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Stationary Logic and Its Friends—II

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Introduction This paper is the successor to "Stationary Logic and Its Friends—I" [10]. The three sections of the paper can be read independently. The first two sections assume some familiarity with stationary logic, denoted $L(aa)$ (see [2]). The third section concerns a closure operation for abstract logic. There a familiarity with [9] would be helpful.

In the first section we define, for regular λ, the λ-interpretation of $L(aa)$, denoted $L(aa^\lambda)$. In this notation, the standard interpretation is $L(aa^\omega)$. The most easily understandable case occurs when $\lambda^{<\lambda} = \lambda$. Then for models with universe λ^+, aa^λ expresses "for all but a nonstationary set of ordinals of cofinality λ". We show if $\lambda^{<\lambda} = \lambda$, then $L(aa^\lambda)$ has the same validities as $L(aa^\omega)$ and $L(aa^\lambda)$ is (λ, ω)-compact.

The second section is devoted to the proof of the consistency of the following approximation to the Δ-closure of $L(Q)$ being contained in $L(aa)$.

> Suppose $L_1 \cap L_2 = L_0$, $\psi_1 \in L_1(Q)$ and $\psi_2 \in L_2(Q)$.
> Further suppose every finitely determinate L_0-structure *either* can be expanded to a model of exactly one of ψ_1 of ψ_2 *or* can be expanded to a finitely determinate model of exactly one of ψ_1 or ψ_2.
> Then there is a sentence $\theta \in L_0(aa)$ such that every finitely determinate model of ψ_1 satisfies θ and no finitely determinate model of ψ_2 satisfies θ. (So θ separates the reducts of finitely determinate models of ψ_1 from those of ψ_2.)

(See Section 2 for the definition of finitely determinate. Of course Q is the quantifier expressing "there exist uncountably many".) In [10] we showed that every consistent $L(Q)$-sentence has a finitely determinate model. So this result establishes the consistency of the Δ-closure of $L(Q)$ being contained in $L(aa)$

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relative to some large class of models. Assuming $MA + \neg CH$ the displayed sentence is false ([8], Theorem 1.3).

The third section is a prolonged observation motivated by the algebraic intuition that homomorphic images are as fundamental as subalgebras. A common requirement on a logic is that one can talk about definable substructures (i.e., via relativization, cf. [1]). However, many common logics, such as $L(Q)$, do not allow us to talk about definable homomorphic images; i.e., modeling out by definable congruences. We show that demanding a logic be congruence closed is innocuous. Every logic has an easily described congruence closure which inherits most of the good properties of the logic.

The results in the first section are due to Shelah. Shelah had a plan for a proof that it is consistent that the Δ-closure of $L(Q)$ is contained in $L(aa)$. In [10] we promised this proof would appear here. Unfortunately the proof was flawed. Mekler realized that using finite determinancy the proof could be altered to work if we weaken the assumption and conclusion. The third section is due to Mekler.

1 The λ-interpretation Assume λ is a regular cardinal. We now describe the semantics of the language $L(aa^{\lambda})$. Fix X. Let $D_{\lambda}(X)$ be the filter on $P_{\lambda^+}(X)$ generated by those filters F such that player II has a winning strategy for the game $G(F)$. Here players I and II alternately choose elements of an increasing λ-chain of elements of $P_{\lambda^+}(X)$. Player II wins if the union of the chain is in F. $D_{\lambda}(X)$ is closed under the intersection of λ many elements and under diagonal intersection; i.e., if $\{F_a\colon a \in X\} \subseteq D_{\lambda}(X)$, then $\{A | A \in F_a \text{ for all } a \in A\} \in D_{\lambda}(X)$ (cf. [5] or [7]).

This filter can be best understood in the case where $\lambda^{<\lambda} = \lambda$. In this case $D_{\lambda}(\lambda^+)$ is just the restriction of the cub filter to the ordinals of cofinality λ. In fact $D_{\lambda}(X)$ has an alternative definition.

1.1 Proposition *Suppose $\lambda^{<\lambda} = \lambda$ and $\boldsymbol{P} = \{f\colon \alpha \xrightarrow{1-1} X\colon \alpha < \lambda^+\}$ ordered by inverse containment. Then $F \in D_{\lambda}(X)$ iff $\mathbf{1} \Vdash^{\boldsymbol{P}} \tilde{g}^{-1}{}'' \check{F}$ contains the intersection of a cub with the ordinals of cofinality λ. (Here $\tilde{g}$ is a name for $\bigcup G$.)*

Define $L(aa^{\lambda})$ (i.e., the λ-interpretation of $L(aa)$) by letting $A \models aa^{\lambda}s\psi(s)$ iff $\{s \in P_{\lambda} = (A)\colon A \models \psi[s]\} \in D_{\lambda}(A)$. Note that the axioms and rules for $L(aa)$ given in [2] are sound for $L(aa^{\lambda})$. So any sentence universally valid for $L(aa^{\omega})$ is also universally valid for $L(aa^{\lambda})$. We will show: if $\lambda = \lambda^{<\lambda}$ then the validities of $L(aa^{\lambda})$ are the same as the validities of $L(aa^{\omega})$. In view of the preceding remark it will suffice to show: if a sentence ψ has a model in the ω-interpretation then it also has one in the λ-interpretation.

One might wonder why we restrict ourselves to one cofinality; i.e., why not use the cub filter on λ^+ rather than restricting to the ordinals of cofinality λ?

1.2 Example Let $L(a^{\lambda})$ denote the logic defined by $A \models a^{\lambda}s\psi(s)$ iff $\{s \in P_{\lambda^+}(A)\colon A \models \psi[s]\}$ contains a cub. Let ψ express: L is a λ^+-like dense linear order $a^{\lambda}s$ (*sup* s exists); for all a and b $cf\{x\colon x < a\} = cf\{x\colon x < b\}$. (Use a ternary function to express the last clause.) Then ψ has a model iff $\lambda = \omega$.

1.3 Theorem *Suppose λ is an uncountable regular cardinal and $\lambda^{<\lambda} = \lambda$.*
(1) *A sentence $\psi \in L(aa)$ has a model in the λ-interpretation iff it has a model in the ω-interpretation.*
(2) *$L(aa^{\lambda})$ is (λ, ω)-compact, i.e., any finitely consistent set of sentences of cardinality λ has a model.*

Proof: (1) The proof is similar to that of Chang's two-cardinal theorem ([3], Theorem 7.7, p. 438). We begin by studying the structure $\langle H(\kappa), \in\rangle$ where κ is some fixed cardinal $>2^{\omega_1}$. By way of explaining our interest in it, note that for any sentence $\psi \in L(aa^{\omega})$ (we assume $L(aa) \in H(\kappa)$) ψ has a model iff $\langle H(\kappa), \in\rangle \models$ "ψ has a model". Enrich the language by adding a binary naming relation R (as in Chang's two-cardinal theorem) and for each formula $\psi(\bar{x}) \in L(aa)$ a relation $R_{\psi}(\bar{x})$ defined so that $\langle H(\kappa), \in\rangle \models \forall \bar{x}\, (R_{\psi}(\bar{x}) \leftrightarrow \psi(\bar{x}))$. For the moment all models will be (first-order) elementarily equivalent to $\langle H(\kappa), \in\rangle$ in the expanded language. To simplify notation we will write for $\psi \in L(aa)$ "$A \models \psi[\bar{a}]$" instead of "$A \models R_{\psi}[\bar{a}\}$". Also we systematically confuse $b \in A$ with $\{x\colon A \models x \in b\}$.

