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On Generic Structures

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Abstract We discuss many generalizations of Fraissé's construction of countable 'homogeneous-universal' structures. We give characterizations of when such a structure is saturated and when its theory is ω-categorical. We also state very general conditions under which the structure is atomic.

1 Introduction In this paper we investigate variations on the classical construction of countable homogeneous-universal structures from appropriate classes of finite structures. The most basic result here is the following theorem of Fraissé [1]:

Theorem 1.1 *Let K be a class of finite structures in a finite, relational language that is closed under isomorphism and substructure. Assume further that K satisfies the joint embedding property and amalgamation. Then,*

1. *there is a unique, countable $\mathfrak{A}$ which is "homogeneous-universal" for K, i.e., $\mathfrak{A}$ is (ultra)-homogeneous and K is precisely the class of finite structures embeddable in $\mathfrak{A}$;*
2. *the complete theory of the structure $\mathfrak{A}$ in (1) is ω-categorical.*

It is easy to see that (1) holds also for countably infinite relational languages provided K contains only countably many isomorphism types, but (2) may fail in this context. If K is not closed under substructure then the same basic argument establishes a variant of (1) in which $\mathfrak{A}$ satisfies a weaker sort of homogeneity (called *pseudo-homogeneity* by Fraissé); here too (2) may fail, even if the language is finite. More recently, Hrushovski [3,4] has used a construction that generalizes the basic construction by replacing substructure by stronger relations.

In this paper we unify all of these variations in a single framework (allowing also functions and constants in the language). We refer to the resulting structures as *generic* rather than as homogeneous-universal. We then investigate some properties of these generics. Ever since Morley-Vaught there has been a tendency to view homogeneous-universal structures as analogues of saturated models. The

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main question we consider is how to determine the conditions under which the generic is actually saturated. We also give various examples where this fails.

Throughout we will assume that the underlying language is always countable, but it may contain function and constant symbols. Whenever we mention a class K, K will be a class of finite L-structures closed under isomorphism. The following definition is the starting point of our discussion.

Definition 1.2 A class $(K, \leq)$ of finite structures, together with a relation $\leq$ on $K \times K$, is called *smooth* if $\leq$ is transitive, $\mathfrak{B} \leq \mathfrak{C}$ implies $\mathfrak{B} \subseteq \mathfrak{C}$, and for all $\mathfrak{B} \in K$ there is a collection $p^{\mathfrak{B}}(\bar{x})$ of universal formulas with $|\bar{x}| = |\mathfrak{B}|$ and for any $\mathfrak{C} \in K$ with $\mathfrak{B} \subseteq \mathfrak{C}$,

$$\mathfrak{B} \leq \mathfrak{C} \Leftrightarrow \mathfrak{C} \models \phi(\bar{b}) \text{ for all } \phi \in p^{\mathfrak{B}}$$

where $\bar{b}$ enumerates the universe of $\mathfrak{B}$. We also require that $p^{\mathfrak{B}} = p^{\mathfrak{C}}$ if $\mathfrak{B} \cong \mathfrak{C}$.

It should be noted that for any class K of finite structures, $(K, \subseteq)$ is always smooth, where $\subseteq$ denotes the usual substructure relation. The reader should note also that the restriction on the formulas being universal is close to being necessary to extend the definition to $(K, \leq)$-unions.

Definition 1.3 Let $(K, \leq)$ be a smooth class of finite structures. A structure $\mathfrak{A}$ is a $(K, \leq)$*-union* if $\mathfrak{A} = \bigcup_{n \in \omega} \mathfrak{C}_n$, where each $\mathfrak{C}_n \in K$ and $\mathfrak{C}_n \leq \mathfrak{C}_{n+1}$ for all $n \in \omega$. If $\mathfrak{A}$ is a $(K, \leq)$-union and $\mathfrak{B} \subseteq \mathfrak{A}$, $\mathfrak{B} \in K$, we define

$$\mathfrak{B} \leq \mathfrak{A} \Leftrightarrow \mathfrak{A} \models \phi(\bar{b}) \text{ for all } \phi \in p^{\mathfrak{B}},$$

where again $\bar{b}$ enumerates the universe of $\mathfrak{B}$. Equivalently, $\mathfrak{B} \leq \mathfrak{A}$ if and only if $\mathfrak{B} \leq C_n$ for some (equivalently for a tail of) $n \in \omega$.

The following definition and the existence and uniqueness theorem that follows are essentially due to Fraissé [1] in the case where $\leq$ is $\subseteq$.

