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The Gentzen–Kripke Construction of the Intermediate Logic LQ

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Abstract The Gentzen-Kripke construction of the semantics for intermediate logic LQ, which is obtainable from the intuitionistic propositional logic H by adding the weak law of excluded middle $\neg A \vee \neg\neg A$, is presented. Our construction spans the Gentzen system and the Kripke semantics for LQ by providing the way from the cut-elimination theorem to model-theoretic results. The completeness and decidability theorems are shown in this method.

1 Introduction An intermediate logic LQ is obtainable from the intuitionistic propositional logic H by adding the weak law of excluded middle $\neg A \vee \neg\neg A$. The logic can also be axiomatized by the axiom schema $(\neg A \rightarrow \neg B) \vee (\neg B \rightarrow \neg A)$. The motivation of LQ is purely technical rather than philosophical in that LQ is one possible extension of H. Our interest thus lies in its characterization in relation to the formalization of H.

The semantics for LQ can be given by the class of directed Kripke frames; see Gabbay [4]. Several authors also proposed the Gentzen systems for LQ, none of which is successful. For example, the cut-elimination theorem cannot be proved in the sequent calculus of Boričić [2] as Hosoi pointed out. Recently, Hosoi [6] gave a Gentzen-type formulation GQ for LQ and proved the cut-elimination theorem.

The purpose of this paper is to develop the Gentzen–Kripke construction of the semantics for LQ using the subformula models, as a modification of the one developed in Akama [1] for intuitionistic predicate logic. The proposed construction spans the Gentzen system and the Kripke semantics for LQ by providing the way from the cut-elimination theorem to model-theoretic results. The completeness and decidability theorems are shown in this method.

2 Intermediate logic LQ and its Gentzen-type formulation GQ The intermediate logic LQ is one of the extensions of the intuitionistic propositional logic H with *the weak law of excluded middle*, i.e. $\neg A \vee \neg\neg A$. The proof and model

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theories for LQ are modified so that the weak law of excluded middle can be accommodated therein. As for model theory, Gabbay [4] presented a directed Kripke model for LQ. For the detailed exposition of a Kripke semantics for intuitionistic logic, the reader is referred to Fitting [3].

In the following, we review a Gentzen-type formulation GQ of LQ as developed by Hosoi [6]. GQ is a variant of LJ due to Gentzen [5] for intuitionistic logic. It requires two additional axioms for beginning sequents:

(BL) $A \to$,
(BR) $\to A$,

where A denotes an arbitrary formula. In addition to the rules in LJ, the following rule should be added:

$$\text{(WEM)} \quad \frac{\Gamma \to \Delta \qquad \Gamma \to \Delta}{\Gamma \to \Delta},$$

where each instance of BL and BR in a proof figure must be related to exactly one application of WEM in the proof figure.

To avoid a contradiction, the rule (WEM) needs the next four conditions; see Hosoi [6]:

(1) All of the sequents in each of $S(\mathrm{WEM}_i)$ are constructed with one and the same formula.
(2) Each instance of BL in $S(\mathrm{WEM}_i)$ is on the left upper branch of WEM_i.
(3) Each instance of BR in $S(\mathrm{WEM}_i)$ is on the right upper branch of WEM_i.
(4) Each instance of BR in $S(\mathrm{WEM}_i)$ has, somewhere between it and the related WEM_i, at least one sequent with an empty succedent.

Here, $S(\mathrm{WEM}_i)$ denotes the set of instances of BL and BR, i.e. $A_i \to$ and $\to A_i$, related to WEM_i.

Intuitively these conditions enable GQ to make a hidden application of the weak law of excluded middle. Thus they should not be confused with the restrictions corresponding to the contraction rule in LJ. It is also to be noticed that we have the classical sequent system if we delete (4) from these conditions.

The following is a proof for the weak law of excluded middle:

$$\text{(WEM)}\ \dfrac{(\to\lor)\ \dfrac{(\to\neg)\ \dfrac{\overset{\text{(BL)}}{A \to}}{\to \neg A}}{\to \neg A \lor \neg\neg A} \qquad (\to\lor)\ \dfrac{(\to\neg)\ \dfrac{(\neg\to)\ \dfrac{\overset{\text{(BR)}}{\to A}}{\neg A \to \$}}{\to \neg\neg A}}{\to \neg A \lor \neg\neg A}}{\to \neg A \lor \neg\neg A}$$

where the symbol \$ in the empty succedent shows the relation mentioned in Condition (4). In the above proof, both beginning sequents BL and BR are related to the rule WEM at the end sequent.