1.4 Lemma
(1) *Suppose $A = \bigcup_{i\in\alpha} A_i$(an elementary chain) where: $\alpha < \lambda^+$; each A_i is saturated; for all i, $\omega^{A_i} = \omega^{A_0}$; and for all i, $|A_i| = \lambda$. Then there is a saturated model $B \succ A$ such that $|B| = \lambda$ and $\omega^B = \omega^{A_0}$.*
(2) *Suppose A is saturated, $\bar{a} \in A$, $A \models$ stat $s\, \psi(s, \bar{a})$ and $|A| = \lambda$. Then A can be extended to a saturated model B such that: $|B| = \lambda$; $A \in B$; $B \models \psi[A, \bar{a}]$; and for $\bar{b} \in A$ and $\theta \in L(aa)$ if $B \models \theta[A, \bar{b}]$ then $A \models$ stat $s\, \theta(s, \bar{b})$.*

Proof (of Lemma): (1) This is just the key step in the proof of Chang's two-cardinal theorem.

(2) Fix an enumeration $\{a_{\alpha}\colon \alpha < \lambda\}$ of A which we will treat as a set of variables. Let $X = \{x_{\alpha}\colon \alpha < \lambda\}$ be a set of variables. We will define an increasing sequence $p_{\alpha}(\alpha < \lambda)$ of complete types over the empty set and an increasing sequence $A_{\alpha}(\alpha < \lambda)$ of subsets of A of cardinality $<\lambda$ so that: (a) for all α the variables of p_{α} are $X_{\alpha} \cup A_{\alpha}$ where $X_{\alpha} = \{x_{\beta}\colon \beta < \alpha\}$ and $\{a_{\beta}\colon \beta < \alpha\} \subseteq A_{\alpha}$; (b) for all α, $Th(A)_{A_{\alpha}} \subseteq p_{\alpha}$; (c) for all α and $y \in A \cup X$, $(y \in x_0) \in p_{\alpha}$ iff there is $a \in A_{\alpha}$ so that $(a = y) \in p_{\alpha}$; (d) $\psi(x_0, \bar{a}) \in p_1$; and (e) for all $\theta \in L(aa)$ $\forall \bar{x}((\theta(x_0, \bar{x}) \wedge \bar{x} \in x_0) \rightarrow$ *stat* $s\, \theta(s, \bar{x})) \in p_1$. A consequence of (e) is: (f) $\forall x(x \in \omega \rightarrow x \in x_0) \in p$, since $(H(\kappa), \epsilon) \models$ *aas* $\forall x\, (x \in \omega \rightarrow s(x))$.

We will define an equivalence relation on $A \cup X$ by $y \equiv z$ iff $(y = z) \in \bigcup_{\alpha<\lambda} p_{\alpha}$. Then we let $B = A \cup X/\equiv$ where the relations are defined in the obvious way. Some care must be taken to ensure B is a saturated model. Since this is a matter of routine enumeration we will just describe how the iteration is done.

Let A_0 be the domain of a countable elementary submodel of A which includes the constants of ψ and a_0. Let $p_0 = Th(A)_{A_0}$.

Claim $p_1' = p_0 \cup \{\psi(x_0, \bar{a})\} \cup \{a \in x_0\colon a \in A_0\} \cup \{\forall \bar{x}(\theta(x_0, \bar{x}) \wedge \bar{x} \in x_0) \rightarrow$ *stat* $s\, \theta(s, \bar{x}))\colon \theta \in L(aa)\}$ *is consistent.*

Proof (of claim): By diagonal intersection for all $\theta \in L(aa)$ $(H(\kappa), \in) \models$ *aas*

$\forall \bar{x}(s(\bar{x}) \to (aa\, t\, \theta(t, \bar{x}) \to \theta(s, \bar{x})))$. Also $(H(\kappa), \in) \models aa s\, \exists y\, \forall z(s(z) \leftrightarrow z \in y)$. Since A is elementarily equivalent to $(H(\kappa), \in)$, p_1' is finitely satisfiable in A.

Let p_1 be any complete extension of p_1'. Suppose $p_\alpha(\alpha > 1)$ has been chosen and $q_\alpha(x_\alpha)$ is some type over $X_\alpha \cup A_\alpha$ containing p_α. (The q_α should be chosen so that B will turn out to be a saturated model.) Note that $p'_{\alpha+1} = q_\alpha \cup Th(A)_{A_\alpha \cup \{a_\alpha\}} \cup \{a_\alpha \in x_0\}$ is consistent. In fact any finite subset already occurs in q_α, where the variables are replaced by elements of A_0. If $(x_\alpha \notin x_0) \in q_\alpha$, then let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$ and $p_{\alpha+1}$ be any complete (consistent) extension of $p'_{\alpha+1}$. Suppose $(x_\alpha \in x_0) \in q_\alpha$. Let p''_α be a complete extension of p'_α. Since A is saturated, p''_α is realized in A where the elements of $A_\alpha \cup \{a_\alpha]$ are interpreted as themselves. Choose $a \in A$ so that x_α is interpreted as a in such a realization. Let $p_{\alpha+1}$ be a complete extension of $p''_\alpha \cup \{x_\alpha = a\}$ and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha, a\}$. At limit ordinals take unions.

