Notre Dame Journal of Formal Logic
Volume 27, Number 1, January 1986

Russell's First Theory of Denoting and Quantification

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Russell presented his first theory of denoting in his 1903 work, *The Principles of Mathematics* (*PoM*) [13] (unless otherwise indicated, all parenthetical page references in this paper are to that work). Russell's theory poses a considerable puzzle for the modern reader. It is clear that a principal role of the theory of denoting is that of providing an analysis of sentences containing expressions of generality—expressions of the form "any A", "an A", "some A", and so on. Unfortunately, there is little agreement about how it is supposed to do this. Jager, the author of one of the most detailed analyses of Russell's philosophy [10], proposes the following interpretation:

> Russell's theory of denotation (1903) may be characterized as one which envisages, in place of the now standard two quantifiers (universal and existential) *three* independent operators. His three operators correspond respectively to the English terms *all* (i.e. 'each and every'), *a* (i.e. 'some or other'), and *some* (i.e. 'some particular'). [10], p. 146

Jager is sympathetic to what he takes to be Russell's purpose in developing his first theory of denoting. His assessment of that theory stands in sharp contrast with that of Geach [8]. Geach understands Russell to be proposing a version of the medieval theories of different types of *suppositio*, or mode of reference, again for the purpose of analyzing sentences containing expressions of generality. Geach argues that this sort of analysis is radically mistaken, and considers Russell's use of *denoting concepts* and *denoted objects* to be unwarranted "metaphysical speculation, which we may henceforth ignore as irrelevant to logic" [8], p. 62.

I find Jager's interpretation of Russell's theory implausible, and Geach's dismissal of it too hasty. In this paper I want to re-examine the connections

*I am indebted to Robert Brandom, Joseph Camp, Charles Parsons, and especially to Richmond Thomason, for their numerous and helpful suggestions about various topics considered in this paper. The comments of an anonymous referee for this Journal have also been of considerable help.

Received July 23, 1984

between Russell's first theory of denoting and quantificational logic. The difficulty in setting out these connections is partly due to the fact that most of what Russell has to say about the analysis of sentences containing expressions of generality *antedates* the final development of his theory of denoting. But it is also due to the fact that Russell has two quite different accounts of the nature of denoting and of denoted objects. These complications have gone completely unnoticed by most critics of Russell's views, with the result that much of what has been written about Russell's theory of denoting, and about its application to the analysis of sentences containing denoting phrases, is either misleading or simply mistaken.[1] In this paper I hope to clear this muddle up. My plan is as follows:

I begin with a concise and careful summary of the relevant features of Russell's theory. I am especially concerned to distinguish the early and late formulations of the theory, and the two different versions of the early theory on which the later portions were superimposed. In Section 2, I set out what I take to be Russell's analysis of sentences containing expressions of generality. Support for my interpretation requires a detailed study of Russell's claims and examples: this is the topic of Section 3. What emerges from this discussion is a better understanding of the central features of Russell's theory of denoting, of its connections with quantification, and of what is of value in Russell's analysis of general sentences. In Section 4 I set out the syntax for a language suited to the representation of that analysis; I develop a formal semantics for each version of Russell's theory; and I show that these interpreted systems are equivalent, in a sense to be made precise, to standard first-order logic. My conclusion is that Russell's first theory of denoting *can* be reconstructed in a logically cogent manner.

1 The theories of denoting Russell completed a draft of Part I of *PoM*—in which most of the properly philosophical discussion in that work is to be found—in May 1901.[2] The final manuscript of Part I was completed in May 1902. Study of these drafts shows that Russell had already worked out a formulation of his theory of denoting in 1901. And it shows that even though he made significant changes in this early formulation, he actually retained most of it in the considerably supplemented formulation of the following year. Russell tells us repeatedly in *PoM* that the theory of denoting is central to his philosophy; and accordingly he makes impressive claims on its behalf. He tells us that the notion of denoting lies at the bottom of the subject-predicate logic, of all theories of substance, of the opposition between things and ideas and between discursive thought and immediate perception (p. 53). As regards mathematics specifically, he tells us that "the whole theory of definition, of identity, of classes, of symbolism, and of the variables is wrapped up in the theory of denoting" (p. 54). And a constant theme throughout the work is Russell's claim that denoting provides "the inmost secret of our power to deal with infinity" (p. 73; *cf.* pp. 66, 145). Now most critics of Russell's theory simply identify the theory of denoting and the revised 1902 formulation without even noticing the presence in the text of the earlier formulation. Since many of the applications of the theory of denoting do depend essentially on the changes characteristic of the 1902 account, for some purposes this identification is not inconvenient. But those applications with which we shall be concerned in this paper—roughly, those that have to do with logic—*do not depend* on the peculiarities of the 1902 account, and indeed were taken over wholesale from the earlier manuscript. Thus, at the very least, identification of the theory of denoting with the 1902 formulation is apt to be misleading, and may even make a clear understanding of the role of that theory, in Russell's analysis of sentences containing expressions of generality, impossible. In what follows we shall therefore have to be careful to distinguish and set out the *general framework* within which Russell worked, the *two formulations* of the theory of denoting, and the *two versions* of that theory common to both formulations.

According to the position Russell adopts in *PoM*, all words and phrases in a meaningful sentence have a meaning "in the sense that they are symbols which stand for something other than themselves" (p. 47). We may follow Russell's usage and call meaning in this linguistic sense *indication*: so in the scheme of *PoM* every word and phrase indicates, and indicates what it means. Further, propositions—the nonlinguistic entities indicated by sentences—are composed of the indications of the words and phrases in the corresponding sentences.[3]

Among these expressions are what Russell calls *denoting phrases*: grammatically correct phrases beginning with one of the six words "all", "every", "any", "a(n)", "some", and "the", "or some synonym of one of them" (p. 56). The chief change introduced in the 1902 formulation of the theory of denoting concerns Russell's treatment of the linguistic meaning—that is, indication—of denoting phrases. According to the revised account, denoting phrases indicate *denoting concepts*; and denoting concepts can be characterized by appealing to the role they play in the propositions in which they occur. Russell puts it this way:

> A concept *denotes* when, if it occurs in a proposition, the proposition is not *about* the concept but about the term connected in a certain peculiar way with the concept. (p. 53)

(*Aboutness* is a primitive notion not explicated by Russell. By definition a proposition is about precisely its *logical subjects*.) So a concept is a denoting concept if any proposition in which it occurs is not about the concept but about an entity related to that concept, namely its denotation. Such *denotata* Russell calls 'objects', an expression he uses in a technical sense (p. 55 n.*).

In *PoM* Russell states emphatically that the denoting relation is a logical and not a linguistic relation,[4] holding between concepts and entities and not between words and entities (pp. 47, 53). But in a simple, derivative sense, we may also speak of the denotation of a denoting phrase, meaning by this the object denoted by the indication of that phrase; and we may therefore say that denoting phrases denote objects. This manner of speaking allows us to highlight the chief difference between the first and second formulations of the theory of denoting. As the passages that Russell took over from the 1901 manuscript make clear, in his first formulation of the theory of denoting Russell understood the linguistic meaning of denoting phrases (except, apparently, definite descriptions) to be the denoted objects themselves.[5] So according to the *first* formulation of the theory of denoting the linguistic meaning of a denoting phrase is a denoted *object*, while according to the *second* formulation of the theory the linguistic meaning of a denoting phrase is a denoting *concept*. Since in the finished manuscript of *PoM* Russell retained his first analysis of sentences containing expressions of generality, we have here an explanation for the otherwise puzzling fact that denoting concepts have no important role to play in that analysis (pp. 56–61). Figure 1 contrasts the two formulations:

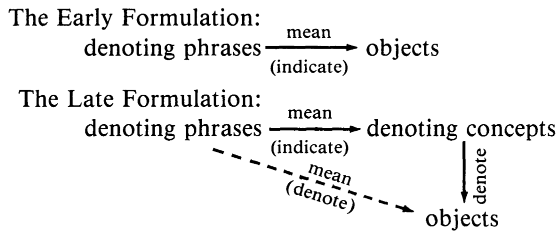

Figure 1. The two formulations of the theory of denoting.

Russell's theory of denoting is made even more confusing by his tendency to give different and incompatible accounts of the denoting *relation* (whether this be understood as primarily a logical (1902) or a linguistic (1901) relation), and different accounts of the nature of denoted *objects*. Moreover, Russell does not parcel his views out neatly: he runs both accounts together, moving uneasily between them, endorsing first one, then the other. To make matters clearer, I shall explicitly distinguish two versions of the theory of objects, which I label the "official" and the "unofficial" theories. (I call the first version the "official" theory because it is the one commonly attributed to Russell.) It should be noted that Russell's vacillation between the official and unofficial theories *antedates* the revisions of 1902, and is quite independent of them. It is in fact as much present in the first formulation of the theory of denoting as it is in the second. The differences between the official and unofficial theories can therefore be brought out within either formulation of the theory of denoting; but it will prove most convenient to contrast the two versions of the theory by using the resources and terminology of the 1902 account. (In most cases the reader can construct a completely accurate reformulation within the 1901 account simply by replacing "concept", where it occurs, with "phrase".) For simplicity, I shall also exclude definite descriptions from the discussion.

Russell's official view is that there is a single denoting relation holding between denoting concepts and the objects they denote (p. 65). Consequently, if A is a class-concept (say, the meaning of a common-noun phrase), then the various denoting concepts deriving from A (i.e., *all A's*, *every A*, *any A*, *an A*, and *some A*) stand in precisely the same logical relation to the objects they denote. Moreover, *each* of these objects is a *complex* or *combination of all and only* the terms in the extension of the class-concept A; but the combinations are supposed to be different for the objects denoted by the different denoting concepts. We have, then, corresponding to the five types of denoting concept we are considering, five different types of denoted object. Taking the class-concept *horse* as an example, the combinations Russell gives are these: the denoting concept *all horses* denotes a *numerical conjunction* of horses; the denoting concept *every horse* denotes a *propositional conjunction* of horses; the concept *any horse* denotes a *variable conjunction* of horses; *a horse* denotes a *variable disjunction* of horses; and *some horse* denotes a *constant disjunction* of horses (*cp.* pp. 57–58).

Russell applies his theory of denoting to pure logic, and considers an unrestricted variable to be the object denoted by the concept *any term*. That is, Russell understands individual variable letters to be symbols indicating objects (which he also calls "variables"); and he treats (the object) the variable as that combination of absolutely all terms which is the denotation of the denoting concept *any term* (p. 91). He also thinks that an account of the quantifiers presupposes an account of denoting, because in giving an explanation of the role of the quantifiers we have to make use of denoting phrases (at least those of the form 'any *A*') (p. 92). This thesis about the relation of the quantifiers to denoting phrases is the precise reversal of Russell's position in 'On Denoting' [18], pp. 105–106, in which he accounts for the truth conditions of sentences containing denoting phrases by appealing to a primitive quantificational operator, 'is always true', which binds free variables in open sentences.[6]

It is quite difficult to give an intuitive model for the official theory's objects. One way to attempt to do this is to appeal to the Leibnizian notion of a *heap* or *mere aggregate:* a nonsubstantial complex of terms, that is, one with no internal unifying principle. This seems to capture part of what Russell has in mind when he says that the various combinations are effected without the use of relations (p. 58)—they are completely extrinsic combinations. But perhaps a better sense of the nature of Russellian objects can be obtained by considering the case of a numerical conjunction, the object denoted by a concept *all A's*. When Russell contrasts classes taken in intension and classes taken in extension, the contrast he has in mind is that between an attribute and the terms to which that attribute applies. Now an attribute can apply to many entities, and since the entities to which it applies *are* its extension, the extension of a class-concept is essentially plural, a complex of terms and not a single term. This complex of terms is Russell's class as many, and the class as many is the object denoted by a concept *all A's* (p. 80). It is of the class as many that we predicate number: hence the name, *numerical* conjunction, and hence the object's "essential plurality".