Definition 1.4 Suppose that $(K, \leq)$ is a smooth class of finite structures. A structure $\mathfrak{A}$ is $(K, \leq)$*-generic* if

1. $\mathfrak{A}$ is a $(K, \leq)$-union.
2. For each $\mathfrak{B} \in K$ there is $\mathfrak{B}' \leq \mathfrak{A}$, $\mathfrak{B} \cong \mathfrak{B}'$ (i.e., $\mathfrak{B}$ embeds strongly into $\mathfrak{A}$).
3. If $\mathfrak{B}, \mathfrak{C} \in K$, $\mathfrak{B}, \mathfrak{C} \leq \mathfrak{A}$ and f is an isomorphism of $\mathfrak{B}$ onto $\mathfrak{C}$, then f extends to an automorphism of $\mathfrak{A}$.

A standard back-and-forth argument shows that $\mathfrak{A}$ is $(K, \leq)$-generic if and only if conditions (1), (2), and (3*) hold, where

3* If $\mathfrak{B} \leq \mathfrak{A}$, $\mathfrak{B} \leq \mathfrak{C}$ and $\mathfrak{B}, \mathfrak{C} \in K$ then there is $\mathfrak{C}' \leq \mathfrak{A}$ and an isomorphism $f: \mathfrak{C} \to \mathfrak{C}'$ so that $f \restriction B = \text{id}$.

Recall that a class $(K, \leq)$ satisfies the joint embedding property (JEP) if for every $\mathfrak{B}_1, \mathfrak{B}_2 \in K$ there is $\mathfrak{C} \in K$ and isomorphic embeddings $f_i: \mathfrak{B}_i \to \mathfrak{C}$ so that $f_i(\mathfrak{B}_i) \leq \mathfrak{C}$ for $i = 1,2$. $(K, \leq)$ satisfies the amalgamation property (AP) if for any $\mathfrak{A}$, $\mathfrak{B}_1, \mathfrak{B}_2 \in K$ with $\mathfrak{A} \leq B_1$ and $\mathfrak{A} \leq B_2$ there is $\mathfrak{C} \in K$ and isomorphic embeddings $f_i: \mathfrak{B}_i \to \mathfrak{C}$ so that $f_i(\mathfrak{B}_i) \leq \mathfrak{C}$ for $i = 1,2$ and $f_1 \restriction A = f_2 \restriction A$.

Theorem 1.5 *Suppose $(K, \leq)$ is a smooth class of finite structures.*
- *There is a $(K, \leq)$-generic structure if and only if K contains only countably many isomorphism types and $(K, \leq)$ satisfies (JEP) and (AP).*
- *If $\mathfrak{A}$ and $\mathfrak{A}'$ are each $(K, \leq)$-generic then $\mathfrak{A} \cong \mathfrak{A}'$.*

2 Generic structures when $\leq$ is $\subseteq$

This section will be devoted to a discussion of the model-theoretic properties of a $(K, \leq)$-generic structure when $\leq$ is simply $\subseteq$. We will be particularly interested in characterizing when the generic structure is saturated. As this setting is a special case of the general theory, however, we will need to anticipate theorems from the next section in our discussion.

We call a structure $\mathfrak{A}$ locally finite if for all finite $X \subseteq A$ there is a finite substructure $\mathfrak{B}$ of $\mathfrak{A}$ containing X. A theory T is locally finite if every model of T is locally finite. Let

$$K(T) = \{\text{all finite substructures of models of } T\}.$$

The following remark follows easily from the definitions and the fact that any model of $T_\forall$ can be extended to a model of T.

Remark 2.1

1. T is locally finite iff $T_\forall$ is locally finite.
2. $K(T) = K(T_\forall)$.
3. If $\mathfrak{A}$ is K-generic then $K \subseteq K(\mathrm{Th}(\mathfrak{A}))$.

We next distinguish two nice subclasses of $K(T)$. A subclass K of $K(T)$ is *cofinal* if for any $\mathfrak{B} \in K(T)$ there is $\mathfrak{C} \in K$ with $B \subseteq \mathfrak{C}$. We call a subclass K of $K(T)$ *large* if any countable model of T is contained in a union of an increasing chain of elements of K.

Certainly K a large subset of $K(T)$ implies K is cofinal and T is locally finite. However, the following example shows that the converse does not hold, even in the case where a K-generic structure exists.

Example 2.2 A locally finite theory T and a cofinal subclass K of $K(T)$ that is not large.