1.5 Lemma *There is a model B so that for all $\psi(\bar{x}) \in L(aa)$ and $\bar{b} \in B$, B satisfies $\psi[\bar{b}]$ in the λ-interpretation (denoted $B \models {}^{\lambda}\psi[\bar{b}]$) iff $B \models R_\psi[\bar{b}]$.*

Proof: We use the same strategy as in [2]. Partition $\{\alpha: \alpha < \lambda^+$ and $cf(\alpha) = \lambda\}$ into λ^+ disjoint stationary subsets $S_{\psi[s,\bar{b}]}(\psi(s, \bar{x}) \in L(aa)$ and $\bar{b} \in \lambda^+)$. Choose a continuous elementary chain $A_\alpha(\alpha < \lambda^+)$ of models so that: the underlying set of each A_α is some ordinal $<\lambda^+$; for all α, $\omega^{A_0} = \omega^{A_\alpha}$; for all α, $A_{\alpha+1}$ is saturated of cardinality λ; if $cf(\alpha) = \lambda$, $\bar{a} \in A_\alpha$, $\psi(s, \bar{x}) \in L(aa)$ and $A_\alpha \models aa s\, \psi(s, \bar{a})$, then $A_{\alpha+1} \models \psi[A_\alpha, \bar{a}]$; $(A_\alpha \in A_{\alpha+1})$; and if $\alpha \in S_{\psi[s,\bar{b}]}$ and $A_\alpha \models stat\, s\, \psi[s, \bar{b}]$ then $A_{\alpha+1} \models \psi[A_\alpha, \bar{b}]$. Lemma 1.4 is exactly what is needed to show such a chain exists. Let $B = \bigcup_{\alpha<\lambda} A_\alpha$. Next we show that if $cf(\alpha) = \lambda$ then $A_\alpha \in B$. By the construction $A_\alpha \in A_{\alpha+1}$ and $A_{\alpha+1} \models |A_\alpha| = \omega$ (since $A_\alpha \models$ "$aa s(|s| = \omega)$"). So there is $f \in A_{\alpha+1}$ such that $A_{\alpha+1} \models$ "f is an onto function from ω to A_α". Since $\omega^B = \omega^{A_0}$, $A_\alpha^{A_{\alpha+1}} = A_\alpha^B$. An easy induction on the construction of formulas shows B is the desired model.

To finish the proof we drop our convention about the use of "model". In $(H(\kappa), \in)$ there is a connection between internal and external satisfaction: namely $(H(\kappa), \in) \models$ "$M \models aa s\, \psi(s)$" iff $(H(\kappa), \in) \models aa s$ "$M \models \psi(s \cap M)$". Now we turn our attention to B (of Lemma 1.5). Suppose $B \models$ "M is a structure". (Recall that we confuse M with $\{x: B \models x \in M\}$ equipped with the obvious functions and relations.)

Claim *For all $\psi(\bar{s}, \bar{x}) \in L(aa)$, $\bar{A} \in B$ and $\bar{m} \in M$, $M \models^\lambda \psi[\overline{A \cap M}, \bar{m}]$ iff $B \models$ "$M \models \psi[\overline{A \cap M}, \bar{m}]$".*

Proof: The proof is by induction on the construction of formulas. The only interesting case occurs when ψ is $aa\, t\, \psi(t, \bar{s}, \bar{x})$. Suppose $B \models$ "$M \models aa\, t\, \psi[t, \overline{A \cap M}, \bar{m}]$". By Lemma 1.5 $B \models^\lambda aa\, t$ "$M \models \psi[t \cap M, \overline{A \cap M}, \bar{m}]$". Also $B \models^\lambda aa\, t\, \exists x\, \forall z\, (x \in z \leftrightarrow t(x))$. So there is $F \in D_\lambda(B)$ such that for all $C \in F$, $C \in B$ and $B \models^\lambda$ "$M \models \psi[C \cap M, \overline{A \cap M}, \bar{m}]$". By the induction hypothesis $M \models^\lambda \psi[C \cap M, \overline{A \cap M}, \bar{m}]$. Hence $M \models^\lambda aa\, t\, \psi[t, \overline{A \cap M}, \bar{m}]$. The other direction is much the same.

Finally, to prove that if $\psi \in L(aa)$ has a model in the ω-interpretation then it has a model in the λ-interpretation, note that: ψ has a model in the ω-

interpretation iff for some $M \in A_0$, $A_0 \models$ "$M \models \psi$" iff $M^B \models^\lambda \psi$. (The notation is from the proof of Lemma 1.5.) As has already been noted the other direction follows from [2].

(2) Suppose λ is uncountable, $T \subseteq L(aa)$, T is finitely consistent, and $|T| = \lambda$. We can assume $T \subseteq A_0$ (from the proof of Lemma 1.5). Add a constant M to the language. Since $Th(H(\kappa), \in) \cup \{$"$M \models \psi$": $\psi \in T\}$ is consistent, this theory has a saturated model C of cardinality λ. The reduct of C to the language of $(H(\kappa), \in)$ is isomorphic to A_0. Hence there is some $M \in A_0$ so that for all $\psi \in T$, $A_0 \models$ "$M \models \psi$". Hence $M^B \models^\lambda T$.

1.6 Corollary *Suppose λ is regular and $\lambda^{<\lambda} = \lambda$. Then $L(aa^\lambda)$ satisfies $LS(\lambda^+)$; i.e., any consistent sentence has a model of cardinality at most λ^+.*

2 Determinant Δ-pairs Say $L(Q)$-sentences ψ_1 and ψ_2 form a *Δ-pair for finitely determinate structures* if *either* every finitely determinate structure in the common language can be expanded to a model for exactly one of ψ_1 or ψ_2 *or* every finitely determinate structure in the common language can be expanded to a finitely determinate model of exactly one of ψ_1 or ψ_2. In this section we will show it is consistent that if $L(Q)$-sentences ψ_1 and ψ_2 form a Δ-pair for finitely determinate structures then there is an $L(aa)$-sentence in the common language which separates the reducts of finitely determinate models of ψ_1 from those of ψ_2 (i.e., there is an $L(aa)$-sentence θ in the common language such that every finitely determinate model of ψ_1 satisfies θ and no finitely determinate model of ψ_2 satisfies θ). Call such a θ a *determinate interpolant* between ψ_1 and $\neg\psi_2$.

Given a pair of $L(Q)$-sentences ψ_1 and $\neg\psi_2$ with no determinate interpolant we will define three notions of forcing. All these will turn out to be the same as adding ω_1 Cohen reals. We will use this fact to show that if we add ω_1 Cohen reals then ψ_1 and ψ_2 are not a Δ-pair for finitely determinate structures. Because of the completeness theorem for $L(aa)$ "having no determinate interpolant" is an absolute property. So we'll have to show that after adding ω_1 Cohen reals if ψ_1 and $\psi_2 \in L(Q)$ are a Δ-pair for finitely determinant structures then ψ_1 and $\neg\psi_2$ have a determinate interpolant.

To begin, fix L_1, L_2 and $L_0 = L_1 \cap L_2$ and sentences $\psi_1 \in L_1(Q)$, $\psi_2 \in L_2(Q)$. Assume no $\theta \in L_0(aa)$ is a determinate interpolant between ψ_1 and $\neg\psi_2$. So if we let $\Sigma = \{\theta \in L_0(aa)\colon DET_1 \cup \{\psi_1\} \vdash \theta\}$ then $\Sigma \cup DET_2 \cup \{\psi_2\}$ is consistent. Here DET_l denotes the determinancy scheme for $L_l(aa)$:

$$aa\, s_1 \ldots aa\, s_n\, \psi(s_1, \ldots, s_n) \vee aa\, s, \ldots, aa\, s_n \neg\psi(s_2, \ldots, s_n)$$

where $\psi(s_1, \ldots, s_n)$ is an $L_l(aa)$-formula. Recall a model is finitely determinate if it satisfies DET.