The numerical conjunction is the paradigm denoted object for Russell. Furthermore, it is the only sort of denoted object common to both the official and unofficial theories. So it is important that, however peculiar its nature, a set-theoretical explanation of numerical conjunctions is available. Naturally this explanation does not tell us everything about denoted objects. According to Russell the various denoted objects are supposed to be different combinations of terms in the extension of the corresponding class-concept. Now there are two independent ways in which such combinations differ. First, they can be either *conjunctions* or *disjunctions* of terms; second, they can be either *constant* or *variable* combinations of terms. These distinctions need to be explained; in the next section I will account for them by appealing to the truth conditions of sentences containing the appropriate denoting phrases. For the moment, however, I shall turn to Russell's second version of the theory of denoted objects.

Alongside the offical theory in the text of *PoM* is an alternative account of denoted objects and the denoting relation, which I have called Russell's *unofficial theory*. According to this version of the theory, (some) denoting concepts do *not* denote an "assemblage of terms" in the extension of the associated class-concept; instead, they denote an unspecified, *ambiguous* term, or perhaps better put, they *ambiguously denote* a term in the extension of that concept (pp. 58, 59, 91, 94 *inter alia*). Russell himself seems not to have made any clear distinction between (ordinary) denotation of an ambiguous entity and ambiguous denotation of an (ordinary) entity.[7] He uses these expressions interchangeably, but since he more frequently speaks of ambiguous denotation than of ambiguous terms, in what follows I shall say that according to the unofficial theory denoting concepts 'ambiguously denote' terms.

Since the term 'ambiguously denoted' by a denoting concept is just a term in the extension of the corresponding class-concept, it might seem that the unofficial theory completely does away with the official theory's metaphysically puzzling complexes of terms. In support of this conclusion one could cite Russell's heading for section 75 of *PoM*: "*Every, any, a* and *some* each denote one object, but an ambiguous one" (p. xxiii). It would, however, be a mistake to draw this conclusion. Throughout *PoM* Russell remains convinced that there are classes as many and extensions of class-concepts, and that the class as many is essentially plural, a complex of terms and not a single term. So even in the revised, unofficial version of the theory of objects, the denotation of a concept *all A's* is considered to be a complex or combination of terms. It follows that in the unofficial theory there are different types of *denotata* (terms and complexes of terms), different types of *denoting relation*, or both: either some denoting concepts denote complexes while other concepts 'ambiguously' denote individuals, or some denoting concepts denote ambiguously, while others denote uniquely determined combinations of terms. Figure 2 contrasts the two versions of the theory:

The Official Version:

denoting phrase/concept		denoted entity
All A's Every A Any A An A Some A	$\xrightarrow[\text{(indicates/denotes)}]{\text{means}}$	numerical conjunction propositional conjunction variable conjunction variable disjunction constant disjunction

The Unofficial Version, Preferred Interpretation:

denoting phrase/concept		denoted entity
All A's	$\xrightarrow[\text{(indicates/denotes)}]{\text{means}}$	numerical conjunction
Every A Any A An A Some A	$\xrightarrow[\text{(ambiguously denotes)}]{\text{means}}$	term

Figure 2. The two versions of the theory of denoting.

It seems both striking and quite puzzling that Russell should run such different accounts together, as indeed he does throughout Part I of *PoM*, at one point espousing the official theory (e.g., pp. 58, 62) and at another the unofficial theory (e.g., pp. 58–59, 77). In order to understand how this could happen, we must begin by realizing that the original and central purpose of the early formulation of the theory of denoting is to account for sentences containing expressions of generality.[8] Now the introduction of denoting concepts in 1902 certainly allows for *additional* applications of the theory of denoting,[9] applications which were to become dominant in Russell's later work. But this change does not essentially affect the problem of analyzing sentences containing denoting phrases, since at most it merely reformulates the problem by shifting from the linguistic level to the logical level: by replacing, that is, the analysis of sentences containing denoting phrases with the analysis of propositions containing denoting concepts. (This point is discussed in more detail at the end of the present section.) The root causes of Russell's uncertainty are therefore best examined within the context of the 1901 formulation of the theory of denoting, according to which denoting phrases mean (indicate) objects. The reasons for Russell's uncertainty can be illustrated by considering sentences (1) and (2):

(1) Frege is astute.
(2) Every logician is astute.

Sentences (1) and (2) appear to have the same structure; yet, as consideration of their inferential force makes plain, they convey quite different sorts of claim. *Prima facie*, this leaves us with several choices:

A. We may conclude that (surface) grammar notwithstanding, (1) and (2) have comparably different *logical* forms.

We may, however, prefer a different explanation. We may decide that our grammatical intuitions *are* correct – (1) and (2) are both simple subject-predicate sentences – and conclude that the difference between the two is due to the different behavior of their subjects.

B. This difference might be explained by positing a single semantic relation, a *naming* relation, and assigning to the phrase "every logician" an ontologically unusual *nominatum*, a *conjunctive* entity.

Or

C. We might complicate our semantics to benefit our ontology, by supposing that unlike the name "Frege", the phrase "every logician" 'distributively' or 'ambiguously' names ontologically unexceptional things. The same sort of choice confronts us when we compare, say,

(3) Frege does not despise Russell.

with each of the sentences in the pairs (4), (5) and (6), (7):

(4) Frege does not despise any logician.
(5) Frege does not despise every logician.

(6) Frege does not despise some logician.
(7) Frege does not despise a logician.

Once again we can account for the different claims made by these sentences by attributing different logical forms to them, or by assigning them all the same form as (3), say, and concluding that the values of the denoting phrases must differ among themselves. This second option leads either to different types of combinations of terms, or to different ways of naming (denoting) terms. Now to someone who resolves, as Russell did, that "grammar, though not our master, will yet be taken as our guide" (p. 42), the first option, A, would have seemed implausible, and some version of the theory of objects would appear as natural and perhaps unavoidable. But treating denoting phrases as names of entities does not decide between the remaining alternatives B and C: hence Russell's vacillation.

For there are arguments in favor of each view. For one thing, objects as complexes of terms seem to be required by other parts of Russell's general framework—by his set-theoretical foundational program, for instance, with its class as many. If anything, Russell's discovery in June 1901 of the Contradiction (the Russell Paradox concerning non-self-membered classes) seems to have reinforced his belief in the existence of classes as many, at the expense of classes as one. That is to say, it reinforced his tendency to distinguish the extension of a class-concept and the single set (the Russellian 'whole') of the terms in that extension, since the Contradiction seemed to show that some properties may determine an extension but no corresponding set. Second, the official theory was to be recommended because it called for a single semantic relation, that of 'standing for', or indicating, albeit at the price of having some expressions stand for what are indeed "very paradoxical" (p. 62) entities.

On the other hand, Russell's official theory faces obvious difficulties.[10] For instance, it seems that given a class-concept A and a method of combination for all the terms in the extension of A, the corresponding denoted object is uniquely determined. But there can only be one object denoted by any given denoting phrase, if this is right; yet two occurrences of the same denoting phrase in a sentence may well signal the presence of *independent* variables. Consider for example.

(8) If some American is a mother then some American is not a mother.

The unofficial theory helps solve this difficulty, for if denoting phrases denote ambiguously, then different occurrences of the same denoting phrase in a sentence may well denote different terms, and this makes an explanation of independent variables possible (p. 94). Essentially for this reason, in his unpublished manuscripts written in 1904 and 1905 ([14], [15], [16]) Russell came to prefer the unofficial version of his theory to the official version. Furthermore, the unofficial version has the advantage of disposing of different *types* of complexes of terms, albeit at the price of introducing different (linguistic) meaning relations. But the unofficial theory remains committed to classes as many, which are 'collective' entities. What is surprising is therefore not so much Russell's vacillation between the official and unofficial versions of this theory of denoting as his apparent complacency about this vacillation.

Now this vacillation between the two versions of the theory antedates and survives the introduction of denoting concepts in 1902. In fact in this area the revisions of 1902 bring about apparently promising but ultimately superficial changes. According to Russell a sentence—any sentence—means the proposition that is composed of the indications of the words and phrases making up the sentence. The introduction of denoting concepts in 1902 may therefore seem to make possible a uniform, univocal and unproblematic analysis of the meaning of sentences like (1) and (2). If denoting phrases mean (indicate) denoting concepts instead of denoted objects, then it appears that we can indeed respect our grammatical intuitions, treat denoting phrases as names, and do so without commitment either to different and suspect types of linguistic meaning or to such "paradoxical" entities as different complexes of terms. Thus the theory of meaning depends only on the presence of denoting concepts, and does not require additional denoted objects. However, as Russell's characterization of denoting concepts makes clear, we still need a theory of denoted objects if we are to give an *adequate* analysis of sentences containing denoting phrases, since such sentences are *about* the denoted objects and *not about* the (indicated) denoting concepts. One way to bring out what is lacking without denoted objects is to note that in order to have an adequate analysis of sentences containing denoting phrases we need to know not only what such sentences *mean*—what propositions they indicate—but also under what circumstances they are *true*. But the truth of such a sentence, or of the proposition it indicates, depends not on the properties of the indicated concept but on those of the denoted object, something Russell unquestionably understood (pp. 53-54; *cf.* [15], p. 5). This means that a theory of denoted objects is still needed, and moreover, precisely in order to account for the difference in truth conditions to be found in sentences like (1) and (2). But then both options open to the 1901 formulation of the theory, and all the problems to be faced in choosing between those options, simply reappear in the 1902 formulation. In short, the addition of denoting concepts in the 1902 formulation of the theory of denoting neither eliminates nor solves the problems facing the 1901 formulation. We may therefore henceforth ignore the additions of 1902 and concentrate exclusively on the common core represented by the official and unofficial versions of the 1901 formulation.

2 What the theory of denoting is supposed to do It is clear that one of Russell's principal concerns in Chapter V of *PoM*, where he discusses denoting, is to give an analysis of sentences containing denoting phrases. In 1901 Russell believed that he could explain the truth conditions for such sentences by having the various denoting phrases indicate different objects. Thus a key goal of Russell's theory of denoting is to give an account of sentences containing one or more denoting phrases; and the *work* in this account is supposed to be done by the properties of the different denoted objects. But as the fact that Russell gives different and incompatible theories of denoted objects makes quite clear, it is vital for us to distinguish *which* features of such sentences Russell wanted to explain, and *how* he thought that his theory explained them. It would be a serious mistake to identify differentiating theses of either the official or the unofficial versions of the theory of denoting with Russell's application of the theory to the analysis of sentences containing denoting phrases. Instead, to the extent that this is possible, we should seek to discover details of that application that are *indifferent* to the choice of one version rather than the other. In short, we should seek to discover what the theory of denoting is supposed to *do*.

The terminology of the official theory is especially helpful in this regard. Russell's argument for the distinctness of the five "different combinations" of terms involves an appeal to the different truth conditions of sentences containing the corresponding phrases (pp. 56–58). I will argue that Russell's distinction between *conjunctions* and *disjunctions* of terms corresponds to differences in the analysis of sentences containing the appropriate denoting phrases, depending on whether their translations in predicate logic use universal or existential quantification. Accordingly, Russell groups "any" and "every" phrases together as having conjunctive (universal) force, and groups "some" and "a" phrases together as having disjunctive (existential) force.