Let $L = \{S\}$, S a binary relation, and let T be the theory of a successor function, i.e., T says every element has a unique successor and a unique predecessor. Let K denote the finite *models* of T. Then $\mathfrak{A}$, the K-generic structure, just consists of infinitely many disjoint copies of every finite cycle. However K is not large, as $(\mathbf{Z}, S)$ is not contained in the union of finite cycles.

As far as the utility of these notions is concerned, the following proposition states that if K' is a cofinal subclass of K and K has a generic then K' has the same generic. Consequently, the interesting case is one in which a large subclass of $K(T)$ satisfies (JEP) and (AP) whereas $K(T)$ does not.

Proposition 2.3 *Assume that K' is a cofinal subclass of a class K of finite structures, and assume that $\mathfrak{A}$ is K-generic. Then $\mathfrak{A}$ is K'-generic as well. In particular K' also satisfies (JEP) and (AP).*

Proof: Conditions (2) and (3*) of the alternate definition of genericity follow trivially from $\mathfrak{A}$ being K-generic and K' being a subclass of K, so it suffices to show that $\mathfrak{A}$ is a union of a chain of elements from K'. To see this, it first follows from the cofinality of K' and condition (3*) of $\mathfrak{A}$ being K-generic that given any $\mathfrak{B} \subseteq \mathfrak{A}$ with $\mathfrak{B} \in K$ there is $\mathfrak{C} \in K'$ with $\mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{A}$. Now suppose $\mathfrak{A} = \bigcup_{n\in\omega} \mathfrak{B}_n$ with $\mathfrak{B}_n \subseteq \mathfrak{B}_{n+1}$ and $\mathfrak{B}_n \in K$ for each $n \in \omega$. We construct a chain $\langle \mathfrak{C}_l : l \in \omega \rangle$ of elements of K' by induction on l as follows: let $\mathfrak{C}_0 \in K'$ be arbitrary such that $\mathfrak{B}_0 \subseteq \mathfrak{C}_0 \subseteq \mathfrak{A}$. Next, given $\mathfrak{C}_l \subseteq \mathfrak{A}$, pick n least such that $C_l \subseteq B_n$ and choose $\mathfrak{C}_{l+1} \in K'$ so that $\mathfrak{B}_n \subseteq \mathfrak{C}_{l+1} \subseteq \mathfrak{A}$ by the note above. Clearly $\mathfrak{A} = \bigcup_{l\in\omega} \mathfrak{C}_l$, so $\mathfrak{A}$ is K'-generic.

Note that $K \subseteq K(T)$ is large if and only if any model of $T_\forall$ is contained in a union of an increasing chain from K, so largeness also depends only on $T_\forall$. We also remark that $K(T)$ is a large subset of $K(T)$ if and only if T is locally finite. Also, it is easy to verify that if K is a large subset of $K(T)$ then K is closed under substructures iff $K = K(T)$.

The following lemma is a generalization of a theorem of Fraissé.

Lemma 2.4 *Suppose $K \subseteq K(T)$ is cofinal, T is locally finite and $\mathfrak{A}$ is K-generic. Then $\mathfrak{A}$ is an e.c. model of $T_\forall$.*

Proof: As $\mathfrak{A}$ is the union of substructures of models of $T_\forall$, $\mathfrak{A} \models T_\forall$ by the usual preservation results. Now assume $\phi(\bar{x}, \bar{a})$ is a quantifier-free formula such that there is some $\mathfrak{B} \supseteq \mathfrak{A}$, $\mathfrak{B} \models T_\forall$, and $\mathfrak{B} \models \exists \bar{x}\phi(\bar{x}, \bar{a})$. By adding dummy variables as needed, we may assume $\bar{a}$ is the universe of some $\mathfrak{A}_0 \in K$. Now let $\mathfrak{B}_0 \subseteq \mathfrak{B}$ be finite with $\mathfrak{B}_0 \models \exists \bar{x}\phi(\bar{x}, \bar{a})$. As $\mathfrak{B}_0 \in K(T)$ there is $\mathfrak{C} \in K$, $\mathfrak{B}_0 \subseteq \mathfrak{C}$. Now we finish by amalgamating $\mathfrak{C}$ into $\mathfrak{A}$ over $\mathfrak{A}_0$.

The main theorem of this section is the following equivalence.

Theorem 2.5 *Assume $\mathfrak{A}$ is K-generic and $T = \mathrm{Th}(\mathfrak{A})$. The following conditions are equivalent:*
1. *$\mathfrak{A}$ is saturated.*
2. *K is a large subclass of $K(T)$ and T is model complete.*
3. *K is a large subclass of $K(T)$ and $T_\forall$ is companionable.*

Proof: (1) $\Rightarrow$ (2). Assume that $\mathfrak{A}$ is saturated. Then for any $\mathfrak{B} \models T$, $\mathfrak{B}$ countable, $\mathfrak{B}$ embeds (elementarily) into $\mathfrak{A}$, which is a union of elements of K, so K is a large subclass of $K(T)$.