Just as in the proof of Craig's interpolation theorem from Robinson's consistency theorem we can prove:

2.1 Lemma *There are complete consistent theories $T_l \in L_l(aa)$ $(l \leq 2)$ such that $T_0 \subseteq T_1$, T_2; $\psi_l \in T_l$ $(l = 1, 2)$; and $DET_l \subseteq T_l(aa)$ $(l \leq 2)$.*

To avoid degenerate cases we will assume every model of T_0 has a definable subset which is both uncountable and co-uncountable. (If not, add unary predicates U, V, and W. Replace T_l by the theory expressing: "U and V partition the model into disjoint subsets"; each axiom of T_l relativized to U; and "W is an uncountable co-uncountable subset of V".)

Expand the languages L_l by adding unary predicates $\{U_i: i \in \eta\}$. Here η is the order type of the rationals. Let T_l' be the deductive closure of $T_l \cup \{\psi(U_{i_1}, \ldots, U_{i_n}) \leftrightarrow aa\, s_1 \ldots aa\, s_n \psi(s_1, \ldots, s_n)$: $\psi \in L_l(aa)$ and $i_1 < \ldots < i_n \in \eta\}$.

2.2 Lemma *Each T_l' is complete and consistent. Further $T_0' \subseteq T_1'$, T_2'.*

Proof: Clearly each T_l' is consistent since any model of T_l can be expanded to a model of any finite subset of T_l'. Consider any sentence $\psi(U_{i_1}, \ldots, U_{i_n}) \in L_l'(aa)$ (where all the new predicates are displayed). Since T_l is complete and contains DET_l, either $aa\, s_1 \ldots aa\, s_n \psi(s_1, \ldots, s_n) \in T_l$ or $aa\, s_1 \ldots aa\, s_n \neg\psi(s_1, \ldots, s_n) \in T_l$. So either $\psi(U_{i_1}, \ldots, U_{i_n}) \in T_l'$ or $\neg\psi(U_{i_1}, \ldots, U_{i_n}) \in T_l'$. Similarly $T_0' \subset T_1'$, T_2'.

Having constructed $T_l'(l \leq 2)$ we now consider only the first-order part and construct new theories $S_l(l \leq 2)$. Choose λ so that every countable (first-order) theory has a saturated model of cardinality λ. (As usual an absoluteness argument can be found to eliminate the need for assuming such a λ exists.) For $l \leq 2$, let $(A_l, U_i^l)_{i \in \eta}$ be a saturated model of T_l' of cardinality λ. For $i \in \eta$ and $l \leq 2$, let $V_i^l = \bigcup_{j<i} U_j^l$. Note: if $i < j \in \eta$ then $U_i^l \subsetneqq U_j^l$. Also each U_i^l is closed under the operations of A_l. Let B_l be the two-sorted model $B_l = (A_l \cap (\bigcup V_i^l)$, $\{V_i^l: i \in \eta\}, \epsilon)$. We will use x, y, z to denote variables of the first sort; s, t, u to denote variables of the second sort; and E to denote the membership relation. Let S_l be the theory of B_l and K_l the language of S_l.

2.3 Lemma

(1) *For all $l \leq 2$ and $i \in \eta$, $V_i^l = \bigcup_{j<i} V_j^l$ and $V_j^l \neq V_i^l$ $(i < j)$.*

(2) $S_0 \subset S_1$, S_2

(3) *For any formula $\psi(x_1, \ldots, x_m, s_1, \ldots, s_n, t_1, \ldots, t_k) \in K_l$, the following formula is in S_l:*

$$\forall \bar{x} \forall \bar{s} \forall \bar{t} \forall \bar{u}((\wedge(x_j E t_1 \wedge x_j E u_1) \wedge (s_1 \subset \ldots \subset s_n \subset t_1 \ldots \subset t_k) \wedge (s_1 \subset \ldots \subset s_n \subset u_1 \subset \ldots \subset u_k)) \rightarrow (\psi\ (\bar{x}, \bar{s}, \bar{t}) \leftrightarrow \psi(\bar{x}, \bar{s}, \bar{u})))\ .$$

Proof: (1) Clear.

(2) Since $(A^l, U_i^l)_{i \in \eta} \restriction L_0'$ is saturated of cardinality λ and elementarily equivalent to $(A^0, U_i^0)_{i \in \eta}$, $(A^l, U_i^l)_{i \in \eta} \restriction L_0' \cong (A^0, U_i^0)$. Any such isomorphism induces an isomorphism of $B_l \restriction K_0$ with B_0.

(3) Fix $l \leq 2$. (To simplify notation we drop the l.) Consider $i_1 < \ldots < i_n < j_1 < \ldots < j_k$, $i_n < j_1' < \ldots < j_k'$ and $a_1, \ldots, a_m \in V_{j_1}$, $V_{j_1'}$. Choose $i_n < i_{n+1} < j_1, j_1'$ so that $a_1, \ldots, a_m \in V_{i_{n+1}}$. Choose σ an automorphism of η fixing all $i \leq i_{n+1}$ so that $\sigma(j_r) = j_r'$ $(1 \leq r \leq k)$. Since $T \supset DET$, $(A, \bar{a}, U_i)_{i \in \eta} \equiv (A, \bar{a}, U_{\sigma(i)})_{i \in \eta}$. By saturation $(A, \bar{a}, U_i)_{i \in \eta} \cong (A, \bar{a}, U_{\sigma(i)})_{i \in \eta}$. So there is an automorphism of B fixing $a_1, \ldots, a_m$, $V_{i_1}, \ldots, V_{i_n}$ such that for $1 \leq r \leq k$ the image of V_{j_r} is $V_{j_r'}$.

Choose $\kappa > 2^\omega$. Let $N \prec (H(\kappa), \in)$ so that N is countable and $S_l \in N(l \leq 2)$. Let $M(\nu)$ denote $\omega \cdot (1 + \nu)$. For $l \leq 2$ the conditions of $\boldsymbol{Q}_l$ are the $p[\alpha_1, \ldots,$

α_m, $M(0)$, $M(\nu_1), \ldots, M(\nu_n)]$ such that: α_i, $\nu_j < \omega_1$; $p(\bar{x}, \bar{s})$ is a complete S_l-type; $p \in N$; $\alpha_i \neq \alpha_j$ iff $x_i \neq x_j \in p$; $a_i \in M(\nu_i)(M(0))$ iff $x_i E s_j \in p$; and if $\alpha_i \in M(\nu + 1) \backslash M(\nu)$ then for some j, j' $\nu = \nu_j$ and $\nu + 1 = \nu_{j'}$. Order $\boldsymbol{Q}$ by containment.

