic extending PRM which has the variable-sharing property for \rightarrow is included in PRM_2 .

(c) *An axiomatization of PRM_2 is obtained by adding to PRM the scheme $A \vee (A \rightarrow B)$.*

(d) *PRM_2 has only two proper extensions: the classical calculus and RM_3 (the set of sentences valid in the Sugihara matrix $M_3 = \{-1, 0, 1\}$) (see, e.g., [1], p. 470).*

Proof: (a) This is proved exactly as in the PRM case.

(b) Let L be an extension of PRM having the sharing-of-variable property for \rightarrow . Then $\nVdash R^+(A, B)$. Repeating the argument in the proof of Theorem 5, we can find an augmented Sugihara matrix M in which all theorems of L are valid but $R^+(A, B)$ is not. Since $R^+(A, B)$ is valid in any ordinary Sugihara matrix, M contains both 0_1 and 0_2 and so A_2 is a submatrix of M . Hence all theorems of L are valid in A_2 and $L \subseteq PRM_2$.

(c) It is easy to check that $A \vee (A \rightarrow B)$ is valid in A_2 . On the other hand, if $PRM + \{A \vee (A \rightarrow B)\} \nVdash \phi$, then, by repeating the argument of Theorem 5, we can find an augmented Sugihara M in which $A \vee (A \rightarrow B)$ is valid but ϕ is not. Clearly $A \vee (A \rightarrow B)$ can be valid in M iff it contains exactly one undesignated value. Hence M must be a submatrix of A_2 , and so ϕ is not valid in A_2 .

(d) By the proof of (b), if L is an extension of PRM_2 , then $\vdash R^+(A, B)$. This means, by (c) and Theorem 7(c), that $RM + \{A \vee (A \rightarrow B)\} \subseteq L$. But it is known that this system, called RM_3 in [1], has only one proper extension, namely, the classical calculus.¹³

II. (C) Fragments of PRM and PRM_2

Theorem 11 *R_{fde} is the first-degree-entailment fragment of both PRM and PRM_2 .*

Proof: By Theorem 1, $R_{fde} \subseteq PRM \subseteq PRM_2$. On the other hand in [1], 15.3 it is shown that $\vdash_{R_{fde}} A \rightarrow B$ iff $V(A) \leq V(B)$ for any valuation V in the Smiley matrix, so (see the note after Definition 5) $\vdash_{R_{fde}} A \rightarrow B$ iff $A \rightarrow B$ is valid in A_2 , iff $\vdash_{PRM_2} A \rightarrow B$.

Theorem 12 *PRM and PRM_2 have the same $\{\sim, \rightarrow\}$ and $\{\rightarrow\}$ fragments. These fragments are proper extensions of RMI_{\supset} and RMI_{\neg} , respectively. Moreover, these fragments are relevantly maximal logics in the sense that they have the sharing-of-variable property and they include any other extension of RMI_{\supset} (or RMI_{\neg}) having this property. (So no proper extensions of them in their language have this property.)*

Proof: In [3] we call the above fragments of PRM_2 (i.e., the sets of sentences in the $\{\sim, \rightarrow\}$ and $\{\rightarrow\}$ language valid in A_2) “ RMI_{\supset}^2 ” and “ RMI_{\neg}^2 ” respectively. We show there that a complete axiomatization of RMI_{\supset}^2 (RMI_{\neg}^2) is obtained by adding to RMI_{\supset} (RMI_{\neg}) the scheme: $[(P_1 \rightarrow P_2) \rightarrow P_3] \rightarrow [((P_1 \rightarrow P_3) \rightarrow P_2) \rightarrow (((P_2 \rightarrow P_3) \rightarrow P_1) \rightarrow (P_1 \rightarrow P_1) + (P_2 \rightarrow P_2) + (P_3 \rightarrow P_3))]$.¹⁴ Now, it is not difficult to check that this formula is valid in AS and so provable in PRM . Hence, RMI_{\supset}^2 (RMI_{\neg}^2) is contained in PRM , and since $PRM \subseteq PRM_2$, RMI_{\supset}^2 (RMI_{\neg}^2) is exactly the $\{\sim, \rightarrow\}$ ($\{\rightarrow\}$) fragment of both systems. The other parts of the theorem contain properties of RMI_{\supset}^2 and RMI_{\neg}^2 that were proved in [3].