As a first step toward showing model completeness we show that if $\bar{a}, \bar{b}$ are from A and $\mathrm{tp}_\exists(\bar{a}) \subseteq \mathrm{tp}_\exists(\bar{b})$ then $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$. Suppose $\bar{a}$ and $\bar{b}$ are as above. Choose $\mathfrak{C} \subseteq \mathfrak{A}$, $\mathfrak{C} \in K$ with $\bar{a}$ from C. Let

$$q(\bar{x}, \bar{y}) = \{\alpha(\bar{x}, \bar{y}) : \mathfrak{A} \models \alpha(\bar{a}, \bar{d}),\ \alpha \text{ q.f.}\},$$

where $\bar{d}$ enumerates $C \setminus \bar{a}$. Since $\mathrm{tp}_\exists(\bar{a}) \subseteq \mathrm{tp}_\exists(\bar{b})$, $q(\bar{b}, \bar{y})$ is consistent, hence realized in $\mathfrak{A}$ by some $\bar{e}$. Now defining f by $f(\bar{a}\bar{d}) = \bar{b}\bar{e}$, f is an isomorphism of elements of K so f extends to an automorphism of $\mathfrak{A}$, which implies that $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$ as desired.

To complete the proof of model completeness, suppose $\phi(x)$ is any formula. It follows from the paragraph above and the saturation of $\mathfrak{A}$ that for any $\bar{a}$ from A so that $\mathfrak{A} \models \phi(\bar{a})$, there is a single existential formula $\theta_{\bar{a}}(\bar{x})$ such that $\mathfrak{A} \models \theta_{\bar{a}}(\bar{a})$ and $T \models \forall \bar{x}[\theta_{\bar{a}}(\bar{x}) \rightarrow \phi(\bar{x})]$. But now, by the saturation of $\mathfrak{A}$ again, it follows that $\mathfrak{A} \models \forall \bar{x}[\phi(\bar{x}) \leftrightarrow \psi(\bar{x})]$, where ψ is a finite disjunction of $\theta_{\bar{a}}$'s. So ϕ is equivalent to an existential formula, which implies T is model complete.

(2) $\Rightarrow$ (3). As T is model complete, T is the model companion of $T_\forall$.

(3) $\Rightarrow$ (1). Assume that S is the model companion of $T_\forall$. Now as K is cofinal and T is locally finite, $\mathfrak{A}$ is an e.c. model of $T_\forall = S_\forall$ by Lemma 2.4. Thus $\mathfrak{A} \models S$, so T is model complete. However, as noted above, every countable model of $\mathrm{Th}(\mathfrak{A})$ embeds isomorphically, hence elementarily into $\mathfrak{A}$. That is, $\mathfrak{A}$ is a universal model of T, so $\mathfrak{A}$ is saturated by Proposition 3.1.

If we assume K is closed under substructures we obtain the following corollary.

Corollary 2.6 *Assume K is closed under substructures, $\mathfrak{A}$ is K-generic and $T = \mathrm{Th}(\mathfrak{A})$. Then $\mathfrak{A}$ is saturated if and only if T is locally finite, $K = K(T)$, and $T_\forall$ is companionable.*

Proof: Assume that $\mathfrak{A}$ is saturated. Then $\mathfrak{A}$ is universal. However, as $\mathfrak{A}$ is K-generic, it is locally finite, so T must be locally finite. Also, by the theorem above, K is a large subclass of $K(T)$, so $K = K(T)$ and $T_\forall$ is companionable.

For the converse we need only recall that T locally finite implies $K(T)$ is a large subclass of itself and apply the theorem.

We conclude this section with two examples of classes K each having a generic structure that is not saturated. In the first example K is not a large subclass of $K(\mathrm{Th}(\mathfrak{A}))$, and in the second $\mathrm{Th}(\mathfrak{A})$ is not model complete.

Example 2.7 A class K closed under substructure with a K-generic structure $\mathfrak{A}$ such that $\mathrm{Th}(\mathfrak{A})$ admits elimination of quantifiers, yet $\mathfrak{A}$ is not saturated (in fact $\mathrm{Th}(\mathfrak{A})$ does not have a prime model).