2.4 Lemma
(1) *For all* $l \leq 2$ *and* $\alpha < \omega_1$, $\{q\colon \alpha \in dom\ q\}$ *is dense in* $\boldsymbol{Q}_l$. (*For* $q = p[\alpha_1, \ldots, \alpha_m, M(0), M(\nu_1), \ldots, M(\nu_n)]$ *dom* $q = \{\alpha_1, \ldots, \alpha_m, M(0), M(\nu_1), \ldots, M(\nu_n)\}$. *We ignore the fact that* $M(\nu)$ *is also an ordinal.*)
(2) *For all* $l \leq 2$ *and* $\nu < \omega_1$, $\{q\colon M(\nu) \in dom\ q\}$ *is dense in* $\boldsymbol{Q}_l$.

Proof: This is straightforward. The only things to note are that $\forall s \forall t \exists u(s \subset t \rightarrow s \subset u \subset t) \in S_0$ and for all n $\forall s \forall t \exists^{>n} x(s \subset t \rightarrow (xEt \wedge (\neg x\ Es))) \in S_0$.

2.5 Lemma *For all* $l \leq 2$, *forcing with* $\boldsymbol{Q}_l$ *is the same as adding* ω_1 *Cohen reals.*

Proof: This is much like the proof of Lemma 1.5 in [10]. For limit ordinals $\lambda < \omega_1$, let $\boldsymbol{P}_\lambda$ denote $\{q\colon \alpha \in dom\ q$ implies $\alpha \in M(\lambda)$ and $M(\nu) \in dom\ q$ implies $\nu < \lambda\}$. Since we only allow countably many types, $\boldsymbol{P}_\lambda$ is countable. (This is the reason for the choice of N.) At limit stages we take unions. So to finish the proof it suffices to show for all limit λ: if $q \in \boldsymbol{Q}_l$ there is $q' \in \boldsymbol{P}_\lambda$ so that for all $r \in \boldsymbol{P}_\lambda$ extending q', r and q are compatible. (The hypothesis on S_0 guarantees the appropriate posets are nontrivial.)

Suppose q is $p[\bar{\alpha}, \bar{\beta}, M(0), M(\nu_1), \ldots, M(\nu_n), M(\tau_1), \ldots, M(\tau_k)]$ where $\alpha_j \in M(\lambda)$, $\nu_j < \lambda$, $\beta_j \notin M(\lambda)$ and $\tau_j \geq \lambda$. Let $q' = p'\{\bar{\alpha}, M(0), M(\nu_1), \ldots, M(\nu_n)]$ (where $p'(\bar{x}, \bar{s})$ is the obvious restriction of $p(\bar{x}, \bar{y}, \bar{s}, \bar{t})$). Let $p_1(\bar{x}, \bar{z}, \bar{s}, \bar{u}) \in N$ be any complete type extending p'. Suppose $r = p_1[\bar{\alpha}, \bar{\gamma}, M(0), M(\nu_1), \ldots, M(\nu_n), M(\sigma_1), \ldots, M(\sigma_m)]$. For any $\psi(\bar{x}, \bar{y}, s, \bar{t}) \in p$, $\forall \bar{t} \exists \bar{y}((\wedge(s_i \subset t_j)) \rightarrow \psi(\bar{x}, \bar{y}, \bar{s}, \bar{t})) \in p'$. Hence $\exists \bar{t} \exists \bar{y}((\wedge(s_i \subset t_j)) \wedge (\wedge(u_i \subset t_j))) \wedge \psi(\bar{x}, \bar{y}, \bar{s}, \bar{t})) \in p_1$. So $r \cup q$ can be extended to a condition.

For $l = 1$, 2 and $q \in \boldsymbol{Q}_l$, let $q \restriction \boldsymbol{Q}_0$ denote $\{\psi[\bar{\alpha}, M(0), M(\nu_1), \ldots, M(\nu_n)] \in q\colon \psi(\bar{x}, \bar{s}) \in K_0\}$. Also for $G \subseteq \boldsymbol{Q}_l$, let $G \restriction \boldsymbol{Q}_0 = \{q \restriction \boldsymbol{Q}_0\colon q \in G\}$.

2.6 Lemma *Suppose* $l = 1, 2$. *If* G *is* $\boldsymbol{Q}_l$*-generic, then* $G \restriction \boldsymbol{Q}_0$ *is* $\boldsymbol{Q}_0$*-generic. Also for all* $q \in \boldsymbol{Q}_0$ *there is* $q' \in \boldsymbol{Q}_l$ *so that* $q' \restriction \boldsymbol{Q}_0 = q$.

Proof: This lemma is true since $S_0 \subseteq S_l$. Here we use the ability to quantify over variables of both sorts.

Suppose now G_l is $\boldsymbol{Q}_l$-generic ($l \leq 2$). Let $(A_l, M(\nu), \in)_{\nu < \omega_1}$ be the K_l-structure with universe ω_1 whose diagram is determined by G_l (i.e., $A_l \models R[\alpha_1, \ldots, \alpha_m]$ iff there is some $q \in G_l$ such that $R[\alpha_1, \ldots, \alpha_m] \in q$). We now inductively associate with each $L_l(aa)$-formula ψ a K_l-formula ψ^*. If ψ is atomic then ψ^* is ψ. (We confuse $s(x)$ and xEs.) Also $(\neg\psi)^*$ is $\neg(\psi^*)$, $(\psi \wedge \theta)^*$ is $\psi^* \wedge \theta^*$, and $(\exists x\theta)^*$ is $\exists x\theta^*$. Suppose ψ is $aas\theta$ and the free variables of ψ are $x_1, \ldots, x_n, s_1, \ldots, s_m$. Let ψ^* be $\forall s((\wedge(x_i Es) \wedge \wedge(s_i \subset s)) \rightarrow \theta^*)$.

2.7 Lemma *Let* A_l *and* G_l *be as above* ($l \leq 2$). *Fix* $\alpha_1, \ldots, \alpha_m$, $M(\nu_1), \ldots, M(\nu_n)$ *and* $\psi(x_1, \ldots, x_m, s_1, \ldots, s_n) \in L_l(aa)$. $A_l \models \psi[\alpha_1, \ldots, \alpha_m, M(\nu_1), \ldots,$

$M(\nu_n)]$ iff for some $q \in G_l$, $\psi^*[\alpha_1, \ldots, \alpha_m, M(\nu_1), \ldots, M(\nu_n)] \in q$. Further $A_l \models DET_l$.