II. (D) A subsystem with a more intuitive semantics Theorem 5 shows that *PRM* and *RM* differ slightly: while in *RM* essentially all “paradoxical” sentences are equivalent, in *PRM* we can have exactly two different such sentences. It may seem more intuitive to have a potentially infinite number of “paradoxical” sentences, irrelevant (and so non-equivalent) to each other in pairs. This can be done, but only with the price of weakening the distribution axiom. This is not too high a price, though. The justification of this axiom is not clear anyway, and it is known to be the source of many unpleasant properties of the relevance system.¹⁵

As a first step we introduce a new connective, definable in *PRM*, and prove a theorem about it. Both the connective and the theorem are important on their own right.

Definition 6 $A \supset B =_{df} (A \rightarrow B) \vee B$.¹⁶

Lemma $A, A \supset B \vdash_{PR} B$.

Proof: Since $\vdash_{PR} A \rightarrow ((A \rightarrow B) \rightarrow B)$, it follows that $A \vdash_{PR} (A \rightarrow B) \rightarrow B$. Also, $\vdash_{PR} B \rightarrow B$, so $A \vdash_{PR} (A \supset B) \rightarrow B$. Hence the lemma.

Theorem 13 The deduction theorem *Let T be an L -theory where L is any extension of *PRM* (having the same rules of inference), then $T, A \vdash_L B$ iff $T \vdash_L A \supset B$.*

Proof: The lemma gives the “if” part. For the “only if” it is enough to check that the following three schemes are valid in *AS* and so provable in *PRM*: (i) $A \supset A$; (ii) $(A \supset B) \supset [(A \supset (B \rightarrow C)) \supset (A \supset C)]$; (iii) $(A \supset .B \rightarrow C) \supset [(A \supset .B \rightarrow D) \supset (A \supset .B \rightarrow (C \wedge D))]$.

We generalize now the semantics of *PRM*:

Definition 7

(a) By a *generalized Sugihara matrix* we mean any structure which is either an ordinary Sugihara matrix, or results from a normal Sugihara matrix S , having a first designated value 1 by adding to it a set of neutral values $\{I_i\}_{i \in J}$, all taken to be designated. We further define: $\sim I_i = I_i \rightarrow I_i = I_i \wedge I_i = I_i \vee I_i = I_i$, and if $i \neq j$ then $I_i \rightarrow I_j = I_i \wedge I_j = -1$, $I_i \vee I_j = 1$.

(b) *ES*, the *canonical generalized Sugihara matrix* is obtained from the integers (without 0) by adding to them a *countable* set of neutral values in the manner described in (a).

Obviously, a sentence A is valid in any generalized Sugihara matrix iff it is valid in *ES*.

Theorem 14 *Let PRM^* be the system obtained by replacing the distribution axiom of *PRM* by the following relevant version:*

(RD) $[R \wedge (A, B) \vee R \wedge (A, C) \vee R \wedge (B, C)] \supset [(A \vee B) \wedge (A \vee C) \rightarrow .A \vee (B \wedge C)]$.

Then PRM^ is complete for the set of sentences valid in *ES*.*

Proof: It is easy to check that all theorems of *PRM^** are valid in *ES*. For the completeness part of the theorem we note first that *PRM^** has all the proper-

ties of *PRM* listed in Theorem 4. This is so because one can easily check that, whenever the distribution axiom is used in the proof of Theorem 4,¹⁷ we can apply previous parts of that theorem and the lemma after Definition 6 for justifying the use of *RD* instead of distribution. Granting this, we can repeat now the proof of Theorem 5 for *PRM** word by word. There is just one exception: in one place in that proof, distribution is used directly (i.e., not through applications of Theorem 4): when we show that there are at most two neutral values. In *PRM** we do not have this limitation and so we just have to omit this step.

The next theorem summarizes those properties of *PRM* that *PRM** has as well. The proofs are exactly as those of the corresponding theorems for *PRM* (Theorems 3, 4, 6, 7, 8, 13) and are left to the reader.

Theorem 15 *PRM** has the sharing-of-variable property for both \rightarrow and \wedge , it is decidable, the deduction theorem for \supset holds in it, and it has all the proof-theoretical properties stated in Theorem 4. Further, adding to it either $R^{\wedge}(A, B)$ or $R^{\vee}(A, B)$ as an axiom, or either *adj* or γ as a rule of inference, we get *RM*.