Hosoi discussed the above formulation in detail and proved the cut-elimination theorem for GQ.

3 Subformula models for LQ Before providing a new construction of the semantics for LQ, several preliminaries are in order.

Definition 3.1 (Subformula) A *subformula* of the formula A is defined inductively as follows:

1. If A is a formula, then A is a subformula of A.
2. If A and B are formulas, then the subformulas of A and B are subformulas of $A \,\&\, B$, $A \vee B$, and $A \supset B$.
3. If A is a formula, the subformulas of A are subformulas of $\neg A$.

If Γ and Δ are finite (possibly empty) sets of formulas, then $\Gamma \rightarrow \Delta$ is called a *sequent*. Instead of $\Gamma \rightarrow \Delta$ we also use the pair (Γ, Δ). The pair (Γ, Δ) is *dual consistent* if $\Gamma \rightarrow \Delta$ is not provable in the intermediate logic LQ.

We write $S(A)$ for the (finite) set of all subformulas of the formulas A. A dual consistent pair (Γ, Δ) is *complete* with respect to $S(A)$ if $\Gamma \cup \Delta = S(A)$. The symbol $\vdash$ ($\models$) denotes provability (validity) in LQ.

The following results concerning the above pair are needed for the subsequent arguments.

Lemma 3.2 *If* (Γ, Δ) *is dual consistent then either* $(\Gamma \cup \{A\}, \Delta)$ *or* $(\Gamma, \Delta \cup \{A\})$ *is dual consistent for any formula* A.

Proof: Assume that both $(\Gamma \cup \{A\}, \Delta)$ and $(\Gamma, \Delta \cup \{A\})$ are dual inconsistent. By this assumption, $\Gamma \cup \{A\} \rightarrow \Delta$ and $\Gamma \rightarrow \Delta \cup \{A\}$ are provable in LQ. It follows that the sequent $\Gamma \rightarrow \Delta$ is provable in LQ contrary to the hypothesis that (Γ, Δ) is dual consistent.

As a corollary to this, we can obtain the following:

Lemma 3.3 *Every dual consistent pair* (Γ, Δ), *where* $\Gamma, \Delta \subseteq S(A)$, *has a complete and dual consistent extension with respect to* $S(A)$.

Given a formula A, $G(A)$ is the collection of all the sets $\Gamma \subseteq S(A)$ where $(\Gamma, S(A) - \Gamma)$ is a dual consistent and complete pair.

Lemma 3.4 $G(A) \neq \varnothing$.

Proof: From the consistency of LQ, $\rightarrow$ (the null sequent) is not provable in LQ. Thus, $(\varnothing, \varnothing)$ is dual consistent. By Lemma 3.3, $(\varnothing, \varnothing)$ has a complete extended pair (Γ, Δ) such that $\Gamma \subseteq S(A)$.

Theorem 3.5 *Given a subformula B of A, it belongs to* $\Gamma \in G(A)$ *iff* $\Gamma \rightarrow \{B\}$ *is provable in LQ.*

Proof: Necessity is obvious. Sufficiency can be proved by *reductio ad absurdum*. Suppose B does not belong to Γ, then B belongs to $S(A) - \Gamma$. Since $\Gamma \in G(A)$, $(\Gamma, S(A) - \Gamma)$ is dual consistent. $(\Gamma, \{B\})$ is also dual consistent, too. Therefore, $\Gamma \rightarrow \{B\}$ is not provable in LQ.

A *directed Kripke model* is a triple $\langle G, R, V \rangle$, where G is a nonempty set of worlds, R is a directed relation on G, and V is a function which maps every propositional variable A to a subset of G satisfying $\forall \Gamma^*(\Gamma \in V(A) \Rightarrow \Gamma^* \in V(A))$. Here Γ^* means a successor of Γ. The *directed relation* is a partial ordering with the property $\exists\Delta\forall\Gamma(\Gamma R\Delta)$. The subformula model simulates the standard directed Kripke model in relation to the subformula property in the sequent calculus GQ.