Proof: The proof is by induction on the construction of formulas. The only difficulties occur when dealing with the quantifiers. To handle the existential case it suffices to show: if $\exists x \psi[x, \bar{\alpha}, M(0), M(\nu_1), \ldots, M(\nu_n)] \in q$, then for some β and $r \supseteq q$ $\psi[\beta, \bar{\alpha}, M(0), M(\nu_1), \ldots, M(\nu_n)] \in r$. (This implies the relevant density property.) We may assume q is $p[\bar{\alpha}, M(0), M(\nu_1), \ldots, M(\nu_n)]$. Further, since p is a complete type we may assume ψ includes "$x \neq \alpha_i$" for all i and ψ either includes "$xEM(\nu_i)$" or "$\neg xEM(\nu_i)$". There are various possibilities. We will only do one representative case. Assume $0 < \nu_1 < \ldots < \nu_n$, ν_1 is a limit ordinal and ψ includes $xE(M(\nu_1) \backslash M(0))$.

By 2.3.1 we can assume $\psi[x, \bar{\alpha}, M(0), M(\nu_1), \ldots, M(\nu_n)]$ includes $\exists s(M(0) \subset s \subset M(\nu_1) \wedge xEs)$. So $p(\bar{y}, \bar{t})$ can be expanded to a type (in N) $r(x, \bar{y}, \bar{t}, s)$ which includes $\psi(x, \bar{y}, \bar{t})$, $t_0 \subset s \subset t_1$, and xEs. Choose $\beta \in M(1) \backslash M(0)$. So $r[\beta, \bar{\alpha}, M(0), M(\nu_1), \ldots, M(\nu_n), M(1)]$ is the required condition.

Consider a formula $aa s \theta[\alpha_1, \ldots, \alpha_m, M(\nu_1), \ldots, M(\nu_n), s]$. Consider any condition $q = p[\alpha_1, \ldots, \alpha_m, M(0), M(\nu_1), \ldots, M(\nu_n)]$. By Lemma 2.3(3) either $\forall s((\wedge \alpha_i Es) \wedge \wedge(M(\nu_i) \subset s)) \rightarrow \theta^*(\alpha_1, \ldots, \alpha_m, M(\nu_1), \ldots, M(\nu_n), s)) \in q$ or $\forall s((\wedge(\alpha_i ES) \wedge \wedge(M(\nu_i) \subset s)) \rightarrow \neg\theta^*(\alpha_1, \ldots, \alpha_m, M(\nu_1), \ldots, M(\nu_n), s)) \in q$. In the first case for all large enough τ and $r \supseteq q$ if $M(\tau) \in dom\, r$ then $\theta^*(\alpha_1, \ldots, \alpha_m, M(\nu_1), \ldots, M(\nu_n), M(\tau)) \in r$. So if $q \in G_l$ then $A_l \models aa s \theta[\alpha_1, \ldots, \alpha_m, M(\nu_1), \ldots, M(\nu_n)]$. In the other case $A_l \models aa s \neg\theta[\alpha_1, \ldots, \alpha_m, M(\nu_1), \ldots, M(\nu_n), s]$. This completes the proof as well as showing $A_l \models DET_l$.

2.8 Lemma *For any $L_l(Q)$-sentence $\psi \in T_l$, $A \models \psi$. In particular for $l = 1$, 2, $A_l \models \psi_l$.*

Proof: View ψ as an $L(aa)$-sentence; i.e., $Qx\theta$ is replaced by $aa\, s\, aa\, t\, \exists x(t(x) \wedge \neg s(x) \wedge \theta)$. It is not hard to see that if $\psi \in T_l$ then $\psi^* \in S_l$.

Remark: We have a way to construct finitely determinate models of any consistent $L(Q)$ sentence which also works for L^{POS}. The reader might wonder if these methods apply to $L(aa)$-sentences consistent with *Det*. To see that they fail suppose T contains the sentence expressing "$<$ is an ω_1-like order". Then S will contain a sentence expressing "for all s *sup* s does not exist". Here S and T play the roles of S_l and T_l above.

2.9 Theorem *It is consistent, assuming the consistency of ZF, that if ψ_1, $\psi_2 \in L(Q)$ form a Δ-pair for finitely determinate structures then ψ_1 and $\neg\psi_2$ have a determinate interpolant.*

Proof: Let $\boldsymbol{P}$ be the poset for adding ω_1 Cohen reals. Assume H is $\boldsymbol{P}$-generic and that ψ_1 and ψ_2 form a Δ-pair for finitely determinate structures in $V[H]$. Suppose $\psi_1 \in L_1(Q)$ and $\neg\psi_2 \in L_2(Q)$ have no determinate interpolant in $L_0(aa)$, where $L_1 \cap L_2 = L_0$. Let $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2 (\in V)$ be as above. By Lemma 2.5 there is $G \in V[H]$ so that G is $\boldsymbol{Q}_0$-generic and $V[G] = V[H]$. Let $\tilde{A}_l$ be the canonical name for A_l (where A_l is as in Lemmas 2.7 and 2.8). Since ψ_1 and ψ_2 form a determinate Δ-pair and $1 \Vdash^{\boldsymbol{Q}_0}$ "$\tilde{A}_0$ is finitely determinate" $(\tilde{A}_0)_G$ can be expanded to a model of ψ_1 or a model of ψ_2 (which, depending on how ψ_1 and ψ_2 form a determinate Δ-pair, we may be able to assume is finitely determinate.) For definiteness assume $(\tilde{A}_0)_G$ can be expanded to a model of ψ_1 which is finitely determinate.

Choose $p \in G$ so that $p \Vdash$ "$\tilde{A}_0$ can be expanded to a model of ψ_1". Choose $q \in \boldsymbol{Q}_2$ so that $q \upharpoonright \boldsymbol{Q}_0 = p$. Choose G_2 $\boldsymbol{Q}_2$-generic so that $V[G_2] = V[H]$ and $q \in G_2$. (The existence of such G_2 is implied by the homogeneity of $\boldsymbol{P}$.) Let $G_0 = G_2 \upharpoonright \boldsymbol{Q}_2$. Note that $(\tilde{A}_2)_{G_2} \upharpoonright L_0 = (\tilde{A}_0)_{G_0}$. Also $p \in G_0$. So $V[H] \models$ "$(\tilde{A}_0)_{G_0}$ can be expanded to a finitely determinate model of ψ_2" (namely $(\tilde{A}_2)_{G_2}$). Since $p \in G_0$, $V[G_0] \models$ "$(\tilde{A}_0)_{G_0}$ can be expanded to a finitely determinate model of ψ_1". But $\omega_1^V = \omega_1^{V[G_0]} = \omega_1^{V[H]}$. So any finitely determinate model of ψ_1 with universe ω_1 in $V[G_0]$ remains a finitely determinate model of ψ_1 in $V[H]$. So $V(H) \models$ "$(\tilde{A}_0)_{G_0}$ can be expanded to a finitely determinate model of ψ_1 and to a finitely determinate model of ψ_2".

Remark: Under $MA + \neg CH$ there are sentences ψ_1, $\psi_2 \in L(Q)$ forming a Δ-pair with no determinate interpolant between ψ_2 and $\neg\psi_2$. Indeed the usual example where ψ_1 says "T is a special Aronzajn tree" and ψ_2 says "T is not an ω_1-tree or T has a branch of length ω_1" suffices ([8], Theorem 1.3).