Note that by Proposition 3.4, the underlying language must be infinite. Let $L = \{P_s : s \in {}^{<\omega}2\}$, where each P_s is a unary predicate. Let $\mathfrak{B}$ be the structure with universe ${}^{\omega}2$ and $P_s^{\mathfrak{B}} = \{f \in {}^{\omega}2 : s \subseteq f\}$ for each $s \in {}^{<\omega}2$ and let $T = \mathrm{Th}(\mathfrak{B})$. As is well-known, T has no prime model and T admits elimination of quantifiers.

Let $\mathfrak{A}$ be any countable model of T such that

$$\left|\bigcap \{P_s^{\mathfrak{A}} : s \subseteq f\}\right| \leq 1 \quad \text{for every } f \in {}^{\omega}2.$$

Let K be the class of all finite L-structures embeddable in $\mathfrak{A}$. It is easy to check that $\mathfrak{A}$ is K-generic.

Example 2.8 A theory T in a finite, relational language and a large subclass K of $K(T)$ so that there is a K-generic structure $\mathfrak{A}$ that is not saturated.

Let $L = \{E, R\}$, where E and R are both binary predicates. Fix (as in Henson [2]) a countable collection $(\langle G_n, R_n \rangle)_{n \in \omega}$ of mutually non-embeddable finite tournaments (i.e., a directed graph with an edge in some direction between any two vertices).

Let T be the following collection of universal axioms:

- E is an equivalence relation
- On each equivalence class, R defines a directed graph
- $R(x,y) \to E(x,y)$
- For each $n \neq m$ an axiom stating that if G_n embeds into an equivalence class then G_m does not embed into the same class.

Let K consist of all $\mathcal{B} \in K(T)$ such that some G_n embeds in each $E^{\mathcal{B}}$-class (with the n allowed to vary among the classes). The verification that K satisfies (JEP) and (AP) is as in [2]. Let $\mathcal{A}$ denote the K-generic structure. Clearly $\mathrm{Th}(\mathcal{A})$ is not ω-categorical since there are infinitely many 1-types ("the equivalence class of x embeds G_n"). However, it is easy to verify (or one can invoke Proposition 3.4) that $\mathcal{A}$ is atomic, so it cannot be saturated.

Finally, to see that K is a large subclass of $K(T)$, let $\mathcal{B}$ be a countable model of T. Then there is a countable model $\mathcal{C}$ of T extending $\mathcal{B}$ with every $E^{\mathcal{C}}$-class embeds some G_n, and $\mathcal{C}$ can be written as a union of a chain from K as desired.

3 The smooth case In this section we wish to study general facts about $(K,\leq)$-generic structures for an arbitrary smooth class of finite structures. Our first result requires only that $\leq$ be type-definable.

Proposition 3.1 *Assume that $(K,\leq)$ is smooth and that the $(K,\leq)$-generic structure $\mathcal{A}$ is weakly saturated (i.e., $\mathcal{A}$ realizes every pure type consistent with* $\mathrm{Th}(\mathcal{A})$). *Then $\mathcal{A}$ is saturated.*

Proof: We first show that every model $\mathcal{B}$ of $\mathrm{Th}(\mathcal{A})$ which is a $(K,\leq)$-union can be elementarily embedded in $\mathcal{A}$. Say that $\mathcal{B}$ is the union of the $(K,\leq)$-chain $\{\mathcal{B}_n\}_{n\in\omega}$. Since $\mathcal{A}$ is $(K,\leq)$-generic we may assume that $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B}_n \leq \mathcal{A}$ for all $n \in \omega$. We show that $\mathcal{B} \preceq \mathcal{A}$. Suppose $\mathcal{A} \models \phi(\bar{b})$ where $\bar{b} \subseteq B$. We may assume that $\bar{b}$ enumerates some B_n. We may further take $p^{\mathcal{B}_n}(\bar{x})$ to include the open diagram of $\mathcal{B}_n$. In order to show $\mathcal{B} \models \phi(\bar{b})$ it suffices to show that $\mathrm{Th}(\mathcal{A}) \models \bigwedge p^{\mathcal{B}_n}(\bar{x}) \to \phi(\bar{x})$. However, if this failed then by the weak saturation of $\mathcal{A}$ we could find some $\bar{a} \subseteq A$ realizing $p^{\mathcal{B}_n}(\bar{x}) \cup \{\neg\phi(\bar{x})\}$. But then by genericity there would be an automorphism of $\mathcal{A}$ taking $\bar{a}$ to $\bar{b}$, which is a contradiction.

