144
Notre Dame Journal of Formal Logic
Volume 33, Number 1, Winter 1992

A Note on Some Weak Forms of the Axiom of Choice

GARY P. SHANNON

Abstract Erdös and Tarski proved that in ZFC, if $(P, \leq)$ is a quasi-order that has antichains of cardinality θ for all $\theta < \kappa$, and if κ is singular or $\kappa = \aleph_0$, then $(P, \leq)$ has an antichain of cardinality κ. Some variations of this result are developed as weak forms of the Axiom of Choice.

This note contains some variations of a result of Erdös and Tarski [1] which are developed as weak forms of the Axiom of Choice (AC).

Definition Let $(P, \leq)$ be a quasi-order (i.e., $\leq$ is reflexive and transitive).

Two elements x, y of P are said to be *incompatible* if there does not exist $z \in P$ such that $z \leq x$ and $z \leq y$ (otherwise x and y are said to be *compatible*). A subset I of P is said to be an *antichain* if any two elements of I are incompatible.

A partial order $(P, \leq)$ is a *tree* iff for all $x \in P$, $\{y \in P : y \leq x\}$ is well-ordered by $\leq$. If $(P, \leq)$ is a tree and $x \in P$, then the *height* of x ($ht(x)$) is the order type of $\{y \in P : y \leq x\}$. For each ordinal α, the αth level of P ($lev_\alpha(P)$) is $\{x \in P : ht(x) = \alpha\}$. The *height* of P is the least α such that the αth level of P is empty. A *branch* of P is a maximal chain. Henceforth it will be assumed that all trees are single-rooted (that is, $|lev_0(P)| = 1$).

If $(P, \geq)$ is an upside-down tree then a *strong antichain* is an antichain that has at most one element from each level of P.

$SH(\mu)$ is the hypothesis that no μ-Souslin tree exists.

Erdös and Tarski [1] proved that in ZFC, if $(P, \leq)$ is a quasi-order that has antichains of cardinality θ for all $\theta < \kappa$, and if κ is singular or $\kappa = \aleph_0$, then $(P, \leq)$ has an antichain of cardinality κ. The question of to what extent converses of the result of Erdös and Tarski can be obtained will be somewhat considered. That is, in ZF, is the statement "if $(P, \leq)$ has antichains of cardinality θ for all

Received March 2, 1990; revised June 12, 1991

$\theta < \kappa$, then $(P, \leq)$ has an antichain of cardinality κ" equivalent to a form of the Axiom of Choice?

If no restrictions are put on P or on κ, then the statement "if $(P, \leq)$ has antichains of cardinality θ for all $\theta < \kappa$, then $(P, \leq)$ has an antichain of cardinality κ" does not hold in ZF. For example, let $(P, \leq)$ be the upside-down tree of height $\aleph_1$ defined as follows: for each $x \in P$ of $ht < \omega$, assume that x has exactly two successors of height $ht(x) + 1$; assume that there is only one $x \in P$ of $ht\ \omega$, and assume that $\{x \in P : \omega \leq ht(x) < \aleph_1\}$ is a chain. Then $(P, \leq)$ has antichains of cardinality θ for all $\theta < \aleph_1$, but $(P, \leq)$ has no antichain of cardinality $\aleph_1$.

If $(P, \leq)$ is an upside-down tree, then with some additional conditions a weak form of AC will be obtained.

Theorem *In ZF, the Axiom of Choice for families of cardinality κ (AC^κ), κ a well-orderable cardinal, is equivalent to the following assertion: If $(P, \leq)$ is a tree of height $\lambda \leq \kappa$ which has fewer than λ branches of height less than λ, then one of the following holds for the upside-down tree $(P, \geq)$:* (i) *there exists a cardinal $\mu < \lambda$ such that $(P, \geq)$ does not contain a strong antichain of cardinality μ;* (ii) *$(P, \geq)$ contains a strong antichain of cardinality λ; or* (iii) *$(P, \geq)$ contains a chain of cardinality λ.*

Proof: Assume AC^κ. If λ is singular then the result is done by the proof of Theorem 1 in [1] (as given, Theorem 1 of [1] is done in ZFC, but the proof can be done in ZF, assuming AC^κ). If λ is regular, then suppose that P has neither a strong antichain of cardinality λ, nor a chain of cardinality λ. P is the union of its branches, thus P is a union of $<\lambda$ sets of cardinality $<\lambda$, and thus (since λ is regular) $|P| < \lambda$. This contradicts that the height of $(P, \leq)$ is λ, and thus either (ii) or (iii) must hold.

Assume that if $(P, \leq)$ is a tree of height $\lambda \leq \kappa$ which has fewer than λ branches of height smaller than λ, then one of the following holds for the upside-down tree $(P, \geq)$: (i) there exists a cardinal $\mu < \lambda$ such that $(P, \geq)$ does not contain a strong antichain of cardinality μ; (ii) $(P, \geq)$ contains a strong antichain of cardinality λ; or (iii) $(P, \geq)$ contains a chain of cardinality λ.