Similarly I maintain, and will show in the next section, that Russell's distinction between *constant* and *variable* combinations of terms corresponds to a difference in the *scope* of the corresponding denoting phrases in sentences in which they occur (*wide* and *narrow* scope respectively). So, as I read him, Russell groups "any" and "some" phrases together as having wide scope in sentences in which they occur, and groups "every" and "a" phrases together as having narrow scope in sentences in which they occur.

Supposing that this classificatory scheme accurately captures Russell's intentions, I believe that Russell's application of his theory of denoting to sentences containing denoting phrases embodies an interesting and significant insight. There is of course no scope difference to be made out in sentences in which a *noncomplex* predicate is attached to a denoting phrase. (We can think of noncomplex predicates as being just the primitive predicates of a formal language. For our purposes, the same point can be made by referring to sentences containing no sentential connectives, though this certainly oversimplifies the general case.) If "$G(\ )$" is a noncomplex predicate, then "$G(\text{every } A)$" and "$G(\text{any } A)$", for example, are equivalent. But in, say, the schema "If $F(\underline{a})$, then p", we have a choice. (Here "$\underline{a}$" is a place holder for terms.) We can regard a sentence of that form as constructed by applying the conditional connective to the subsentences "$F(\underline{a})$" and "p"; or as constructed by first forming the complex predicate "If $F(\ )$ then p" and then applying it to the argument $\underline{a}$. The difference in constructional history has no effect on the interpretation of the sentence if the argument $\underline{a}$ is a proper name. But in a language without devices for marking scope distinctions, sentences of this kind will manifest scope ambiguity when the argument is an expression of generality. Suppose that in such a language "αA" is a phrase expressing universality. Then a sentence of the form "If $F(\alpha A)$ then p" has *two* readings, corresponding to the two constructional histories given above. On the first, the scope of the universal quantifier includes just the antecedent; on the second, the scope of the universal quantifier includes the whole conditional. This problem will arise whenever an expression of 'conjunctive' force (that is, one expressing universality) is embedded in a complex sentence. As I understand it, Russell's thesis is that English does not have this problem because it employs a disambiguating convention. When a sentence of the form in question is to be read as a conditional, we use "every"; and when it is to be read as a predication, of the complex predicate

"If $F(\ )$ then p" of its argument, we use "any". So English speakers should have no difficulty in distinguishing the claims made by (9) and (10), if Russell is right:

(9) If Frege despises every logician, then Frege is a cad.
(10) If Frege despises any logician, then Frege is a cad.

Naturally a similar ambiguity would arise for phrases having 'disjunctive' force, and here too I understand Russell to be saying that English employs a disambiguating convention. "Some" phrases are treated as having wide scope in sentences in which they occur, and "a" phrases are treated as having narrow scope in sentences in which they occur. This analysis is appropriate for (11) and (12):

(11) Frege does not despise some logician.
(12) Frege does not despise a logician.[11]

Of course this disambiguating scheme does not yet tell us how to interpret all sentences which exhibit multiple generality—that is, sentences containing two or more denoting phrases. Let us call a sentence (*quantificationally*) *homogenous* if it contains only denoting phrases of the same quantificational force (that is, only "any" and "every" phrases or "some" and "a" phrases). A sentence is (*quantificationally*) *heterogenous* if it is not homogenous. Then Russell's scheme as I have presented it yields unambiguous though rigid analyses of all homogenous sentences.[12] But it does not tell us how to interpret heterogenous sentences. Consider

(13) Any man loves some woman.

Obviously it makes a difference if we take "any" as having wide scope over "some" instead of the reverse. (On the first reading the sentence is true iff no man is a misogynist, while on the second it is true iff there is at least one universally loved woman.) Now the scope rules I have given to this point do not tell us what to do in a sentence like (13). We cannot give both denoting phrases wide scope. So if Russell were really concerned to distinguish scope differences among the various denoting phrases, then we should expect him to give us rules for interpreting such heterogenous sentences as (13). I believe he does. In terms of strength, stronger getting wider scope in a sentence, I understand Russell's theory as giving decreasing strength in the order given to phrases beginning with "some", "any", "a(n)", and "every". (This claim too will be defended in the next section.) Russell therefore reads (13) as saying that there is a woman who is loved by every man. To get the 'other' reading, we have to use "a" in place of "some":

(14) Any man loves a woman.

is understood by Russell as saying that no man is a misogynist.

Now the use of "any" and "every" as scope indicators in complex sentences is familiar and widely recognized (see, for example, [12], pp. 70 ff and [5], pp. 217 ff). But the interpretation of English sentences which I recommend we see Russell as proposing seems too rigid to represent actual usage. "Some" and "any" apply both to count nouns and to mass nouns, for instance, while "a" and "every" apply only to count nouns. So we cannot always rely on the choice of "some" instead of "a" or of "any" instead of "every" to mark scope distinctions over connectives, but must sometimes rely on other devices, for instance passive constructions. Similar remarks apply when "some" and "any" are used with plurals. On the other hand, the language Russell wants, and the one in which he poses most of his examples and illustrations, is one which is adequate for mathematics. So some of these problems, for instance those connected with mass noun phrases, are not immediately relevant to the appropriateness of his disambiguating conventions. More important, Russell explicitly acknowledges that his account of denoting phrases does not follow ordinary usage (p. 56, n.*). So he should be understood as proposing an interpretation of sentences containing denoting phrases which involves considerable *regimentation.* (This is not a significant concession: *any* application of a formal theory of quantification to a natural language will require some regimentation.) Given the great complexity of the linguistic data (cf. [19], pp. 199 ff and [23], Chapter 3), it is quite likely that this regimentation will have only a restricted application. This does not diminish the logical importance of the phenomena towards which, on my interpretation, Russell's theory is directed.

Another advantage of this interpretation is that it makes it clear that the various distinguishing features of the different denoted objects in the official theory are postulated by Russell *solely for the purpose of* accounting for certain features of sentences containing denoting phrases. And it is partly because they are introduced just for that purpose that Russell is so casual about the use of the unofficial theory in this same connection. Russell has no independent reason for hypothesizing different complexes of terms. His objective is simply to account for the features of sentences containing denoting phrases, and any account that allows him to do this therefore meets his principal objective.

The evidence for my interpretation is indirect. Russell does not talk *explicitly* of the 'scope' of denoting phrases, nor even of their 'quantificational force'. Indeed, as we should expect, he often speaks of the "nature and properties" of denoted objects instead of the "nature and properties" of denoting phrases. Moreover, Russell's official terminology does not conform rigidly to the scheme I have proposed. Russell does not call the denotations of "any" phrases *constant* conjunctions, as my interpretation suggests he should, since I claim that he treats "any" phrases as having wide scope in sentences in which they occur. Instead he calls them *variable* conjunctions. Similarly, he calls the denotations of "every" phrases *propositional* instead of variable conjunctions. These discrepancies need to be explained.

I believe that Russell misstates his own theory. My interpretation locates the *point* of Russell's distinctions between denoting phrases of the same quantificational force in a general feature of sentences containing such phrases, namely their scope ambiguity.[13] So it is vital that I account for what I take to be Russell's misstatement of his own theory, and that I show that the overwhelming majority of his examples, as well as the general thrust of his discussion, both argue strongly in favor of my interpretation.

3 Analysis of the text My interpretation of Russell's analysis of sentences containing denoting phrases is in conflict with the text at two points. First, on my reading of it Russell's theory gives wide scope to "any" phrases and not to "every" phrases. Therefore a sentence containing an "any" phrase is generally equivalent to a conjunction of sentences, while one containing an "every" phrase is not generally equivalent to a conjunction of sentences. But Russell tells us that he calls the object denoted by an "every" phrase a *propositional* conjunction because "the proposition in which it occurs is equivalent to a conjunction of propositions" (p. 57). Second, Russell tells us that an "any" phrase "seems half-way between a conjunction and a disjunction" (p. 57), and accordingly claims that "*any* $\underline{a}$ denotes $\underline{a}_1$ or $\underline{a}_2$ or . . . or $\underline{a}_n$" (p. 59) – supposing that "$\underline{a}_1$, $\underline{a}_2, \ldots, \underline{a}_n$" is a complete list of proper names for the $\underline{a}$'s. But as I have presented it, in Russell's theory an "any" phrase should denote some sort of *conjunctive* entity.

In both cases, I believe, Russell's claims are the result of a bad choice of examples. Consider first the claim that a sentence containing an "every" phrase is equivalent to a conjunction of sentences. I pointed out in Section 2 that if "$G(\ )$" is a noncomplex predicate, then "$G(\text{any } A)$" and "$G(\text{every } A)$" will be equivalent: so in particular "$G(\text{every } A)$" will – in this special case – be equivalent to a conjunction of sentences. Unfortunately, when Russell first discusses the role of "every" phrases in sentences, his choice of a canonical example is a sentence of just this type (p. 56), and so he is led to a poor choice of terminology.

Similarly, Russell's initial choice of a sentence containing an "any" phrase explains his claim that "*any* $\underline{a}$ denotes $\underline{a}_1$ or $\underline{a}_2$ or . . . or $\underline{a}_n$". As his canonical example Russell chooses a conditional whose antecedent contains an "any" phrase (p. 56). Russell reads a sentence of the form

(15) If $F(\text{any } A)$, then p.

this way:

(16) $(x)(\text{If } A(x), \text{ then if } F(x) \text{ then } p)$.

Suppose that "$\underline{a}_1$, $\underline{a}_2, \ldots, \underline{a}_n$" is a complete list of proper names for the A's. Then (16) is equivalent to

(17) If $F(\underline{a}_1)$ then p, and if $F(\underline{a}_2)$ then p, and . . . and if $F(\underline{a}_n)$ then p.

(using $A(\underline{a}_i)$), and in turn this is equivalent to

(18) If $F(\underline{a}_1)$ or $F(\underline{a}_2)$ or . . . or $F(\underline{a}_n)$, then p.

But for Russell (18) is equivalent to

(19) If $F(\underline{a}_1$ or $\underline{a}_2$ or . . . or $\underline{a}_n)$, then p.

Hence his claim, and so too his decision to call the object denoted by an "any" phrase a *variable* conjunction.[14]

In both cases, then, an unfortunate initial choice leads Russell to false generalizations. But Russell *does not go on to apply* these generalizations to the examples he gives as illustrations of his analysis of sentences containing denoting phrases (pp. 59–61). So it seems proper to conclude that Russell misstates his theory, and that my interpretation corrects his faulty formulation.