Next, since $\mathcal{A}$ is weakly saturated and is itself a $(K,\leq)$-union, we know that for every type $q(\bar{x})$ consistent with $\mathrm{Th}(\mathcal{A})$ there is some $\mathcal{B} \in K$ such that $q(\bar{x}) \cup p^{\mathcal{B}}(\bar{x},\bar{y})$ is consistent. It follows from this that any countable model of $\mathrm{Th}(\mathcal{A})$ has an elementary extension that is a $(K,\leq)$-union. Thus $\mathcal{A}$ is universal. In particular $\mathrm{Th}(\mathcal{A})$ is small, so we can find $\mathcal{A}'$, a countably saturated elementary extension. From this we can form a chain $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \mathcal{A}_2 \preceq \ldots$ where $\mathcal{A}_{2n}$ is isomorphic to $\mathcal{A}$ and $\mathcal{A}_{2n+1}$ is countably saturated. Let $\mathcal{A}^*$ be the union of this chain. Now $\mathcal{A}^*$ is saturated, so it suffices to show that $\mathcal{A}^*$ is $(K,\leq)$-generic. The first two clauses are easily checked as every $\mathcal{B} \in K$ embeds strongly in $\mathcal{A}$ and $\mathcal{A} \preceq \mathcal{A}^*$, so $\mathcal{B}$ embeds strongly in $\mathcal{A}^*$. Finally, if $f\colon \mathcal{B} \to \mathcal{C}$ is an isomorphism of substructures of $\mathcal{A}^*$ with $\mathcal{B},\mathcal{C} \in K$ and $\mathcal{B},\mathcal{C} \leq \mathcal{A}^*$ then there is $\mathcal{A}_{2k}$ so that $\mathcal{B}$ and $\mathcal{C}$ are substructures of $\mathcal{A}_{2k}$ and $\mathcal{B},\mathcal{C} \leq \mathcal{A}_{2k}$. Thus f extends to an automorphism of $\mathfrak{A}_{2k}$, so B and C (as sets) have the same complete type over $\emptyset$. So, as $\mathfrak{A}^*$ is saturated, f extends to an automorphism of $\mathfrak{A}^*$.

Recall that $\mathfrak{A} \leq_1 \mathfrak{B}$ if and only if $\mathfrak{A} \subseteq \mathfrak{B}$ and

$$\mathfrak{A} \models \phi(\bar{a}) \Leftrightarrow \mathfrak{B} \models \phi(\bar{a})$$

for any universal formula $\phi(\bar{x})$ and any $\bar{a}$ from A. We call a theory T *1-model complete* if for any two models $\mathfrak{A}, \mathfrak{B}$ of T, $\mathfrak{A} \leq_1 \mathfrak{B}$ implies $\mathfrak{A} \leq \mathfrak{B}$.

Theorem 3.2 *Assume that $(K, \leq)$ is a smooth class of finite structures and that $\mathfrak{A}$ is $(K, \leq)$-generic. Let $T = \mathrm{Th}(\mathfrak{A})$. Then the following are equivalent:*
1. *$\mathfrak{A}$ is saturated.*
2. (a) $\mathrm{Th}(\mathfrak{A})$ *is 1-model complete,*
   (b) *every countable model of T can be embedded as a 1-substructure of some $(K, \leq)$-union, and*
   (c) *$\mathfrak{A}$ realizes every universal type consistent with T.*

Proof: The proof that if $\mathfrak{A}$ is saturated then T is 1-model complete is exactly analogous to the proof of model completeness in Theorem 2.5. The other two clauses of (2) are immediate consequences of the saturation of $\mathfrak{A}$.

Conversely, by Proposition 3.1 it suffices to show that (2) implies that every model $\mathfrak{B}$ of T that is a $(K, \leq)$-union is elementarily embeddable in $\mathfrak{A}$. So let $\mathfrak{B} \models T$ be the union of the $(K, \leq)$-chain $\{\mathfrak{B}_n\}_{n \in \omega}$. As before, we may assume that $\mathfrak{B} \subseteq \mathfrak{A}$ and each $\mathfrak{B}_n \leq \mathfrak{A}$. In view of (2a) it suffices to show that $\mathfrak{B} \leq_1 \mathfrak{A}$. If this fails then there is some universal formula $\theta(\bar{x})$ and some $\bar{b} \subseteq B$ such that $\mathfrak{B} \models \theta(\bar{b})$ but $\mathfrak{A} \models \neg\theta(\bar{b})$. Again we may assume that $\bar{b}$ enumerates some B_n and that $p^{\mathfrak{B}_n}(\bar{x})$ includes the complete open diagram of $\mathfrak{B}_n$. But now by (2c) there is some $\bar{a} \subseteq A$ realizing $p^{\mathfrak{B}_n}(\bar{x}) \cup \{\theta(\bar{x})\}$, contradicting the genericity of $\mathfrak{A}$.