3 Congruence closed logics In this section we will define what it means for a logic to be congruence closed. Being congruence closed like being closed under relativization is a natural requirement to place on a logic. However $L(Q)$ is not congruence closed. We will see that the congruence closure of a logic is easily constructed and preserves many of the good properties of the logic. We first review what we mean by a logic. (We follow [9].) For simplicity assume no language has function symbols. To a class K of structures of a fixed similarity type, closed under isomorphism, is associated a quantifier Q^K. To illustrate, suppose K is a class of structures closed under isomorphism with a binary relation R and a constant symbol c. A typical formula $\chi(\bar{w}, y)$ beginning with Q^K is $Q^K x, \langle x_0, x_1\rangle\ (\psi(x, \bar{w}), \phi(x_0, x_1, \bar{w}), y)$, where $\psi(x, \bar{w})$, $\phi(x_0, x_1, \bar{w})$ are formulas whose free variables include x and x_0, x_1, respectively. For all structures, A, $A \models \chi[\bar{a}, b]$ iff $((\psi(x, \bar{a}))^A$: $\phi^A(x_0, x_1, \bar{a}), b) \in K$. A logic $\mathcal{L}$ is obtained by adding to $L_{\omega\omega}$ some quantifiers $Q^K (K \in \mathcal{K})$ and giving $\mathcal{L} = L(Q^K$: $K \in \mathcal{K})$ the obvious syntax and semantics.

A logic is *congruence closed* if for any formula $\phi(x_1, x_2, \bar{y})$ and $\psi(\bar{y})$ there is a formula ψ^* such that: for all A and $\bar{a} \in A$, $A \models \psi^*[\bar{a}]$ iff $\phi(x_1, x_2, \bar{a})^A$ is a congruence relation with respect to the relations used in ψ and $A/\equiv \models \psi[\bar{a}/\equiv]$. Here $\equiv$ denotes the equivalence relation $\phi(x_1, x_2, \bar{a})$. A Δ-closed logic is obviously congruence closed. The congruence closure of a logic is easily described. For K a class of structures closed under isomorphism, let K^* be the class of structures $\langle A, \equiv\rangle$ such that $\equiv$ is a congruence relation on A and $A/\equiv \in K$. For convenience assume $\equiv$ is the first relation in the similarity type of $\langle A, \equiv\rangle$. Given a logic $\mathcal{L} = L(Q^K$: $K \in \mathcal{K})$ let $\mathcal{L}^* = L(Q^{K^*}$: $K \in \mathcal{K})$.

3.1 Proposition *$\mathcal{L}^*$ is the smallest congruence closed logic extending $\mathcal{L}$.*

Proof: First note: $\mathcal{L} \subseteq \mathcal{L}^*$ (i.e., interpretable); e.g., $Q^K x, \langle x_0, x_1\rangle\ (\phi(x, \bar{w}), \psi(x_0, x_1, \bar{w}))$ is equivalent to $Q^{K^*} x, \langle x_0, x_0\rangle, \langle x_1, x_2\rangle\ (\phi(x, \bar{w}), x_0 = x_0, \psi(x_1,$

$x_2, \bar{w})$). Also if $\mathcal{L}'$ is any congruence closed logic extending $\mathcal{L}$, then $\mathcal{L}^* \subset \mathcal{L}'$. It remains to show $\mathcal{L}^*$ is congruence closed. But this is a simple induction.

3.2 Proposition *Suppose $\mathcal{L}$ is a logic and $\mathcal{L}^*$ its congruence closure.*
(1) *If $\mathcal{L}$ is obtained by adding κ quantifiers to $L_{\omega\omega}$ then so is $\mathcal{L}^*$. (Note: κ may be finite.)*
(2) *If $\mathcal{L}$ is axiomatizable, so is $\mathcal{L}^*$.*
(3) *If $\mathcal{L}$ has the Tarski property, so does $\mathcal{L}^*$.*
(4) *If $\mathcal{L}$ is (κ, λ)-compact, so is $\mathcal{L}^*$.*

Proof: (1) is a consequence of the construction of $\mathcal{L}^*$. For (2), see 2.24 in [9]. (3) is an induction on formulas in $\mathcal{L}^*$. (4) follows since $\mathcal{L}^* \subseteq \Delta(\mathcal{L})$ ([9], 2.13).

3.3 Example The congruence closure of $L(Q)$ is $L(Q^E)$ where $Q^E x, \langle x_0, x_1\rangle$ $(\phi(x, \bar{y}), \psi(x_0, x_1, \bar{y}))$ expresses "$\psi(x_0, x_1, \bar{y})$ is an equivalence relation on $\phi(x, \bar{y})$ and $\psi(x_0, x_1, \bar{y})$ has at least $\aleph_1$ equivalence classes". This quantifier is due to Feferman [4]. He introduced this quantifier in order to extend $L(Q)$ to get a logic strong enough to handle Keisler's [6] counterexample to Δ-interpolation in $L(Q)$. To see how Q^E fits in with the discussion above, let K be the class of structures (with no relations or constants) of structures of cardinality at least $\aleph_1$. Then $L(Q)$ is $L(Q^K)$ and Q^E is Q^{K^*}.

Recall Keisler's example. Let A_0 be a model with an equivalence relation with $\aleph_0$ uncountable equivalence classes. Let A_1 be a model with an equivalence relation with $\aleph_1$ uncountable equivalence classes. Then A_0 and A_1 are $L(Q)$-equivalent but not $L(Q^E)$-equivalent. So Keisler's example shows $L(Q)$ is not congruence closed. Note that A_0 and A_1 are ω-homogeneous models of an ω-categorical theory. (By A being *ω-homogeneous* we mean if $\bar{a}$ and $\bar{b}$ are finite sequences satisfying the same type then there is an automorphism of A taking $\bar{a}$ to $\bar{b}$.)

The following proposition is useful for showing models are equivalent in the Beth closure of a logic.

3.4 Lemma *Suppose A, B are ω-homogeneous models of an ω-categorical theory and A is $\mathcal{L}$-equivalent to B. Then A is $Beth(\mathcal{L})$-equivalent to B. ($Beth(\mathcal{L})$ is the Beth closure of $\mathcal{L}$.)*

Proof: We first give a description of $Beth(\mathcal{L})$ (cf. [9], p. 168). Suppose $\psi(P, R_1, \ldots, R_n)$ is a Beth definition in $\mathcal{L}$; i.e., $\forall R_1 \ldots R_n \, \exists^{\leq 1} P \psi(P, R_1, \ldots, R_n)$ is valid. (Here all the relations of $\psi(P, R_1, \ldots, R_n)$ are displayed.)