A full proof of this claim of course depends on a careful examination of Russell's illustrations, to which we now turn. First, an explanation of Russell's set-theoretic terminology.[15] Russell uses "belongs to" to mean "is an element of", "is contained in" to mean"is a subset of", "the logical sum of" to mean "the union of" (given sets), "the logical product of" to mean "the intersection of" (given sets), "common part" to mean "common elements", and "term of" to mean "element of". Russell gives 32 examples, divided into three groups, the first of which is:

> (α) Let $\underline{a}$ be a class, and $\underline{b}$ a class of classes. We then obtain in all six possible relations of $\underline{a}$ to $\underline{b}$ from various combinations of *any*, $\underline{a}$ and *some*...
> (1) Any $\underline{a}$ belongs to any class belonging to $\underline{b}$, in other words, the class $\underline{a}$ is wholly contained in the common part or logical product of the various classes belonging to $\underline{b}$.
> (2) Any $\underline{a}$ belongs to a $\underline{b}$, i.e. the class $\underline{a}$ is contained in any class which contains all the $\underline{b}$'s, or, is contained in the logical sum of all the $\underline{b}$'s.
> (3) Any $\underline{a}$ belongs to some $\underline{b}$, i.e. there is a class belonging to $\underline{b}$, in which the class $\underline{a}$ is contained. The difference between this case and the second arises from the fact that here there is one $\underline{b}$ to which every $\underline{a}$ belongs, whereas before it was only decided that every $\underline{a}$ belonged to a $\underline{b}$, and different $\underline{a}$'s might belong to different $\underline{b}$'s.
> (4) An $\underline{a}$ belongs to any $\underline{b}$, i.e. whatever $\underline{b}$ we take, it has a part in common with $\underline{a}$.
> (5) An $\underline{a}$ belongs to a $\underline{b}$, i.e. there is a $\underline{b}$ which has a part in common with $\underline{a}$. This is equivalent to "some (or an) $\underline{a}$ belongs to some $\underline{b}$."
> (6) Some $\underline{a}$ belongs to any $\underline{b}$, i.e. there is an $\underline{a}$ which belongs to the common part of all the $\underline{b}$'s, or $\underline{a}$ and all the $\underline{b}$'s have a common part. (pp. 59–60)

Russell's use of "$\underline{a}$" and "$\underline{b}$" in these examples is imprecise, since they are sometimes used as names and sometimes used as general terms. In what follows I will therefore replace them with "A" and "B", treated as constants naming a set and a set of sets respectively. I will also treat $\phi\psi$ as a notational variant of $\psi \in \phi$.

In the notation of first-order logic Russell's analyses of (1) and (2) should be rendered as follows:

(1*) $(x)(Ax \supset (y)(By \supset yx))$
(2*) $(x)(Ax \supset (\exists y)(By \mathbin{\&} yx))$

as his paraphrases make clear. Notice that in (2*) the "any" phrase has been given wide scope over the "a" phrase. But (3) is

(3*) $(\exists x)(Bx \mathbin{\&} (y)(Ay \supset xy))$,

giving "some" wide scope over "any". Russell's explanation of the difference between (2) and (3) is precisely along these lines. (4) is

(4*) $(x)(Bx \supset (\exists y)(Ay \mathbin{\&} xy))$

(as Russell's paraphrase makes clear), while (6) – in which "an" is replaced by "some" in (4) – is analyzed as

(6*) $(\exists x)(Ax \mathbin{\&} (y)(By \supset yx))$.

Again "some" is given wide scope over "any", and "any" has wide scope over "an". Finally, Russell reads (5) as

(5*) $(\exists x)(Ax \mathbin{\&} (\exists y)(By \mathbin{\&} yx))$,

and correctly observes that we may replace "an" or "a" with "some" to get an equivalent sentence.

Russell next gives six more examples exactly along the lines of the first group. His explanation of the differences among the six cases is of interest, however, so I will reproduce them without comment:

> (β) Let $\underline{a}$, $\underline{b}$ be two series of real numbers; then six precisely analogous cases arise.
> (1) Any $\underline{a}$ is less than any $\underline{b}$, or, the series $\underline{a}$ is contained among numbers less than every $\underline{b}$.
> (2) Any $\underline{a}$ is less than a $\underline{b}$, or, whatever $\underline{a}$ we take, there is a $\underline{b}$ which is greater... It does not follow that some term of the series $\underline{b}$ is greater than all the $\underline{a}$'s.
> (3) Any $\underline{a}$ is less than some $\underline{b}$, or, there is a term of $\underline{b}$ which is greater than all the $\underline{a}$'s. This case is not to be confounded with (2).
> (4) An $\underline{a}$ is less than any $\underline{b}$, i.e. whatever $\underline{b}$ we take there is an $\underline{a}$ which is less than it.
> (5) An $\underline{a}$ is less than a $\underline{b}$, i.e. it is possible to find an $\underline{a}$ and a $\underline{b}$ such that the $\underline{a}$ is less than the $\underline{b}$...
> (6) Some $\underline{a}$ is less than any $\underline{b}$, i.e. there is an $\underline{a}$ which is less than all the $\underline{b}$'s. This was not implied in (4), *where the $\underline{a}$ was variable, whereas here it is constant.* ((p. 60)—the emphasis in (6) is mine.)

Russell's final group of examples helps distinguish "every" and "any" phrases. He gives twenty sentences with appended analyses:

> (γ) Let $\underline{a}$ and $\underline{b}$ be two classes of classes...
> (1) Any term of any $\underline{a}$ belongs to every $\underline{b}$, i.e., the logical sum of $\underline{a}$ is contained in the logical product of $\underline{b}$.
> (2) Any term of any $\underline{a}$ belongs to a $\underline{b}$, i.e., the logical sum of $\underline{a}$ is contained in the logical sum of $\underline{b}$.
> (3) Any term of any $\underline{a}$ belongs to some $\underline{b}$, i.e., there is a $\underline{b}$ which contains the logical sum of $\underline{a}$.
> (4) Any term of some (or an) $\underline{a}$ belongs to every $\underline{b}$, i.e., there is an $\underline{a}$ which is contained in the product of $\underline{b}$.
> (5) Any term of some (or an) $\underline{a}$ belongs to a $\underline{b}$, i.e., there is an $\underline{a}$ which is contained in the sum of $\underline{b}$.
> (6) Any term of some (or an) $\underline{a}$ belongs to some $\underline{b}$, i.e., there is a $\underline{b}$ which contains one class belonging to $\underline{a}$.
> (7) A term of any $\underline{a}$ belongs to any $\underline{b}$, i.e., any class of $\underline{a}$ and any class of $\underline{b}$ have a common part.
> (8) A term of any $\underline{a}$ belongs to a $\underline{b}$, i.e., any class of $\underline{a}$ has a part in common with the logical sum of $\underline{b}$.
> (9) A term of any $\underline{a}$ belongs to some $\underline{b}$, i.e., there is a $\underline{b}$ with which any $\underline{a}$ has a part in common.

(10) A term of an $\underline{a}$ belongs to every $\underline{b}$, i.e., the logical sum of $\underline{a}$ and the logical product of $\underline{b}$ have a common part.
(11) A term of an $\underline{a}$ belongs to any $\underline{b}$, i.e., given any $\underline{b}$, an $\underline{a}$ can be found with which it has a common part.
(12) A term of an $\underline{a}$ belongs to a $\underline{b}$, i.e., the logical sums of $\underline{a}$ and $\underline{b}$ have a common part.
(13) Any term of every $\underline{a}$ belongs to every $\underline{b}$, i.e., the logical product of $\underline{a}$ is contained in the logical product of $\underline{b}$.
(14) Any term of every $\underline{a}$ belongs to a $\underline{b}$, i.e., the logical product of $\underline{a}$ is contained in the logical sum of $\underline{b}$.
(15) Any term of every $\underline{a}$ belongs to some $\underline{b}$, i.e., there is a term of $\underline{b}$ in which the logical product of $\underline{a}$ is contained.
(16) A (or some) term of every $\underline{a}$ belongs to every $\underline{b}$, i.e. the logical products of $\underline{a}$ and $\underline{b}$ have a common part.
(17) A (or some) term of every $\underline{a}$ belongs to a $\underline{b}$, i.e. the logical product of $\underline{a}$ and the logical sum of $\underline{b}$ have a common part.
(18) Some term of any $\underline{a}$ belongs to every $\underline{b}$, i.e. any $\underline{a}$ has a part in common with the logical product of $\underline{b}$.
(19) A term of some $\underline{a}$ belongs to any $\underline{b}$, i.e. there is some term of $\underline{a}$ with which any $\underline{b}$ has a common part.
(20) A term of every $\underline{a}$ belongs to any $\underline{b}$, i.e. any $\underline{b}$ has a part in common with the logical product of $\underline{a}$. (pp. 60–61)

Russell's analyses of the twenty sentences are therefore as follows:

(1*) $(x)(y)((Ax \,\&\, xy) \supset (z)\ (Bz \supset zy))$,

that is

$$(x)(y)(z)((Ax \,\&\, xy) \supset (Bz \supset zy)) \ .$$

(2*) $(x)(y)((Ax \,\&\, xy) \supset (\exists z)(Bz \,\&\, zy))$

which is to say, $\bigcup A \subseteq \bigcup B$. Notice that Russell gives "a $\underline{b}$" narrow scope.

(3*) $(\exists x)(Bx \,\&\, (y)(z)((Ay \,\&\, yz) \supset xz))$.

(3) results from (2) by substituting "some" for "a". Notice that "some" gets wide scope over "any".

(4*) $(\exists x)(Ax \,\&\, (y)(yx \supset (z)(Bz \supset zy)))$.

Russell analyzes (4) correctly. Notice that "some" gets widest scope, followed by "any", and "every" has smallest scope.

(5*) $(\exists x)(Ax \,\&\, (y)(xy \supset (\exists z)(Bz \,\&\, zy)))$.
(6*) $(\exists x)(\exists y)(Ax \,\&\, By \,\&\, (z)(xz \supset yz))$.

Russell analyzes (5) and (6) correctly.

(7*) $(x)(y)((Ax \,\&\, By) \supset (\exists z)(xz \,\&\, yz))$.

The "any" phrases have to be given wide scope over the "a" phrase.

(8*) $(x)(Ax \supset (\exists y)(\exists z)(By \,\&\, (xz \,\&\, yz)))$.

Notice, as Russell's paraphrase makes clear, that the "any" phrase is to be given wide scope over the "a" phrases.

(9*) $(\exists x)(Bx \,\&\, (y)(Ay \supset (\exists z)(yz \,\&\, xz)))$.

As Russell's paraphrase makes clear, (9) must be understood so that "some" has widest scope, and "any" has wide scope over "a".

(10*) $(\exists x)(\exists y)(Ax \,\&\, (xy \,\&\, (z)(Bz \supset zy)))$.

That is, an element of an element of A is an element of each element of B: $\bigcap A \cap \bigcup B \neq \varnothing$, as Russell claims. The "every" phrase has smallest scope.

(11*) $(x)(Bx \supset (\exists y)(\exists z)(Ay \,\&\, (yz \,\&\, xz)))$.

So the "any" phrase has wide scope over the "a" phrases. Notice that (11) results from (10) by substituting "any" for "every". The result is a sentence 'about' any B.

(12*) $(\exists x)(Ax \,\&\, (\exists y)(xy \,\&\, (\exists z)(Bz \,\&\, zx)))$.
(13*) $(x)((y)(Ay \supset yx) \supset (z)(Bz \supset zx))$.

Russell analyzes (12) and (13) correctly.

(14*) $(x)((y)(Ay \supset yx) \supset (\exists z)(Bz \,\&\, zx))$.

This is the correct translation for Russell's paraphrase, and as expected it gives widest scope to "any".

(15*) $(\exists x)(Bx \,\&\, (y)((z)(Az \supset zy) \supset xy))$.

This is the correct translation of Russell's paraphrase. The "some" phrase gets widest scope. Notice that (15) results from (14) by substituting "some" for "a", producing a quantifier shift.

(16*) $(\exists x)((y)(Ay \supset yx) \,\&\, (z)(Bz \supset zx))$.

Both "a" and "some" phrases have wide scope over "every" phrases.

