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Book Review

S. Shelah, *Classification Theory and the Number of Non-Isomorphic Models*. North-Holland, Amsterdam, 1990. 705 pages.

This book contains the proof of the main theorem to date in the branch of model theory known loosely as "Classification theory". The result in question is

The Main Gap Theorem *Let T be a countable first-order theory.*
(1) *If T is not superstable or (is superstable) deep or with the* DOP *or the* OTOP, *then for every uncountable* λ, $I(\lambda, T) = 2^{\lambda}$.
(2) *If T is shallow superstable without the* DOP *and without the* OTOP, *then for every* $\alpha > 0$, $I(\aleph_{\alpha}, T) < \beth_{\omega_1}(|\alpha|)$.

In this review I will attempt to define some of the terms in the theorem, give a rough outline of the proof (which is several hundred pages long), and explain why Shelah sees this as a completion of the classification problem for countable first-order theories. Although this revised edition was not published until 1990, Shelah has not included any results since 1983 in the main body of the book. I will try to indicate where doing so can simplify the proof. I am assuming in this review that the reader understands the basic notions of model theory (as found in, e.g., Chang and Keisler [3]).

The book contains thirteen chapters, of which nine appeared in the original 1978 edition. As there were no essential changes made to the original chapters I will say little about them. (The material is well-described in Lascar [7] and Baldwin [1].) There are other sources for many of the results contained in the first five chapters reflecting changes in viewpoint which have come about in the past ten years. (See, e.g., Lascar [8], Baldwin [2], Poizat [10], Pillay [9] and Hrushovski [6].) Chapter IV on isolation relations contains many results which will probably not be used elsewhere. There are now a handful of isolation relations which suffice in all known settings. Chapter VI on ultraproducts does not play a role in the proof of the Main Gap Theorem. Chapters VII and VIII contain the so-called many-model arguments. Assuming that the theory somehow codes an order or a complicated tree, the maximum number of models is constructed in any sufficiently large cardinality. The proofs of these theorems are probably the least known of all important results in model theory. The proofs often involve com-

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plicated set-theoretic arguments, for which there are no other sources in the literature.

Shelah's program (for non-elementary as well as elementary classes) is to show that there is a clear dividing line between the theories for which there is a structure theorem and those for which there is no structure theorem. *Very* roughly, a structure theorem is a result about the models of a particular theory which assigns a simple cardinal-like invariant to each model. By *invariant* we mean that models are isomorphic if and only if they have the same invariants. Of course, the isomorphism type of a model is an invariant of the model, but it fails the criterion of being cardinal-like. An example of an acceptable invariant is the dimension of a vector space: two vector spaces (over the same field) are isomorphic if they have the same dimension. For another example, let T be the theory with two equivalence relations such that E_1 refines E_2, each E_1-class is infinite, and there are infinitely many such classes in each E_2-class. This theory does have a good structure theorem. A model of cardinality $\aleph_1$, for example, is assigned an invariant as follows. Tag each E_2-class with the number of countable and the number of uncountable E_1-classes it contains. The invariant for the model is the number of E_2-classes with a given tag. As this example shows, an invariant may involve a nesting of simpler invariants. It is difficult to define precisely a notion of invariant which captures all of the examples for which we agree there is a structure theorem. (This is done to *some* satisfaction in Chapter X, Section 1.) Close approximations are given in [1] and Shelah [12]; however, the definitions there are known to be slightly flawed. Even today there remains some fuzziness as to what counts as a structure theorem for an elementary class. An excellent discussion of this problem is found in Hodges [5].