Theorem 3.2 is the result in the general smooth case that corresponds to Theorem 2.5, with (2b) being the correct generalization of the condition that K is a large subclass of $K(T)$. The presence of (2c) seems to be a defect in this result. We do not know if this condition can be deleted.

The following example shows that 1-model completeness in the theorem above cannot be improved, even when the language is nice and the types defining $\leq$ are simply formulas and the theory of the generic is ω-categorical.

Example 3.3 A smooth class $(K, \leq)$ in a finite, relational language whose generic has a theory that is ω-categorical but not model complete.

Let $L = \{E\}$ and let T be the ω-categorical theory specifying that E is an equivalence relation with at most two elements in each class, there are infinitely many classes with two elements and infinitely many classes with only one element. Let $K = K(T)$ and for $\mathfrak{B}, \mathfrak{C} \in K$ define $\mathfrak{B} \leq \mathfrak{C}$ if and only if $\mathfrak{B} \subseteq \mathfrak{C}$ and $\mathfrak{C}$ does not expand any $\mathfrak{B}$-equivalence class. It is easy to check that the $(K, \leq)$-generic structure is the countable model of T and that T is not model complete.

Theorem 3.2 is not entirely satisfactory as given a smooth class $(K, \leq)$ it may be very hard to determine if the clauses of (2) hold. We obtain a more useful characterization of saturation if we restrict both the language and the complexity of the definition of $\leq$.

Suppose L is a finite language (i.e., has only finitely many nonlogical symbols). For any finite L-structure $\mathfrak{B}$, fix an open formula $\theta_{\mathfrak{B}}(\bar{x})$ so that for any L-structure $\mathfrak{A}$ and any $\bar{a}$ from A,

$$\mathfrak{A} \models \theta_{\mathfrak{B}}(\bar{a}) \text{ if and only if the function } f: \bar{a} \to \bar{b} \text{ is an isomorphism,}$$

where $\bar{b}$ is a fixed enumeration of the universe of $\mathfrak{B}$.

The proof of the following proposition is straightforward, but the result is somewhat surprising as it applies to a large number of known examples of generic structures.

Proposition 3.4 *Suppose L is a finite language and $(K, \leq)$ is a smooth class of finite L-structures such that for every $\mathfrak{B} \in K$ the set of formulas $p^{\mathfrak{B}}$ defining $\leq$ consists of a single universal formula $\psi^{\mathfrak{B}}$. If $\mathfrak{A}$ is $(K, \leq)$-generic then $\mathfrak{A}$ is atomic.*

Proof: Suppose $\bar{a}$ is any finite subset of A and choose $\mathfrak{B} \in K$, $\mathfrak{B} \leq \mathfrak{A}$ such that $\bar{a} \subseteq B$. It now follows from the genericity of $\mathfrak{A}$ that $\text{tp}(\bar{a})$ is isolated by the formula $\exists \bar{y}(\theta_{\mathfrak{B}}(\bar{x}, \bar{y}) \wedge \psi^{\mathfrak{B}}(\bar{x}, \bar{y}))$, where $\theta_{\mathfrak{B}}$ and $\psi_{\mathfrak{B}}$ are defined relative to an enumeration of $\mathfrak{B}$ with $\bar{a}$ an initial segment.

In fact, it follows that $\mathfrak{A}$ is the only model of $\text{Th}(\mathfrak{A})$ (up to isomorphism) that can be written as a $(K, \leq)$-union. With the above proposition in hand we are now able to give a nice characterization of when the $(K, \leq)$-generic structure is saturated.

Theorem 3.5 *Suppose that L is a finite language, $(K, \leq)$ is smooth and for every $\mathfrak{B} \in K$, $p^{\mathfrak{B}}$ consists of a single, universal formula $\psi^{\mathfrak{B}}$. Assume further that $\mathfrak{A}$ is $(K, \leq)$-generic. Then the following conditions are equivalent.*

1. *$\mathfrak{A}$ is saturated.*
2. *$\text{Th}(\mathfrak{A})$ is ω-categorical.*
3. *For all n there is N so that if $\bar{a} \in {}^nA$ then there is $\mathfrak{B} \in K$ with $\bar{a} \subseteq B$, $|B| \leq N$ and $\mathfrak{B} \leq \mathfrak{A}$.*
4. *For all n there is N so that for every $\mathfrak{B} \in K$ and every $\bar{b} \in {}^nB$ there are $\mathfrak{B}^*$ and $\mathfrak{C}$ in K with $\bar{b} \subseteq B^*$, $|B^*| \leq N$, $\mathfrak{B} \leq \mathfrak{C}$, and $\mathfrak{B}^* \leq \mathfrak{C}$.*