(17*) $(\exists x)(\exists y)(By \,\&\, (yx \,\&\, (z)(Az \supset zx)))$.
(18*) $(x)(Ax \supset (\exists y)(xy \,\&\, (z)(Bz \supset zy)))$

is the correct translation of Russell's paraphrase. Russell gets this wrong. His paraphrase is the proper analysis of

(18′) A term of any $\underline{a}$ belongs to every $\underline{b}$,

and not of (18), which is equivalent to (16) with "some" for "a".[16]

(19*) $(\exists x)(Ax \,\&\, (y)(By \supset (\exists z)(xz \,\&\, yx)))$.

Russell gets this right. As his paraphrase shows, "some" is to be given widest scope, and "any" has wide scope over "a".

(20*) $(x)(Bx \supset (\exists y)((z)(Az \supset zy) \,\&\, xy))$.

Notice that "every" has smallest scope, "some" has widest scope.

Besides the case of example (18), Russell also makes mistakes in (4), (5), and (6). The paraphrases he gives are correct, but he seems to think that substituting "a" for "some" results in equivalent sentences. This is not right. Making those substitutions produces the following sentences:

(4′) Any term of an $\underline{a}$ belongs to every $\underline{b}$.
(5′) Any term of an $\underline{a}$ belongs to a $\underline{b}$.
(6′) Any term of an $\underline{a}$ belongs to some $\underline{b}$.

In Russell's terminology (4′) says that the logical sum of $\underline{a}$ is contained in the logical product of $\underline{b}$: $\bigcup A \subseteq \bigcap B$. But then the translation of (4′) would be

(4′*) $(x)((\exists y)(Ay \ \& \ yx) \supset (z)(Bz \supset zx))$,

which is equivalent to (1*), as it should be. (5′) says that the logical sum of $\underline{a}$ is contained in the logical sum of $\underline{b}$: $\bigcup A \subseteq \bigcup B$. Thus the translation of (5′) should be

(5′*) $(x)((\exists y)(Ay \ \& \ yx) \supset (\exists z)(Bz \ \& \ zx))$,

which is equivalent to (2*). Finally, (6′) says that some element of $\underline{b}$ contains the logical sum of $\underline{a}$: $\bigcup A \in b'$, for some $b' \in B$. So the translation of (6′) ought to be

(6′*) $(\exists x)(Bx \ \& \ (y)((\exists z)(Az \ \& \ zy) \supset xy))$,

which is equivalent to (3*).

This painstaking and somewhat laborious examination of Russell's illustrations of his theory answers one question, but perhaps raises another. It does show that in virtually all cases Russell follows the scope rules I have given. There are just four exceptions, and in three of these, (4)–(6), Russell correctly analyzes the sentence in question, but incorrectly supposes it to be equivalent to the sentence produced when "an" is substituted for "some". It is clear that Russell is mistaken by his own standards. The scopes given the denoting phrases in (14) and (20) argue in favor of translating (4′) as (4′*) and not as (4*); those given the denoting phrases in (8) and (11) argue that (5′*) is the correct translation of (5′); and those given the denoting phrases in (9) and (19) argue for (6′*) as the translation of (6′). It is likely that Russell was simply careless here.[17] And it therefore seems plausible to conclude that Russell is remarkably sensitive to the scope distinctions marked by the use of different denoting phrases, and that he understands these distinctions along the lines given in the previous section. Now I have sought to account for these distinctions by speaking of the scope of denoting phrases, and by translating into the predicate calculus. Yet Russell does not talk about the scope of denoting phrases, and in analyzing the illustrations reproduced above he does not translate into (even an informal version of) the predicate calculus. It must be granted that Russell *responds* to scope and quantificational differences among denoting phrases; but it may well seem that attributing to him a *theory* about these things is anachronistic. Doesn't the interpretation I have been recommending simply read too much back into Russell's work in *PoM*?

It seems to me that it does not, and for two reasons. While neither of these can be pursued here in any great detail, the question is sufficiently important to warrant at least a schematic answer.

First, Russell's analysis in "On Denoting" [18] of sentences containing denoting phrases is generally held to show that he has a workable version of quantification theory at his disposal, and therefore the resources with which to make the kinds of distinctions I have suggested. But in *PoM* Russell proposes an analysis of such sentences which is strikingly similar to that developed in "On Denoting".[18] Consider, for instance, the following revealing passage:

> explicit mention of *any*, *some*, etc., need not occur in Mathematics: formal implication will express all that is required. Let us recur to an instance already discussed in connection with denoting, where $\underline{a}$ is a class and $\underline{b}$ a class of classes. We have "Any $\underline{a}$ belongs to any $\underline{b}$" is equivalent to "'$\underline{x}$ is an $\underline{a}$' implies that '$\underline{u}$ is a $\underline{b}$' implies '$\underline{x}$ is a $\underline{u}$'"; "Any $\underline{a}$ belongs to a $\underline{b}$" is equivalent to "'$\underline{x}$ is an $\underline{a}$' implies 'there is a $\underline{b}$, say $\underline{u}$, such that $\underline{x}$ is a $\underline{u}$"; "Any $\underline{a}$ belongs to some $\underline{b}$" is equivalent to "there is a $\underline{b}$, say $\underline{u}$, such that '*x* is an $\underline{a}$' implies '$\underline{x}$ is a $\underline{u}$'"; *and so on for the remaining relations considered in Chapter V.* (pp. 89–90, the emphasis in the closing sentence is mine.)

Apart from certain notational eccentricities,[19] the analyses Russell gives are quite respectable translations into (informal) predicate logic, and accord very well with the translations given above. The claim that sentences containing denoting phrases are equivalent to their quantificational paraphrases is repeated more than once in *PoM*: *cf.* pp. 36, 91–92, 264–265. All this strongly suggests that Russell already has the logical resources later prominent in "On Denoting";[20] but something more can and should be said.

It is of course undeniable that Russell's use of these resources in *PoM* is less rigorous and less systematic than in "On Denoting"; but this does not, in and of itself, tell against its accuracy. While it is difficult to say what we should look for in deciding whether a logician has an adequate grasp of quantification, two key steps do stand out. The first is the explicit recognition of the fact that the linear ordering of words in a sentence is not a good guide to its interpretation. The second is the formulation of some analogue of Frege's insight that complex predicates – the *Begriffsschrift's* functions – are formed from sentences by deleting occurrences of a proper name. It is this insight that makes it possible to understand sentences containing expressions of generality as built up in stages; and it is precisely by treating such sentences as built up in stages that an intuitively satisfactory (even if informal) account of quantification is made possible.

Now Russell sees all this. He recognizes that the linguistic ordering of expressions may misrepresent their logical ordering (*cf.* p. 30). He recognizes that predicates are formed by deleting occurrences of proper names in sentences: this is the method whereby he forms *assertions* (pp. 39, 83–84), and, when variables are inserted, *propositional functions* (e.g., p. 84). Consequently he takes quantification to be the application of a variable-binding sentential operator to propositional functions, turning 'real' variables into 'apparent' variables, and so open sentences into closed sentences (p. 13). Finally, he insists that it is vital that we be able to keep track of the stages whereby a multiply general sentence is formed (*cf.* the argument on pp. 83–85), and therefore insists that the formalism adopted must have devices to mark clearly the successive stages of introduction for the appropriate quantifiers (p. 94). In sum, Russell has all the resources needed to develop an intuitively satisfactory account of the predicate calculus.[21]

I do not mean to suggest that Russell is completely clear on all the issues involved here: in several important areas he is badly confused. For instance, as is evident in his discussion of propositional logic (pp. 13–18), Russell tends to run his object and metalanguages together, confusing expressions *of* the formal language and formation and inference rules *for* the language. Similarly, with respect to the theory of denoting, Russell betrays a tendency to build into his formal theory informal rules telling us how to correlate that theory with ordinary-language sentences. This is a serious error: the scope or precedence rules proposed in Section 2 above and defended in the present section are just rules telling us *how to translate* an English sentence in our target language. They are not rough or vague syntactic rules *for* the target language. But confusions of this type do not eliminate what is right in Russell's account; and what is right in Russell's account is sufficient to show that in his analysis of the examples he considers Russell is illustrating a logically cogent *theory* which marks scope and quantificational differences among denoting phrases, along the lines I have indicated.

4 Formalizing Russell's theories: The language L^{**} I have argued that Russell's theory of denoting responds to two features of denoting phrases: their quantificational force and their scope in the sentences in which they occur. Now first-order logic in its standard formulation is entirely adequate to represent these features in a clear and precise manner. The fact is, however, that first-order logic in its standard formulation is not a good choice of language to formalize Russell's theory. To begin, as we have seen, Russell himself considers the (standard) quantificational paraphrase of sentences containing denoting phrases, and rejects it. He concedes that the quantificational paraphrase is equivalent to the original English sentence, but he also insists that the two are not synonymous.

There are several reasons why Russell held this view. For one thing, at the time of *PoM* Russell thought that an account of the quantifiers presupposed his theory of denoting. So he might have argued that standard first-order logic depends on a proper account of sentences containing denoting phrases. Secondly, Russell considered denoting phrases to be *names*, and analyzed sentences containing denoting phrases as expressing propositions which are *about* the objects denoted by those phrases. To capture this part of Russell's theory what is needed is a language in which denoting phrases are treated as terms, and not paraphrased away in context by means of quantification.

In this section I develop the syntax for a language of this sort. It must be emphasized that Russell himself has no such language in *PoM.* I have claimed that Russell has an adequate grasp of quantification theory, and a sound albeit incomplete understanding of its formalization. I do not mean to suggest that he also has a different formalization of first-order logic in mind, a formalization in which denoting phrases are treated as terms. The language L^{**}, which I develop in this section, should be thought of as representing and extending his intentions, as formalizing his *informal* account of English sentences containing expressions of generality, and not as making *tidier* a formalization he already has. Accordingly in developing L^{**} I will choose, among Russell's claims, those that most centrally express his considered view.

Now Russell's 1902 theory treats denoting phrases as having both an indication and a denotation, and therefore seems to call for a higher-order formalization. But as we have seen the application Russell made of his theory of denoting, to the analysis of sentences containing denoting phrases, antedates the late May 1902 formulation of the theory, and comes from the earlier account, according to which denoting phrases stand for objects. The result is that Russell's treatment of sentences containing denoting phrases has no specially close connection with the thesis that denoting phrases mean concepts; and it is better represented in a first-order manner.

A second complication can be dealt with in a similar way. According to the full-blown official theory, each denoting phrase (except a definite description) denotes a different type of complex or combination of objects. I have argued that Russell's principal reason for this conclusion is his reaction to the scope and quantificational differences he detected in sentences containing such phrases. This being so, it follows that in formalizing his account we need to posit no more than two types of complex term, namely a 'conjunctive' and a 'disjunctive' term, and so only two kinds of term-forming operator, one for each type of term.[22] The legitimacy of this conclusion is supported both by Russell's admission that denoting phrases are interdefinable (p. 92), and by his move away from the official theory.

The language L^{**} is derived from a notational variant of a language developed by Stalnaker [20], and obtained by him from work by Stalnaker and Thomason [21], subsequently influenced by Dummett [7]. L^{**} differs from the standard formulation of first-order logic in several respects. First, it has a predicate-forming sentential operator, *. Next, it has three term-forming predicate operators: T (which is meant to form terms with universal, 'conjunctive' force); Q (which forms terms with existential, 'disjunctive' force); and ι (which represents the definite description operator). Finally, it treats the universal quantifier as a sentence-forming predicate operator, instead of the standard variable-binding sentential operator. The syntax of L^{**} is as follows:

The Language L^{**}:
Alphabet:
Primitive Terms: a denumerable set of individual variables; a denumerable set of individual constants.
Primitive Predicates: for each $n \geq 1$, a denumerable set of n-place predicates.
Logical Constants: $\sim$, $\supset$, $\forall$, *, $=$, T, Q, ι.
Grouping Indicators: (,).