How, then, do we proceed to settle the classification problem if we do not have a precise definition of a structure theorem? Shelah's approach is through spectrum functions. For T a theory the *spectrum function of T* is the function $I(-, T)$ such that for λ an infinite cardinal $I(\lambda, T)$ is the number of models of T of cardinality λ up to isomorphism. At least intuitively the more complicated a theory is the more models we expect it to have. Furthermore, whatever we settle on as an acceptable notion of isomorphism invariant, there should be relatively few of them associated to models of a fixed cardinality. In order to include all of the known structure theorems for countable first-order theories, we must take as our greatest lower bound on the number of invariants for models of cardinality $\aleph_\alpha$, $\beth_{\omega_1}(|\alpha|)$. (Absolute value denotes the cardinality of a set.) Thus, we can approximate the statement " T has a structure theorem" with "for every $\alpha > 0$, $I(\aleph_\alpha, T) < \beth_{\omega_1}(|\alpha|)$". (The restriction to uncountable cardinals is necessary as the class of countable models of a theory can behave very differently from the uncountable models. A structure theorem for the countable models need not imply that there is one which works globally, and conversely. The above statement says, however, that if there is an assignment of invariants to the models of one uncountable cardinal, then there is such at every uncountable cardinal.) Following the same line of reasoning, Shelah approximates "there is no structure theorem for T" with "$I(\lambda, T) = 2^\lambda$, for every uncountable λ" (which is the maximum number of models). It is for this reason that Shelah sees the Main Gap Theorem as a quick way to say that the classification problem for countable first-order theories has been solved. As with many theorems (especially Shelah's) the proof of the Main Gap Theorem contains much more information than what is summarized in the statement. Shelah draws on this to justify the claim that the book does contain a solution to the classification problem in Chapter XIII. I will discuss this further below.

Shelah's approach to the Main Gap Theorem is to isolate properties of theories whose presence implies that T has the maximum spectrum function, i.e., $I(\lambda, T) = 2^\lambda$, for every uncountable λ. (Shelah describes this loosely as "T falls on the side of chaos".) Furthermore, the absence of all of these conditions should imply that the spectrum function of T is bounded as in (2) of the theorem (or "T falls on the side of order"). This is a kind of mathematical geography. We are looking for the divides or gaps between the different watersheds of theories, the main gap being between "order" and "chaos". Chapters X–XII, and parts of VIII, are organized around finding these natural properties which imply that a theory is on the side of chaos. At the end of XII he shows that the absence of these implies order by proving the Main Gap Theorem. I discuss these various conditions below.

For T a theory and M a model of T, $S(M)$ denotes the set of consistent complete types over M. T is called *stable in* λ if for any model M of T of cardinality λ, $|S(M)| = \lambda$. A theory is *stable* if it is stable in some (infinite) cardinality. There are four main categories of theories with respect to stability: totally transcendental theories (those stable in every cardinality), superstable theories (those stable in all sufficiently large cardinals), stable unsuperstable theories (which are still stable in most cardinalities), and unstable theories. Shelah shows that unstable theories interpret a linear order of some kind. Linear orders are very complicated structures which, in sufficiently large cardinality, behave very chaotically. Not surprisingly, Shelah shows in Chapter VII that unstable theories have maximal spectrum functions. In fact, all unsuperstable theories have maximal spectrum functions. (The stable, unsuperstable, theories interpret very complicated trees which allow for much the same many-model arguments as do linear orders [Chapter VIII, 2.1].) So, the first main divide is at superstability: unsuperstable theories are chaotic.

The attention now turns to superstable theories. Here matters become more complicated. Since these theories do not interpret orders as the unstable ones do, we must look for other ways in which maximal spectrum functions can occur. The remainder of the proof is organized around the model-theorist's version of free amalgamation. Stability implies the existence of a fairly well-behaved notion of independence. Indeed, the first five chapters of the book are devoted to developing this notion. Given models $M_0 \subset M_1$, $M_2 \subset M_3$, we think of M_3 as being the free amalgam of M_1 and M_2 if M_1 and M_2 are independent over M_0 and M_3 is prime over $M_1 \cup M_2$. As prime models over arbitrary sets may not exist in superstable theories, Shelah first restricts his attention to the class of $\aleph_1$-saturated models. He proves in Chapter IV that there is an $\aleph_1$-saturated model over any set which is prime over that set among the $\aleph_1$-saturated models. The properties of free amalgams within the class of $\aleph_1$-saturated models is the next gap. The superstable theory T is said to have the *dimensional order property* (or DOP) if there is $M_0 \subset M_1$, $M_2 \subset M_3$, where each M_i is $\aleph_1$-saturated, M_1 and M_2 are independent over M_0, $M_3$is prime over $M_1 \cup M_2$ among the $\aleph_1$-saturated models, and M_3 is not minimal over $M_1 \cup M_2$ among the $\aleph_1$-saturated models.