Besides the properties shared by *PRM* and *PRM**, there are welcomed properties that *PRM* lacks but *PRM** has. The following theorem is a very important example:

Theorem 16 *PRM** is a conservative extension of RMI_{\supset} .

Proof: In [3] we show the submatrix of *ES* consisting of $\{-1, 1, I_1, I_2, \dots\}$ is characteristic for RMI_{\supset} . From this fact and the fact that *ES* is sound for RMI_{\supset} the theorem follows easily.

Note: In [3] we show that any proper extension of RMI_{\supset} is obtained by adding schemes the meaning of which is that there are just a finite number of “paradoxical” statements. Since such a limitation is not intuitive (unless we reject the existence of *any* paradoxical statement) RMI_{\supset} is preferable to any of its extensions (excluding perhaps classical logic). (There are also other reasons to choose RMI_{\supset} as the “true” relevance logic in the $\{\rightarrow, \sim\}$ language. See [3].)

The submatrix of *ES* mentioned in the last proof was called A_{∞} in [3], and it was shown there to be a minimal matrix characterizing RMI_{\supset} . For the full language of *PR* we have the following:

Theorem 17 A complete axiomatization of the set of sentences in the language of *PR* which are valid in A_{∞} is obtained by adding to *PRM** the scheme $A \vee (A \rightarrow B)$.

The proof resembles that of Theorem 10, and is left to the reader.

NOTES

1. Not to mention the ordinary, “official” one which is also sometimes used, e.g., in the proof of the admissibility of (γ).
2. We consider this impossibility as another evidence of the pure relevance character of RMI_{\supset} .

3. We prefer here the name R_{fde} to E_{fde} , which was used in [1].
4. In view of these facts, we think that like RM , R itself may hardly be called “a relevance logic”.
5. This observation is due to the referee.
6. In this paper, whenever we talk about provability from a set of assumptions we assume the ordinary (sometimes called “official” by the relevantists) sense of a “proof”.
7. This definition of a Sugihara matrix is a version of Dunn’s concept of a Sugihara chain appearing in [1], p. 421. The adjective “ordinary” was added to distinguish it from the generalization we introduce below.
8. This is shown in [3], III.8. From now on we shall not give a proof of a sentence in RAI_{\rightarrow} (or RMI_{\rightarrow}) when we claim such a proof to exist, since RMI_{\rightarrow} and RMI_{\rightarrow} have an efficient decision procedure (see [3]).
9. This is the first place we use distribution in the proof of Theorem 4. Hence (a)–(i) are true independently of this axiom.
10. I.e., we define $A \equiv B$ iff $T_0 \vdash_{PRM} A \leftrightarrow B$. (By $A \leftrightarrow B$ we mean either $(A \rightarrow B) \circ (B \rightarrow A)$ or $(A \rightarrow B) \wedge (B \rightarrow A)$. Both are provable iff both $A \rightarrow B$ and $B \rightarrow A$ are provable.) This is an equivalence relation. Denote by $[A]$ the equivalence class of A , and let M be the set of the equivalence classes. Define further: $[A] \leq [B]$ iff $T_0 \vdash A \rightarrow B$, $[A] \wedge [B] = [A \wedge B]$, $[A] \vee [B] = [A \vee B]$, $\sim[A] = [A]$, $[A] \rightarrow [B] = [A \rightarrow B]$. These are all well-defined. Finally let T_M , the set of designated values, be $\{[A] \mid T_0 \vdash_{PRM} A\}$.
11. Note also the similarity with respect to R : R is closed, as a system, under both rules, but they cannot be applied freely in (“official”) R -deduction from R -theories without violating basic relevant principles.
12. In [1] the matrix consists of $\{1, 2, 3, 4\}$. The correspondence $1 \leftrightarrow 1, 4 \leftrightarrow -1, 2 \leftrightarrow 0_1, 3 \leftrightarrow 0_2$ is an isomorphism of the two matrices relative to $\{\sim, \wedge, \vee\}$.
13. This is a consequence of Dunn’s characterization of RM ’s extensions. See [1], 29.4 and p. 470.
14. For the case of RMI_{\rightarrow} , an equivalent formula containing only \rightarrow is given in [3].
15. In [1], p. 313 it is called “a headache for E and related systems” (in some respects).
16. More on this connective and its importance in the context of RM can be found in [2].
17. Actually only parts (j) and (k) depend on distribution (see note 9).

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