Complex Expressions:
Sentences:
1. If F is an n-place predicate and $t_1, \ldots, t_n$ are primitive terms, then $Ft_1 \ldots t_n$ is a sentence.
2. If F is a one-place predicate and t is a complex term, then Ft is a sentence.

3. If t_1, t_2 are primitive terms, then $t_1 = t_2$ is a sentence.
4. If A, B are sentences, then so are $\sim A$, $(A \supset B)$.
5. If F is a one-place predicate, then $\forall F$ is a sentence.

Predicates: If A is a sentence and x is a variable, then x^*A is a one-place predicate.

Terms: If F is a one-place predicate, then TF, QF, and ιF are terms.

There are no other complex expressions.

The *scope* of x^* in an expression of the form x^*A, where A is a sentence and x is a variable, is the entire expression x^*A. A variable x *occurs free* in a sentence A iff x occurs in A and at least one occurrence of x in A is not within the scope of an occurrence of x^* in A.

Semantics for the Official Theory:

I isolate four key claims in Russell's official theory:

01. Denoted objects are 'complexes' or 'combinations' of *just* the terms in the extension of the corresponding predicate.
02. There are *different* types of denoted object associated with a given predicate.
03. Denoting phrases are *names.*
04. Predication is to be treated *uniformly*: a uniform analysis is required, whether the grammatical subject be a proper name or a denoting phrase.

There is no way to keep all four claims. There is, for instance, a simple way to have the values of T and Q expressions be 'complexes' of just the elements in the extension of the one-place predicates to which the operators apply (01); let an interpretation I assign to expressions of the form TF and QF the extension (under I) of the predicate F. But this strategy has a price: paired T and Q terms are assigned the same value (violating 02). Further, to get the right truth conditions predication has to be treated asymmetrically (violating 04). As usual, a sentence of the form $G\underline{a}$ is true just in case the value assigned to the constant $\underline{a}$ by an interpretation I is an element of the value assigned by the interpretation I to the predicate G. By contrast a sentence of the form GTF—say, where for simplicity we may take F and G to be primitive one-place predicates—is true under I just in case the value of TF under I is a *subset* of the value of G under I.

The semantics for the official theory proposed in this section gives a uniform analysis of predication sentences, and assigns different values to T and Q terms. To do this the key is to *counterfeit* the quantificational force of sentences containing T and Q terms by an appropriate choice of values for these terms, and an appropriate analysis of predication. The key idea is derived from Frege's treatment of numbers and of predication of number, as adapted by Montague [11]. Instead of assigning the extension of the one-place predicate F as the value under I of the terms TF and QF, we will assign to TF the set of all supersets of the value of F under I; and we will assign to QF the set of all sets having nonnull intersection with the value of F under I. As worked out below, this semantics for the official theory gives a sense in which denoted objects are complexes of terms; it assigns different values to a one-place predicate and the various complex terms formed from that predicate; it treats predication uniformly; and it has individual variables be complexes of terms. But it fails to restrict the values of T and Q terms to complexes of *just* the elements in the extension of the predicate to which the operators apply, violating 01. This is a serious departure from Russell's official theory, but the resulting semantics captures most of the distinctive features of that theory in a cogent and elegant manner. The details are as follows:

I. A *model* is a pair $M = \langle D, g\rangle$ consisting of a nonempty set D together with a mapping g which takes individual constants into D and primitive n-place predicates into subsets of D^n.

II. A *sequence function* s (on a model M) is any function from the set of primitive terms into D satisfying the condition that for each individual constant t, $s(t) = g(t)$. If s is a sequence function, x a variable and d an element of D, let $s(d/x)$ be the sequence function which agrees with s except that it assigns d to x.

An interpretation I of L^{**} is a function assigning values to the terms, predicates, and sentences of L^{**} relative to a model M and a sequence function s according to the following rules:

1. If F is a primitive predicate, then $I(s, F) = g(F)$.
2. If t is a primitive term, then $I(s, t) = \{D' \subseteq D\colon s(t) \in D'\}$; if t is a complex term of the form ιG, then $I(s, t) = \{D' \subseteq D\colon d \in D'\}$ if for some $d \in D$, $I(s, G) = \{d\}$, and $I(s, t) = \varnothing$ otherwise; if t is a complex term of the form TG, then $I(s, t) = \{D' \subseteq D\colon I(s, G) \subseteq D'\}$; if t is a complex term of the form QG, then $I(s, t) = \{D' \subseteq D\colon I(s, G) \cap D' \neq \varnothing\}$.
3. If A is a sentence of the form $Ft_1 \ldots t_n$ (where F is an n-place predicate and each t_i is a primitive term), then $I(s, A) = 1$ iff $I(s, F) \in \{D' \subseteq D^n\colon \langle s(t_1), \ldots, s(t_n)\rangle \in D'\}$; if A is a sentence of the form Ft (where F is a one-place predicate and t is a complex term), then $I(s, A) = 1$ iff $I(s, F) \in I(s, t)$.
4. If A is a sentence of the form $t_1 = t_2$, then $I(s, A) = 1$ iff $I(s, t_1) = I(s, t_2)$.
5. If A is a sentence of the form $\sim B$ then $I(s, A) = 1$ iff $I(s, B) = 0$; if A is a sentence of the form $(B \supset C)$, then $I(s, A) = 1$ iff $I(s, B) = 0$ or $I(s, C) = 1$.
6. If A is a sentence of the form $\forall F$, then $I(s, A) = 1$ iff $I(s, F) = D$.
7. If A is a sentence, $I(s, A) = 0$ iff $I(s, A) \neq 1$.
8. If F is a complex predicate of the form x^*A, then $I(s, F) = \{d \in D\colon I(s(d/x), A) = 1\}$.

The alphabet of L^{**} is of course redundant: there are more logical constants than necessary. In particular however notice that the universal quantifier $\forall$ can be eliminated from L^{**}, since

(20) $\forall F$ is logically equivalent to $FTx^*\!\sim Fx$,

where the variable x does not occur free in the one-place predicate F [(20) says that everything is an F iff every non-F is an F]. So (unrestricted) quantification is reducible to denoting and predication, a point used by Russell in *PoM* to justify the priority of denoting. The present semantics for L^{**} suffices to substantiate this claim.

Less surprising is the fact that L^{**} is equivalent to the language L^*, which is obtained by eliminating the symbols T, Q, and ι from the alphabet of L^{**}, and deleting rule 2 for sentences of L^{**} together with the clause defining complex terms of L^{**}. L^{**} is equivalent to L^* in the sense that there is a mapping from L^{**} into L^* which associates with each sentence of L^{**} a sentence of L^* that comes out true relative to the same models and sequence functions. Any sentence of L^{**} not containing complex terms is already a sentence of L^*. An adequate translation in L^* of a sentence of L^{**} of the form $G\iota F$ is

$$\sim\forall x^*(Fx \supset (Gx \supset \sim\forall\, y^*(Fy \supset x = y))) \ ,$$

where x is the first variable new to $G\iota F$.[23] (This is just the Russellian analysis of definite descriptions formulated in the notation of L^*.) An adequate translation in L^* of a sentence of L^{**} of the form GTF is

$$\forall x^*(Fx \supset Gx) \ ,$$

where x is the first variable new to GTF. And an adequate translation in L^* of a sentence of L^{**} of the form GQF is

$$\sim\forall x^*(Fx \supset \sim Gx) \ ,$$

where x is the first variable new to GQF.

L^{**} and the standard formulation of first-order logic are also equivalent in this sense. The mapping from standard first-order logic into L^{**} is simple: place a star* after any variable immediately preceded by the quantifier $\forall$. The mapping from L^{**} into standard first-order logic is more complicated because of the presence of complex predicates and terms. The following rules work: reading from left to right,

1. Eliminate the first occurrence of a complex term, following the schemata given above.
2. Delete the first occurrence of a starred variable x^* not immediately preceded by the quantifier $\forall$.
3. Replace each occurrence of x in the scope of that occurrence of x^* by the term t immediately following the sentence to which x^* was prefixed.
4. Delete that occurrence of t.
5. Delete all remaining occurrences of the star *.
6. If the quantifier $\forall$ immediately precedes a one-place predicate F, insert occurrences of the first variable x new to the whole sentence immediately after the quantifier and immediately after the predicate.

Rules 1–4 are to be applied serially, and rules 5 and 6 are to be applied only when rules 1–4 can no longer be applied.[24]

(Although in this sense L^{**} and the standard formulation of first-order logic have the same expressive power, there is another sense in which the two languages are not equivalent. L^{**} and the standard formulation of first-order logic assign different logical forms to sentences containing denoting phrases. As an example, consider sentences (3), (4), and (5):

(3) Frege does not despise Russell.
(4) Frege does not despise any logician.
(5) Frege does not despise every logician.

Letting Fxy translate "x despises y", G translate "is a logician", and $\underset{\sim}{a}$ and $\underset{\sim}{b}$ translate "Frege" and "Russell" respectively, we can translate these sentences as follows:

(3**) $\sim F\underset{\sim}{a}\underset{\sim}{b}$
(4**) $x^* \sim F\underset{\sim}{a}\underset{\sim}{x}TG$
(5**) $\sim x^* F\underset{\sim}{a}\underset{\sim}{x}TG$.

So (3**) and (5**) are negation sentences while (4**) is a predication sentence.)

Semantics for the Unofficial Theory:
According to Russell's unofficial theory of denoted objects,

(U1) A denoting phrase 'ambiguously denotes' a term in the extension of the corresponding predicate instead of denoting a complex of all the terms in the extension of that predicate.

Once again

(U2) Only terms in the extension of the appropriate predicate are involved.
(U3) Denoting phrases are names.
(U4) Predication is to be treated uniformly.

(Notice that the variable is not a complex of terms in this version of the theory.)

I represent the idea of *ambiguous denotation* in the following version of the semantics for L^{**} by making a provisional assignment to T and Q terms. (The basic idea is adapted from van Fraassen [22].)

To the definitions of a model and a sequence function we add

III. Given a model $M = \langle D, g\rangle$, let a *selection function* f be any function from the power set $P(D)$ into D, such that for each $C \in P(D)$, $h(C) \in C$. (So h is a partial function, not defined for the empty set.) If h is a selection function and C a nonempty subset of D, then $h(d/C)$ is the selection function that agrees with h except that it assigns $d \in D$ to C.