Definition 3.6 (Subformula Model) A subformula model for the intermediate logic LQ is a triple $\langle G, \subseteq, V\rangle$ for a formula A such that:

a. $G = G(A)$.
b. $\subseteq = R$.
c. $V(B) = \{\Gamma \mid B \in \Gamma\}$ where B is a propositional variable and Γ ranges over $G(A)$.

The idea in the subformula model is no more than a new construction of a directed Kripke model. For a Kripke semantics for LQ, we have to revise a partial ordering in the intuitionistic Kripke model for validating the weak law of excluded middle. In fact, the partial ordering (set-theoretic inclusion) is not sufficient for our purpose. The point here is how two additional axioms of GQ, i.e. BL and BR, are handled in the proposed construction. As shown in Section 2, both BL and BR are used for the proof of the weak law of excluded middle. In other words, both beginning sequents $A \rightarrow$ and $\rightarrow A$ are required for the application of WEM. Following the conditions on WEM, the partial ordering should incorporate an additional condition, i.e. $\forall\Gamma\forall\Delta\exists\Lambda(\Gamma \subseteq \Lambda \ \&\ \Delta \subseteq \Lambda)$, since both A and $\neg A$ are subformulas of $\neg A \vee \neg\neg A$. It is not self-evident that $\subseteq$ is directed, unless $S(A)$ has a maximal element. But $G(A)$ has a maximal element since we now consider only a propositional logic. This restriction can result in the one in 3.6.b. In this sense, the construction of a countermodel for $\neg A \vee \neg\neg A$ is therefore obvious; see Gabbay [6].

The forcing relation $\models$ can be extended in the usual way to any formula B and any $\Gamma \in G$.

Lemma 3.7 *For any* $\Gamma \in G(A)$ *and any formula* B,

$$\Gamma \models B \text{ iff } B \in \Gamma.$$

Proof: This lemma is proved by induction on the length of formulas. The basis is obvious. The inductive steps are as follows:

$$\begin{aligned}
\Gamma \models B_1 \vee B_2 &\Leftrightarrow \Gamma \models B_1 \text{ or } \Gamma \models B_2 \\
&\Leftrightarrow B_1 \in \Gamma \text{ or } B_2 \in \Gamma \\
&\Leftrightarrow \vdash\Gamma \rightarrow \{B_1\} \text{ or } \vdash\Gamma \rightarrow \{B_2\} \\
&\Leftrightarrow \vdash\Gamma \rightarrow \{B_1 \vee B_2\} \\
&\Leftrightarrow B_1 \vee B_2 \in \Gamma.
\end{aligned}$$

Here, $\vdash\Gamma \rightarrow \{B_1 \vee B_2\} \Leftrightarrow \vdash\Gamma \rightarrow \{B_1\}$ or $\vdash\Gamma \rightarrow \{B_2\}$ requires some steps. For ($\Rightarrow$), suppose neither $\vdash\Gamma \rightarrow \{B_1\}$ nor $\vdash\Gamma \rightarrow \{B_2\}$ holds. By Theorem 3.5, neither $B_1 \in \Gamma$ nor $B_2 \in \Gamma$. Then both B_1 and B_2 belong to $S(A) - \Gamma$. Since $(\Gamma, S(A) - \Gamma)$ is dual consistent, $\Gamma \rightarrow \{B_1, B_2\}$ is not provable in GQ. This is in fact the denial of $\vdash\Gamma \rightarrow \{B_1 \vee B_2\}$. Thus the conclusion follows. The converse ($\Leftarrow$) is straightforward.

$$\begin{aligned}\Gamma \models B_1 \supset B_2 &\Leftrightarrow \forall\Gamma^*(\Gamma^* \models B_1 \Rightarrow \Gamma^* \models B_2)\\ &\Leftrightarrow \forall\Gamma^*(B_1 \in \Gamma^* \Rightarrow B_2 \in \Gamma^*)\\ &\Leftrightarrow \forall\Gamma^*(\vdash\Gamma^* \rightarrow \{B_1\} \Rightarrow \vdash\Gamma^* \rightarrow \{B_2\})\\ &\Leftrightarrow \forall\Gamma^*(\vdash\Gamma^* \rightarrow \{B_1 \supset B_2\})\\ &\Leftrightarrow \forall\Gamma^*(B_1 \supset B_2 \in \Gamma^*)\\ &\Leftrightarrow B_1 \supset B_2 \in \Gamma.\end{aligned}$$