Shelah shows in Chapter X, Sections 1 and 2, that if T has the DOP then it is possible to code rather arbitrary binary relations in ($\aleph_1$-saturated) models of T, hence T has the maximal spectrum function. Surprisingly, the absence of the DOP leads to many positive properties of the theory which suggest that a structure theorem is possible. An arbitrary $\aleph_1$-saturated model is prime over an "independent tree" of $\aleph_1$-saturated models, each of cardinality at most $2^{\aleph_0}$ (having a few additional properties; see Chapter X, Section 3). The next dividing line concerns the possible trees which occur in this manner. If T does not have the DOP and there is such a tree which is not well-founded, T is called *deep*. If T is not deep it is *shallow*. Again, if T is deep it is possible to code binary relations and prove that T has the maximal spectrum function (see Chapter X, Sections 3 and 4). When T is shallow the relevant trees have countable foundation rank. It is possible, then, to show that the spectrum function of T, when restricted to $\aleph_1$-saturated models, satisfies the desired bound.

As this treatment suggests, a subclass of the models of a superstable theory will have the desired spectrum function if there are free amalgams which are minimal. A theory without the DOP has the *($<\infty$,2)-existence property* if there are free amalgams. In Chapter XII Shelah defines a condition called the *omitting types order property* (OTOP), which says that an order can be coded in the models of the theory by omitting types. The presence of the OTOP leads to the maximal spectrum function. The majority of that chapter is devoted to proving that if T does not have the DOP and does not have the ($<\infty$,2)-existence property, then it has OTOP. Drawing all of these results together, Shelah proves the Main Gap Theorem in Chapter XII, Section 6.

With the passage of time many simplifications of these proofs have been found. In Hart [4] an alternative to Chapter XII is expounded. This circumvents the complicated isolation notions discussed in Chapter X, Section 1. Notice I have said nothing about Chapter XI. The need for this material is largely eliminated by [4] and Shelah and Buechler [11]. It should be noted that it is only in proving the OTOP/existence dichotomy that the countability of the theory is used. There is an omitting types argument there. The classification problem for uncountable theories is still open.

Chapter XIII is titled "For Thomas the doubter". As you can guess, it is Shelah's justification of the claim that the Main Gap is a solution to the classification problem. In the first section he defines a precise notion of invariant and proves that the theories under (2) of the theorem do have a structure theorem with these invariants. Unfortunately, these invariants (involving generalized quantifiers in an infinitary language) are hard to define. As I mentioned above there is no simple way to define an abstract notion of invariant which matches the structure theorem represented with (2), so these complications are necessary. However, it is unsatisfying that it is unclear precisely how many invariants there are in some cardinalities. In the second section of the chapter Shelah turns his attention from order to chaos. He shows that the models of theories falling under (1) are, indeed, very chaotic. No conceivable structure theorem could be proved for such theories.

In the third section Shelah uses the methods developed to prove Morley's conjecture. This conjecture states that the spectrum function of a theory is nondecreasing on uncountable cardinals. As the methods do not determine precisely the possible spectrum functions of theories under (2), there is real work involved in this. In the last section the problem of computing the possible spectrum functions is reduced to showing that a particular parameter associated to the theory, denoted $snd(T)$, is either $\aleph_0$ or $2^{\aleph_0}$. As Shelah noted in proof on page xxvi, Hrushovski has proved this. (It is unfortunate that one must read to page 648 in order to understand the manner in which this note is written.)

The results contained in this book are beautiful. As it lays the foundation for all of stability theory (probably the most active branch of model theory today), I could not conceive of being without a copy (even though the price is criminal). However, Shelah's esoteric style of writing makes the book virtually unreadable to all but a select few. That is a pity.

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