An interpretation I of L^{**} is a function assigning values to the terms, predicates and sentences of L^{**} relative to a model M, a sequence function s and a selection function h, according to the following rules:

1. *If* F is a primitive predicate, then $I(h, s, F) = g(F)$.
2. If t is a primitive term, then $I(h, s, t) = g(t)$; if t is a complex term of the form ιG, then $I(h, s, t) = d$ if $I(h, s, G) = \{d\}$, and is undefined otherwise; if t is a complex term of the form TG, then $I(h, s, t) = h(I(h, s, G))$; if t is a complex term of the form QG, then $I(h, s, t) = h(I(h, s, G))$. (So if $I(h, s, G) = \varnothing$, then $I(h, s, TG)$ and $I(h, s, QG)$ are undefined.)
3. If A is a sentence of the form $Ft_1 \ldots t_n$ (where F is an n-place predicate and each t_i is a primitive term), then $I(h, s, A) = 1$ iff $\langle I(h, s, t_1), \ldots,$

$I(h, s, t_n)\rangle \in I(h, s, F)$; if A is a sentence of the form $F\iota G$, then $I(h, s, A) = 1$ iff $I(h, s, G)$ is defined and $I(h, s, G) \in I(h, s, F)$; if A is a sentence of the form FTG then $I(h, s, A) = 1$ iff either $I(h, s, TG)$ is undefined or for all $d \in I(h, s, G)$, $I(h(d/I(h, s, G))$, $s, TG) \in I(h, s, F)$; if A is a sentence of the form FQG, then $I(h, s, A) = 1$ iff $I(h, s, QG)$ is defined and for some $d \in I(h, s, G)$, $I(h(d/I(h, s, G)), s, QG) \in I(h, s, F)$.

4. If A is a sentence of the form $t_1 = t_2$, then $I(h, s, A) = 1$ iff $I(h, s, t_1) = I(h, s, t_2)$.
5. If A is a sentence of the form $\sim B$, then $I(h, s, A) = 1$ iff $I(h, s, B) = 0$; if A is a sentence of the form $(B \supset C)$, then $I(h, s, A) = 1$ iff either $I(h, s, B) = 0$ or $I(h, s, C) = 1$.
6. If A is a sentence of the form $\forall F$, then $I(h, s, A) = 1$ iff $I(h, s, F) = D$.
7. If A is a sentence, then $I(h, s, A) = 0$ iff $I(h, s, A) \neq 1$.
8. If F is a complex predicate of the form x^*A, then $I(h, s, F) = \{d \in D: I(h, s(d/x), A) = 1\}$.

A sentence of L^{**} is assigned the same value relative to any model M, sequence function s, and all selection functions h under this definition of I and under the previous definition of I for the official theory.[25] So Russell's unofficial theory of denoting can also be given a precise and rigorous formalization, once again showing the cogency of Russell's application of the theory of denoting to sentences containing denoting phrases.[26]

NOTES

1. Coffa [4] and Geach [8] are notable exceptions, though Geach [8] contains important errors. Interpretation of *PoM* is made even more difficult by Russell's tendency – particularly prominent in this pivotal and transitional work – to use key technical terms of his theory ambiguously: "proposition", "propositional function", "assertion", "constant", "variable", "meaning", "denoting", *inter alia*, are all of this sort. In the text I attempt to minimize the confusion this sort of ambiguity produces by formulating Russell's views in a vocabulary more narrowly and precisely defined, and more consistently used, than is Russell's own practice. For example, in the text I always use the term "proposition" to mean the nonmental (and nonlinguistic) entities which, Russell tells us, alone are essentially true or false (p. xix); and I always use "sentence" to mean the linguistic forms that express these entities. An unavoidable consequence of this strategy is that my formulations of Russell's views sometimes diverge significantly from the literal text.

2. Blackwell [1] is an important source of information about the composition of *PoM*. My conclusions draw significantly from his paper.

3. In *PoM* Russell never claims that sentences indicate propositions, though he speaks about the propositions expressed by symbols. As was first pointed out by Cassin [3], p. 258, Russell makes the connection between sentences and propositions clear in another work, [17], p. 54.

4. Ultimately, the distinction between linguistic and logical relations depends on a contrast between linguistic and nonlinguistic *facts*, and is therefore no clearer or

more precise than that contrast. But the intuitive core to which Russell is appealing in making the distinction is simple enough. First, unlike a logical relation a linguistic relation invariably holds between linguistic entities (sentences and subsentential components, whether as types or tokens) and other entities. Second, the *relata* in a linguistic relation are related extrinsically and arbitrarily, or more accurately, *conventionally*; whereas a logical relation holds between its *relata* "inherently and logically" (p. 53). The fact that a certain name, say "twenty", stands in the indication (linguistic meaning) relation to a certain particular object—the number twenty—is the result of a convention embodied in the initiation and maintenance of a certain practice. The very same *name* might have stood in that relation to *something else*; the very same *number* might have been related to *another name* in the same way. By contrast, the *logical* meaning relation (paradigmatically the relation of a property to its extension) is *not* the result of convention, but depends instead solely on the nonlinguistic facts.

5. As can be gathered from Blackwell [1] and careful study of *PoM*. In print, Coffa [4] is the only text acknowledging this important point.

6. "Here the notion '$C(x)$ is always true' is taken as ultimate and undefinable, and the others are defined by means of it" [18], p. 105.

7. A nice illustration of Russell's indifference about which formulation to use is to be found on the opening page of "On Denoting": "A phrase may denote ambiguously; e.g. 'a man' denotes not many men but an ambiguous man" [18], p. 103. Incidentally, the use in this sentence of the terminology and resources of the unofficial theory of denoting is due to sloppiness on Russell's part. The entire point of "On Denoting" is to replace the theory of denoting in *PoM* with one that takes the denoting relation to be derivative from other notions.

8. In the text I have preferred to put things this way, and to speak of the application of the theory of denoting to the analysis of sentences containing denoting phrases. In *PoM* Russell just as often speaks of giving an analysis of denoting concepts, or of the propositions containing such concepts. (We must be cautious in interpreting such claims, however. The use of the words "proposition" and "concept" need not indicate an intention on Russell's part to speak precisely about nonlinguistic entities since, as noted above, he uses these and other related expressions ambiguously.) There should in any case be no doubt about two points. First, Russell does frequently and explicitly speak of analyzing denoting words and phrases: Sections 58 and 59 of *PoM* provide a good illustration (pp. 55–56). Second, on *any* interpretation his account yields an analysis of denoting phrases and the sentences containing them, either *directly*, as in the 1901 formulation; or *indirectly*, as in the 1902 formulation, by means of an analysis of denoting concepts and the propositions containing them.

The matter is considerably clearer in "On Denoting" [18], where Russell once again takes up these issues in print. He tells us that denoting phrases are characterized by their grammatical form, and adds that "The interpretation of such phrases is a matter of considerable difficulty; indeed, it is very hard to frame any theory not susceptible of formal refutation. All the difficulties with which I am acquainted are met, so far as I can discover, by the theory which I am about to explain" [18], p. 103. This makes it clear that Russell's purpose in "On Denoting" is to develop a theory of denoting phrases and of the sentences containing them. I have preferred to characterize his objectives in *PoM* in similar terms, even though this glosses over the presence in the earlier work of alternative formulations using propositions and concepts.

I therefore see the account given in "On Denoting" as in a key respect like the 1901 formulation of the theory: in each case the hard work is to be done at the linguistic level. There are of course also profound differences between the two versions of the theory, and it may be useful to sketch some of these here. Roughly, Russell's applications of the theory of denoting in *PoM* can be grouped into three classes. First, there is the linguistic use: the theory provides an analysis of the truth-conditions for sentences containing denoting phrases. Second, the mathematical and logical uses: the theory is supposed to account for such things as classes, variables, formal implication (which we may think of as quantification theory) and so on. Finally, the metaphysical and epistemological uses: the theory analyses the nature and structure of certain propositions, accounts for the possibility of mathematical knowledge, gives a distinction between immediate and discursive knowledge, and so on. In "On Denoting" and later the first and third of these uses are still prominent. But the second is virtually absent, since by this time Russell has reversed his earlier stand and decided to use quantification theory to account for denoting (instead of the other way around). (Putting things this way risks an underestimation of Russell's remarkable accomplishment in "On Denoting": the third set of uses of the theory of denoting *hinges on the addition of denoting concepts*. In "On Denoting" Russell manages to *keep* all these applications of the theory without positing either the 1902 formulation's denoting concepts or the official theory's denoted objects.)

9. Two examples from *PoM*, the second steadily more prominent in Russell's work after 1903. First, it may well be that Russell thought that having denoting phrases mean denoting concepts could be of help in dealing with some of the consequences of the Contradiction, the Russell Paradox concerning non-self-membered classes. Citing Cassin [2], Blackwell [1], p. 10, considers this suggestion and concludes that there is no important application of the 1902 formulation of the theory of denoting to the problems created by the Contradiction. I am less certain. The text is obscure on this point, but it is clear that the Contradiction creates a major problem for the 1901 formulation of the theory. As I point out later in the text, one of the lessons Russell drew from the Contradiction is that some properties (class-concepts) determine an extension but no corresponding whole (i.e., set). Now according to the 1901 formulation a denoting phrase contributes a denoted object to the proposition expressed by a sentence in which it occurs. It follows that some sentences will express propositions containing infinitely complex objects (infinite extensions), and further, objects which *cannot* be considered as being in any sense single entities or wholes. But then propositions containing such objects are infinitely complex, and so not knowable by us (p. 145). Since propositions are the linguistic meanings of sentences, this in turn means that some (intuitively) meaningful sentences are, according to the theory, meaningless. The interpolation of denoting concepts as the meanings of denoting phrases clearly avoids this difficulty. A more detailed discussion of this topic is to be found in Dau [6], pp. 123–133.

   Second, the 1902 formulation of the theory of denoting permits us to distinguish direct and indirect knowledge; to distinguish, that is, what Russell was later to call knowledge by acquaintance and knowledge by description. ("On Denoting" contains an early formulation: "The distinction between *acquaintance* and *knowledge about* is the distinction between the things we have presentations of, and the things we only reach by means of denoting phrases." [18], p. 103). This distinction is certainly implicit in the *PoM* doctrine that a proposition expressed by a sentence containing a denoting phrase contains the indicated denoting concept but is *about* something else, namely a denoted object. And it certainly depends on not treating the referent of a denoting phrase as its linguistic meaning, an objective secured in

1902 by the introduction of denoting concepts, and in 1905 and later by treating such phrases as incomplete symbols.

10. As an example of these difficulties, Geach points to Russell's concession, in *PoM*, pp. 54–55, n., that "is" may be understood as expressing identity in sentences of the form "*a* is an *F*". Geach [8], p. 69, argues that this concession is bound to be problematic for Russell's official theory. I agree with Geach's conclusion; but Geach seems not to have noticed that in that passage Russell explicitly advances his unofficial theory in response to the puzzle created by allowing "is" to express identity in, e.g., "Socrates is a man". Russell's comment is that in that sentence "is" "expresses the identity of Socrates with an ambiguous individual" (*loc. cit.*). Geach gives no sign in his [8] of having noticed that the unofficial theory is a distinct version of Russell's theory of denoting and denoted objects.

11. Russell's claim (as I interpret it) about their disambiguating role seems considerably more plausible in the case of "every" and "any" than in that of "some" and "a". Certainly, sentences (9) and (10) in the text can reasonably be regarded as unambiguous; there will however likely be less agreement about (11) and (12). In particular, many will feel that (11) is still ambiguous.

12. Since $\vdash((x)(y)A(x, y) \equiv (y)\ (x)A(x, y))$,

$$\vdash((\exists x)\ (\exists y)A(x, y) \equiv (\exists y)\ (\exists x)A(x, y))\ .$$

13. My interpretation of Russell's analysis of sentences containing denoting phrases differs significantly from that proposed by Geach [8] in his interesting discussion of Russell's theory. The difference is largely due to a difference in emphasis. Russell's analysis of sentences containing denoting phrases can be broken down into four parts:

    A. Section 59, pp. 56–58 of *PoM:* "Extensional account of *all*, *every*, *any*, *a* and *some*" (xxiii). This section gives a preliminary defense of the thesis that there are different types of denoted object, and introduces the official theory's terminology.
    B. Section 60, pp. 58–59: "Intensional account of the same" (xxiii).
    C. Section 61. pp. 59–61: "Illustrations" (xxiii). This section begins with a tabular summary of the previous discussion, and continues with
    D. 32 examples, divided into three parts.