Here the proof of $\forall\Gamma^*(\vdash\Gamma^* \rightarrow \{B_1\} \Rightarrow \vdash\Gamma^* \rightarrow \{B_2\}) \Leftrightarrow \forall\Gamma^*(\vdash\Gamma^* \rightarrow \{B_1 \supset B_2\})$ is somewhat complicated. For ($\Leftarrow$), $\vdash\Gamma^* \rightarrow \{B_1 \supset B_2\}$ and $\vdash\Gamma^* \rightarrow \{B_1\}$ lead to $\vdash\Gamma^* \rightarrow \{B_2\}$. For proving ($\Rightarrow$), suppose the existence of a world Γ^* such that $\Gamma^* \rightarrow \{B_1 \supset B_2\}$ is not provable in LQ. The deduction theorem still holds for LQ (see Gabbay [4], pp. 64ff). Then, by the deduction theorem, $(\Gamma^* \cup \{B_1\}, B_2)$ is dual consistent. By Lemma 3.3, there is a $\Gamma_1 \in G$ such that $\Gamma^* \cup \{B_1\} \subseteq \Gamma_1$ but $B_2 \notin \Gamma_1$. This implies $\vdash\Gamma_1 \rightarrow \{B_1\}$ and $\nvdash\Gamma_1 \rightarrow \{B_2\}$: this is the denial of $\forall\Gamma^*(\vdash\Gamma^* \rightarrow \{B_1\} \Rightarrow \vdash\Gamma^* \rightarrow \{B_2\})$. Other cases can also be shown without any difficulty.

Next we show the completeness theorem for LQ with respect to subformula models. It can be shown as a consequence of the model existence theorem:

Theorem 3.8 (Model Existence Theorem) *If A is not provable in LQ, then there is a model falsifying the formula A.*

Proof: A model $\langle G, \subseteq, V\rangle$ falsifying the formula A can easily be given as follows. By assumption, $(\varnothing, \{A\})$ is dual consistent. Lemma 3.3 enables us to show the existence of a complete pair $(\Gamma, S(A) - \Gamma)$ with respect to $S(A)$. But $\Gamma \in G(A)$ and $A \notin \Gamma$. We can thus prove $\Gamma \not\models A$ by Lemma 3.7. Thus A is not valid in LQ.

As a consequence of Theorem 3.8, the completeness is obtained.

Theorem 3.9 (Completeness Theorem) *A is valid iff it is provable in LQ.*

Proof: Soundness is trivial from a complete check of the axioms and rules of inference for LQ. The only nontrivial case is about the two additional axioms for beginning sequents. One might point out the following proof to yield a contradiction in GQ:

$$\dfrac{(\rightarrow\neg)\ \dfrac{\overset{\text{(BL)}}{A \rightarrow}}{\rightarrow \neg A} \qquad \overset{\text{(BR)}}{\rightarrow A}}{\rightarrow \neg A \,\&\, A}\ (\rightarrow\&)$$

The above proof is not, however, allowed, since it violates Condition (4), which requires the existence of a sequence with an empty succedent somewhere under BR. For further details, see Hosoi [6]. Completeness easily follows from the model existence theorem.

The next theorem is an analogue of another form of completeness in view of the subformula property of GQ.

Theorem 3.10 *For any formula A in* LQ,

$$\vdash A \text{ iff } \forall\Gamma(\Gamma \in G(A) \Rightarrow \Gamma \models A)$$

where $\vdash$ is a forcing relation in the subformula model defined above.

The decidability of LQ follows as a by-product of the cut-elimination theorem, but the proposed subformula model enables us to show decidability since G is viewed as a finite set $G(A)$.

Theorem 3.11 (Decidability Theorem) *LQ is decidable.*

Proof: It is obvious from the fact that $G(A)$ is finite.

We also show the following theorem from the above results.

Theorem 3.12 *If Γ is a consistent set of formulas, then there is a subformula model $\langle G,\subseteq, V\rangle$ and a $\Delta \in G$ such that $\Delta \models A$ for any $A \in \Gamma$.*

Proof: The pair $(\Gamma, \varnothing)$ is dual consistent since Γ is consistent. From Lemma 3.3, we can find a Δ such that $\Gamma \subseteq \Delta$ and $\Delta \in G$. As a consequence of Lemma 3.7, we reach the conclusion.

In future papers, we hope to apply the technique developed in this paper to other intermediate logics.

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