Let K_ψ be the class of models of the form $(A, R_1, \ldots, R_n, \bar{a})$ such that there is a $P \in A^m$ so that $(A, P, R_1, \ldots, R_n) \models \psi(P, R_1, \ldots, R_n)$ and $\bar{a} \in P$. Let $\mathcal{L}_1 = (Q^{K_\psi}: \psi$ a Beth definition in $\mathcal{L})$. Clearly $\mathcal{L}_1 \subseteq Beth(\mathcal{L})$. Also any Beth definition in $\mathcal{L}$ is equivalent to an explicit $\mathcal{L}_1$-definition. For example, if $\psi(P, R)$ is a Beth definition where P is unary and R is binary then the following formula is valid $P(y) \wedge \psi(P, R) \leftrightarrow Q^{K_\psi} x, \langle x_1, x_2\rangle$ $(x = x, R(x_1, x_2), y)$. For $n < \omega$ analogously define $\mathcal{L}_{n+1}$ from $\mathcal{L}_n$. So $Beth(\mathcal{L}) = \bigcup \mathcal{L}_n$.

To prove the lemma it suffices to show A is $\mathcal{L}_1$-equivalent to B. In fact we will show for any formula $\psi(\bar{x}) \in \mathcal{L}_1$ there is a formula $\psi^*(x) \in L_{\omega\omega}$ such that $A, B \models \psi^*(\bar{x}) \leftrightarrow \psi(\bar{x})$. First note that the definable relations on A are contained in the relations invariant under $Aut(A)$. Since A is ω-homogeneous and ω-categorical, the relations invariant under $Aut(A)$ are exactly the $L_{\omega\omega}$-definable relations. So for any $\psi(\bar{x}) \in \mathcal{L}$ there is $\psi^*(\bar{x}) \in L_{\omega\omega}$ such that $A \models \psi(\bar{x}) \leftrightarrow \psi^*(\bar{x})$. Since A is $\mathcal{L}$-equivalent to B, $B \models \psi(\bar{x}) \leftrightarrow \psi^*(\bar{x})$. We now show an appropriate ψ^* exists by induction on $\psi \in \mathcal{L}_1$. The only interesting case occurs when $\psi(x_0, \ldots, x_n, x_{n+1}, \ldots, x_k)$ is of the form

$$Q^{K_\phi}z, \langle\bar{z}_0\rangle, \ldots, \langle\bar{z}_m\rangle \; (\rho(z, \bar{x}), \theta_0(\bar{z}_0, x), \ldots, \theta_m(\bar{z}_m, \bar{x}), x_0, \ldots, x_n) \; ,$$

where ϕ is a Beth definition in $\mathcal{L}$ of an n-ary relation. By the inductive hypothesis we can assume $\rho, \theta_0, \ldots, \theta_m \in L_{\omega\omega}$.

Consider any $a_1, \ldots, a_k \in A$. Assume there is $R \subseteq A^n$ so that $(\rho(z, \bar{a})^A; R, \theta_0(\bar{z}_0, \bar{a})^A, \ldots, \theta_m(\bar{z}_m, \bar{a})^A) \models \phi$. Since ϕ is a Beth definition, R is invariant under $Aut_{\bar{a}}(A)$. So there is a formula $\chi(\bar{y}, \bar{x}) \in L_{\omega\omega}$ such that $R = \chi(\bar{y}, \bar{a})^A$. This formula depends only on the type $p(\bar{x})$ which $\bar{a}$ satisfies. Let χ_p denote the formula χ. Since A is ω-categorical, we can take p to be a formula of $L_{\omega\omega}$. Also as A is $\mathcal{L}$-equivalent to B, if $B \models p[\bar{b}]$ then $(\rho(z, \bar{b})^B, \chi_p(\bar{y}, \bar{b})^B, \theta_0(\bar{z}_0, \bar{b})^B, \ldots, \theta_m(\bar{z}_m, \bar{b})^B) \models \phi$. Let I be the (finite) set of types p such that $X_p(\bar{y}, \bar{x})$ exists.

Now assume no such $R \subseteq A^n$ exists. This is a property of the type p which $\bar{a}$ realizes. This is expressible in $\mathcal{L}$ by saying "$p(\bar{x})$ implies no formula (there are only finitely many nonequivalent ones) $\chi(\bar{y}, \bar{x})$ defines such an R". Since B is also ω-homogeneous, for all $\bar{b} \in B$ such that $B \models p[\bar{b}]$ there is no such R.

So $A, B \models \psi(\bar{x}) \leftrightarrow \left(\left(\bigvee_{p\in I} p(\bar{x})\right) \wedge \bigwedge_{p\in I} (p(\bar{x}) \to \chi_p(x_1, \ldots, x_n, \bar{x}))\right)$.

3.5 Theorem
(1) *Beth* $(L(Q))$ *is not congruence closed.*
(2) *Beth* $(L(Q^{cf}))$ *is not congruence closed.* (Q^{cf} *is* Q^K *where* K *is the class of linear orders of cofinality* ω.)

Proof: (1) Apply 3.4 to Keisler's example.

(2) Let A_1 be $(\eta, <)$ and A_2 be any ω-homogeneous dense linear order of cofinality ω_1 (e.g., $\eta \cdot \omega_1$). Define $<$ on $A_l \times 2$ $(l = 0, 1)$ by $(a, i) < (a', j)$ iff $a < a'$. Then $A_1 \times 2$ is $L(Q^{cf})$-equivalent to $A_2 \times 2$. (The point is that in $A_l \times 2$ there is no definable infinite set with a definable linear order—even using parameters.)

3.6 Proposition *Suppose* $\mathcal{L}$ *is a congruence closed logic. Then Beth* $(\mathcal{L})$ *and w.Beth* $(\mathcal{L})$ *are congruence closed.* (*w.Beth* $(\mathcal{L})$ *is the weak Beth closure of* $\mathcal{L}$; *i.e., we demand explicit definitions for any relation* P *such that* $\exists^{=1} P\Psi(P, R_0, \ldots, R_m)$ *is valid.*)

Proof: We only give the proof for *Beth* $(\mathcal{L})$. It suffices to show $\mathcal{L}_1$ is congruence closed. Suppose $\psi(P, R_0, \ldots, R_m)$ is a Beth definition of P in $\mathcal{L}$. Let $K = K_\psi$. By 3.1 it is enough to show Q^{K^*} is a quantifier in $\mathcal{L}_1$. Since $\mathcal{L}$ is congruence closed there is a formula $\psi^*(P, E, R_0, \ldots, R_m)$ such that for all A, $(A, E^A, P^A, R_0^A, \ldots, R_n^A) \models \psi^*$ iff E^A is a congruence relation and $(A/E^A, P^A/E^A, R_0^A/E^A, \ldots, R_M^A/E^A) \models \psi$. Clearly, $\psi^*(P, E, R_0, \ldots, R_m)$ is a Beth definition of P.

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