    Geach's interpretation gives relatively more importance to C, mine to D. One result is that we reach different conclusions about the *point* of Russell's analysis. In contrast with my interpretation, Geach [8], p. 78, does not recognize any attempt by Russell to give scope rules for the different denoting phrases, and indeed charges Russell with ignoring the scope of denoting phrases.

    Geach's account deserves a careful look. As indicated, Geach bases his interpretation on C, in which Russell summarizes the relevant distinctions made by his official theory. Russell writes that

    > "In cases where the class defined by a class concept has only a finite number of terms, it is possible to omit the class-concept wholly, and indicate the various objects denoted by enumerating the terms and connecting them by means of *and* or *or* as the case may be." (p. 56)

    And later in the text he writes,

    > "In the case of a class of $\underline{a}$ which has a finite number of terms — say $\underline{a}_1, \underline{a}_2, \underline{a}_3, \ldots, \underline{a}_n$, we can illustrate these various notions as follows:

(1) *All $\underline{a}$'s* denotes $\underline{a}_1$ and $\underline{a}_2$ and . . . and $\underline{a}_n$.
(2) *Every $\underline{a}$* denotes $\underline{a}_1$ and denotes $\underline{a}_2$ and . . . and denotes $\underline{a}_n$.
(3) *Any $\underline{a}$* denotes $\underline{a}_1$ or $\underline{a}_2$ or . . . or $\underline{a}_n$, where *or* has the meaning that it is irrelevant which we take.
(4) *An $\underline{a}$* denotes $\underline{a}_1$ or $\underline{a}_2$ or . . . or $\underline{a}_n$, where *or* has the meaning that no one in particular must be taken, just as in *all $\underline{a}$'s* we must not take any one in particular.
(5) *Some $\underline{a}$* denotes $\underline{a}_1$ or denotes $\underline{a}_2$ or . . . or denotes $\underline{a}_n$, where it is not irrelevant which is taken, but on the contrary some one particular $\underline{a}$ must be taken." (p. 59)

Geach [8], pp. 71–72, gives the following reconstruction.

> "If "$\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, . . ." is a complete list of proper names of the $\underline{A}$'s, then: "f(an $\underline{A}$)" is true iff "f($\underline{a}_1$ or $\underline{a}_2$ or $\underline{a}_3$ or . . .)" is true; "f(some $\underline{A}$)" is true iff "$f(\underline{a}_1)$ or $f(\underline{a}_2)$ or $f(\underline{a}_3)$ or . . ." is true; "f(any $\underline{A}$)" is true iff "$f(\underline{a}_1)$ and $f(\underline{a}_2)$ and $f(\underline{a}_3)$ and . . ." is true; "f(every $\underline{A}$)" is true iff "f($\underline{a}_1$ and $\underline{a}_2$ and $\underline{a}_3$ and . . .)" is true."

So according to Geach's reconstruction of Russell's account,

(E) "Frege does not despise a logician" is true iff "Frege does not despise $\underline{a}_1$ or $\underline{a}_2$ or . . . or $\underline{a}_n$" is true.
(F) "Frege does not despise some logician" is true iff "Frege does not despise $\underline{a}_1$ or Frege does not despise $\underline{a}_2$ or . . . or Frege does not despise $\underline{a}_n$" is true.
(G) "Frege does not despise any logician" is true iff "Frege does not despise $\underline{a}_1$ and Frege does not despise $\underline{a}_2$ and . . . and Frege does not despise $\underline{a}_n$" is true.

and

(H) "Frege does not despise every logician" is true iff "Frege does not despise $\underline{a}_1$ and $\underline{a}_2$ and . . . and $\underline{a}_n$" is true.

It follows from Geach's scheme that a sentence containing an "any" phrase is equivalent to a conjunction of sentences, while one containing an "every" phrase is not generally equivalent to a conjunction of sentences. My interpretation leads to the same conclusion. Both accounts are therefore at odds with what Russell tells us (p. 57). (I trace Russell's claim to a poor choice of initial example: the details are given in Section 3.) My interpretation also agrees with Geach's in its assignment of truth conditions to sentences containing denoting phrases.

In spite of these areas of agreement, I prefer my interpretation for several reasons. First, Geach's scheme places too much emphasis on Russell's tabular summary. Since this summary encapsulates Russell's initial errors in setting out his theory, and since Russell does not repeat these errors in his later discussion, such emphasis seems to me incorrect. Second, Geach's interpretation fails to remain neutral between Russell's official and unofficial accounts of denoted objects. Geach simply excludes Russell's unofficial theory: given Russell's uncertainty, this seems to me to be a mistake. Finally, Geach ignores Russell's warning that his tabular summary is applicable *only* when the extension of a class-concept $\underline{a}$ is finite. When a predicate has infinite extension, Russell claims that the method of enumerating the terms in its extension will not work (e.g., pp. 56, 59, 305–306, 349). Indeed, he claims that "the logical purpose which is served by the theory of denoting" (p. 145) is precisely to replace the enumerative scheme, on which Geach relies, when the predicate has infinite extension. So Geach's generalization of that scheme does not capture Russell's considered view, and may mislead on the purpose served by the theory of denoting.

14. Russell introduces his terminology for denoted objects in Section 56, which also includes his first defense of the claim that there are five different kinds of denoted object. As pointed out in the previous note, Russell considered it possible, when a predicate has finite extension, to indicate the various kinds of denoted object by using complex names instead of denoting phrases. He exploits this point in Section 56, by considering a case where the predicate in question ("suitor of Miss Smith") has a small extension (Brown and Jones). He gives five sentences using "only two forms of words, *Brown and Jones* and *Brown or Jones*", and which are supposed respectively to involve "the objects denoted by *all*, *every*, *any*, *a* and *some*" (p. 56). Therefore the explanation given in the text for how Russell comes to call the objects denoted by "every" and "any" phrases "propositional conjunctions" and "constant disjunctions" respectively, actually follows his discussion of these sentences, and makes use of his willingness to replace the complex names with the appropriate corresponding denoting phrase.

15. *Cf. PoM*, p. 60 and Geach [8], pp. 73 ff.

16. Blackwell [1], p. 7, points out that Russell added examples (11), (18), (19), and (20) very late, "in the proof-reading" of the final manuscript.

17. Geach [8], p. 77, comes to the same conclusion, writing that Russell's claim of equivalence is "clearly attributable to a mere slip on Russell's part."

18. In spite of its considerable interest and importance, Russell's willingness in *PoM* to provide quantificational readings of sentences containing denoting phrases is virtually never mentioned in the literature. The only exception known to me is the recent Hursthouse [9].

19. Russell's use of quotation marks is a case in point: as is his practice throughout *PoM*, Russell uses quotation marks here as *grouping indicators*. See, for instance, the similar use in *PoM*, pp. 16–17.

20. There are two important differences between the treatment of such sentences in *PoM* and in "On Denoting". First, in *PoM* Russell does not attempt to give *translation schemata* for sentences containing the various denoting phrases. Second, in *PoM* Russell grants that the *analysandum* and the *analysans* are logically equivalent, but he insists that they are not synonymous (e.g., pp. 36, 91–92). In "On Denoting" he makes the stronger claim of synonymy. There is little argument given in *PoM* for the claim that *analysandum* and *analysans* are not synonymous, and what there is is not convincing. But there would have been two important stumbling blocks in *PoM* to a move to the stronger claim: first, there is Russell's thesis that quantification presupposes the theory of denoting. Second, and by far more significant, is Russell's lack of a contextual quantificational paraphrase for definite descriptions. In the absence of such a paraphrase, the choice of a primitive quantifier and a derivative theory of denoting would have been ruled out.

21. A detailed defense of the claim that Russell has an adequate grasp of quantification theory in *PoM*, along with an investigation of his criticisms of Frege's treatment of quantification can be found in Dau [6], pp. 55–75.

22. In the material mode, we may put the point by saying that Russell needs to posit only two kinds of object, along with two ways in which objects can occur in propositions.

23. I assume that a standard enumeration of the variables of L^{**} is given. When a complex term is eliminated from a subsentence, the variable introduced should be new

to the sentence as a whole, and not just to the subsentence containing the complex term, to avoid collision of variables. (An alternative procedure for eliminating embedded complex terms is that specified in the rules given in the next paragraph in the text, for translating sentences of L^{**} into standard first-order logic.)

24. The equivalence of L^{**} and the standard formulation of first-order logic follows readily from three lemmas:

    I. *A sentence of L^{**} takes the same value relative to any model and sequence function as its translation in L^{*}.*
    II. *A sentence of L^{**} of the form $x^{*}A(x)t$ takes the same value relative to any model and sequence function as $A(t/x)$.*
    III. *A sentence of L^{**} of the form $\forall x^{*}A(x)$ takes the same value relative to any model and sequence function as $\forall xA(x)$ under the standard interpretation of first-order logic.*

25. The proof is by double induction on the number $\underline{n}$ of occurrences of the star *, the quantifier $\forall$ and connectives in a sentence or predicate. We show:

    I. For arbitrary predicate G, $I(h, s, G) = I(h', s, G)$, for all h, h'.

    This is needed to establish:

    II. For all h, and arbitrary sentence A, $I(h, s, A) = I_0(s, A)$, where I_0 is as defined for the official theory.

26. Since Russell rejected the earlier account of denoting phrases as naming objects in favour of the view that they name (i.e., indicate) denoting concepts, I have had little to say in the text about the representation of conjunctive and disjunctive *names*. In the case of concepts with finite extension, where a complete list of names of the terms in the extension of the concept is in theory possible, Russell held, as pointed out above in Note 13, that it was possible to dispense with the denoting concept and make use instead of "enumerations". The semantics developed above for the official theory can help make Russell's claim here more precise. We need to add a clause to the syntax of L^{**} that permits the formation of two kinds of complex names: disjunctive and conjunctive. We may suppose here for simplicity that only basic individual constants are involved. Given the interpretation of a constant as standing for (having as its value) a set of sets (intuitively, the set of its properties; formally, the set of sets of which it is a member), we have a natural way of interpreting *conjunctive terms*: they will be assigned, as their values, the intersection of the values of the conjoined terms. Thus "Jones and Smith" stands for a set of sets (intuitively, the set of their shared or common properties; formally, the set of sets of which they are both members). Similarly, a disjunctive term, say "Jones or Smith", will have as value the union of the values of the disjoined terms. No other changes need to be made in the semantics to get the right truth conditions for sentences containing such complex names.

    In case "$\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, ..." is a complete list of the names of the A's, notice that the value of "$\underline{a}_1$ and $\underline{a}_2$ and..." is the set of all sets of which every $\underline{a}_i$ is a member. That is, the value of the name is just the semantic value of "any A" (or "every A"). Similarly, the value of "$\underline{a}_1$ or $\underline{a}_2$ or..." will be that of "some A" (or "an A"). Thus, as Russell claims, a sentence of subject-predicate form (i.e., of form $\underline{T} + IV$) will be logically equivalent to one containing the appropriate denoting phrase. (Once again, other differences in truth conditions will be due to scope distinctions over connectives or other operators.)

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