Proof: In light of Proposition 3.4, (1), (2), and (3) are easily seen to be equivalent, and (3) $\Rightarrow$ (4) is clear. Thus it suffices to prove (4) $\Rightarrow$ (3). So fix n and let N be the bound given by (4). Fix $\bar{a} \in {}^nA$ and choose $\mathfrak{A}_0 \leq \mathfrak{A}$ with $\bar{a} \subseteq A_0$. Now apply (4) to obtain $\mathfrak{B}^*$ and $\mathfrak{C}$ in K so that $\bar{a} \subseteq B^*$, $\mathfrak{B}^* \leq \mathfrak{C}$, $\mathfrak{A}_0 \leq \mathfrak{C}$, and $|B^*| \leq N$. As $\mathfrak{A}$ is $(K, \leq)$-generic, $\mathfrak{A}_0 \leq \mathfrak{A}$, and $\mathfrak{A}_0 \leq \mathfrak{C}$, there is $\mathfrak{C}' \leq \mathfrak{A}$ and an isomorphism $f: \mathfrak{C} \to \mathfrak{C}'$ such that $f \upharpoonright A_0 = \text{id}$. So let $\mathfrak{B} = f(\mathfrak{B}^*)$. Now $\mathfrak{B} \leq \mathfrak{C}' \leq \mathfrak{A}$, so $\mathfrak{B} \leq \mathfrak{A}$ by transitivity and $\bar{a} = f(\bar{a}) \subseteq B$, so $\mathfrak{B}$ is as desired in (3).

Fraissé's result, Theorem 1.1, is a consequence of Theorem 3.5 since condition (4) clearly holds in his context. We leave it to the reader to verify that the requirement in 3.4 and 3.5 that the language be finite can be relaxed to the following:

- For each n there are only finitely many inequivalent quantifier-free n-types realized among all elements of K.

Further, this is a generalization of the language being finite at least among classes $(K, \leq)$ where a generic structure exists.

Our final example shows that even in the context of a finite, relational language and $\leq$ being definable by a single formula it is still possible for the $(K,\leq)$-generic structure to have a complicated theory.

Example 3.6 A smooth class $(K,\leq)$ of finite structures in a finite, relational language where, for each $\mathcal{B} \in K$, $p^{\mathcal{B}}$ consists of a single, universal formula yet the theory of the $(K,\leq)$-generic is not small.

Let $L = \{E,\leq,P\}$, where E is ternary, $\leq$ is binary, and P is unary. Let $T = \mathrm{Th}(\mathcal{B})$, where $B = \omega \cup {}^{\omega}2$, $P^{\mathcal{B}} = \omega$, $\leq^{\mathcal{B}}$ is $\leq$ on ω, and $E^{\mathcal{B}} \subseteq \omega \times {}^{\omega}2 \times {}^{\omega}2$ is defined by

$$E(k,f,g) \Leftrightarrow f(k) = g(k).$$

Thus, $(E^{\mathcal{B}}(k,\cdot,\cdot))_{k\in\omega}$ are cross-cutting equivalence relations, each with two classes.

Let $K = K(T)$ and define $\leq$ on $K \times K$ by $\mathcal{C}_1 \leq \mathcal{C}_2$ if and only if both

1. $(P^{\mathcal{C}_2},\leq^{\mathcal{C}_2})$ is an end-extension of $(P^{\mathcal{C}_1},\leq^{\mathcal{C}_1})$
2. For every $k \in P^{\mathcal{C}_2}\backslash P^{\mathcal{C}_1}$, all of C_1 is in the same $E^{\mathcal{C}_2}(k,\cdot,\cdot)$-equivalence class.

It is left to the reader to verify that (JEP) and (AP) hold for $(K,\leq)$ so there is a $(K,\leq)$-generic structure $\mathcal{A}$, which must be the prime model of $\mathrm{Th}(\mathcal{A})$, yet $\mathrm{Th}(\mathcal{A})$ is not small.

We remark that Hrushovski's example [3] of a stable, $\aleph_0$-categorical pseudoplane satisfies the hypotheses of this theorem. By contrast, his example [4] of a new strongly minimal set is $(K,\leq)$-generic where $(K,\leq)$ is smooth but $p^{\mathcal{B}}$ is not definable by a single formula, as can be seen by Theorem 3.4 since the generic is not prime.

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