Let $\underset{\sim}{C}$ be a collection of pairwise disjoint, nonempty sets, $\underset{\sim}{C} \approx \kappa$, say $\underset{\sim}{C} = \{C_\delta : \delta \in \kappa\}$. Let $\underset{\sim}{D}$ be the set of choice functions on $\{C_\delta : \delta < \lambda\}$ for all $\lambda < \kappa$. $\underset{\sim}{D}$ is nonempty since in ZF choice functions on $\{C_\delta : \delta < \lambda\}$ exist for λ finite. Define $\leq$ on $\underset{\sim}{D}$ by $f \leq g$ iff $g \subseteq f$. Then $(\underset{\sim}{D}, \leq)$ is an upside-down tree of $ht \geq \omega$ (assume that $(\underset{\sim}{D}, \leq)$ has an artificial single root). Let $\underset{\sim}{D}_0$ denote the subtree of $\underset{\sim}{D}$ of $ht\ \omega$. Note that $\underset{\sim}{D}_0$ has no finite branches (since if B were a branch of height n then (since AC^{fin} holds in ZF) B could be extended to a branch of height $n + 1$, which contradicts that B is maximal). In ZF there exist strong antichains of arbitrarily large finite cardinality; thus (i) does not hold for the subtree $\underset{\sim}{D}_0$, and thus there exists a strong antichain, I, of cardinality $\aleph_0$, or a chain, H, of cardinality $\aleph_0$ in $(\underset{\sim}{D}_0, \leq)$. If I exists, say $I = \{f_{n_k} : n_k \in J \approx \omega\}$, and f_{n_k} is a choice function on $\{C_0, \ldots, C_{n_k}\}$, then $f_{n_0} \cup \bigcup_{n_{k+1} \in J} f_{n_{k+1}} |_{\{C_{n_k+1}, \ldots, C_{n_{k+1}}\}}$ is a choice function on $\{C_n : n \in \omega\}$. If H exists, then there exists a branch B of cardinality $\aleph_0$ such that B contains H, and then $\cup B$ is a choice function on $\{C_n : n \in \omega\}$. Thus $AC^{\aleph_0}$ holds (and thus $\underset{\sim}{D}$ is nonempty for $\lambda = \aleph_0$).

Assume that the height of $(\underset{\sim}{D}, \leq)$ is at least $\tau + 1$ and that AC^τ holds. Let

$(D_{\tau+1}, \leq)$ be the subtree of $(\underline{D}, \leq)$ of $ht\ \tau + 1$. Then, by using AC^{τ}, there exists a chain of cardinality $\tau + 1$ (since $\tau \approx \tau + 1$).

Assume that the height of $(\underline{D}, \leq)$ is τ, where τ is a limit ordinal, and that AC^{δ} holds for all $\delta < \tau$. Let $(\underline{D}_\tau, \leq)$ be the subtree of $(\underline{D}, \leq)$ of $ht\ \tau$. Note that $\underline{D}_\tau$ has no branches of $ht < \tau$ (since if B is a branch in $(\underline{D}_\tau, \leq)$ of $ht\ \alpha < \tau$, then there exists $\alpha < \delta < \tau$ such that $B \cap lev_\sigma(\underline{D}_\tau) = \emptyset$ for all $\sigma \geq \delta$; but then (since AC^{δ} holds) B can be extended to a branch R which contains B as a proper subset, and thus B is not maximal). By $AC^{<\tau}$ there exist in $\underline{D}$ strong antichains of cardinality δ for all $\delta < \tau$; thus (i) does not hold for the subtree $\underline{D}_\tau$, and thus there exists a strong antichain, I, of cardinality τ, or a chain, H, of cardinality τ. If I exists, say $I = \{f_{\alpha_\delta} : \alpha_\delta \in J \approx \tau\}$, and f_{α_δ} is a choice function on $\{C_\sigma : \sigma \leq \alpha_\delta\}$, then let $J = S \cup L \cup \{0\}$, where S is the set of successor ordinals in J, and L is the set of limit ordinals in J. Then

$$f_{\alpha_0} \cup \bigcup_{\alpha_{\delta+1} \in S} f_{\alpha_{\delta+1}} \Big|_{\{C_\sigma : \alpha_\delta + 1 \leq \sigma \leq \alpha_{\delta+1}\}}$$

$$\cup \bigcup_{\alpha_\delta \in L} f_{\alpha_\delta} \Big|_{\{C_\sigma : \sigma \leq \alpha_\delta \text{ and } \sigma \notin \bigcup_{\beta < \alpha_\delta} \operatorname{dom} f_\beta\}}$$

is a choice function on $\{C_\sigma : \sigma < \tau\rangle$. If H exists, then there exists a branch B of cardinality τ such that B contains H, and then $\cup B$ is a choice function on $\{C_\sigma : \sigma < \tau\}$.

By induction there exists a choice function on $\underline{C}$, and thus AC^{κ} holds—which proves the theorem.

Note that the theorem also holds if the condition that the tree has $<\lambda$ branches of $ht < \lambda$ is replaced by the condition that the number of maximal strong antichains of cardinality $<\lambda$ is $<\lambda$.

Let $T(\kappa)$ denote the following statement: If κ is a well-orderable cardinal and if $(P, \leq)$ is a tree of $ht\ \lambda \leq \kappa$, then one of the following holds for the upside down tree $(P, \geq)$: (i) there exists a cardinal $\mu < \lambda$ such that $(P, \geq)$ does not contain a strong antichain of cardinality μ; (ii) $(P, \geq)$ contains a strong antichain of cardinality λ; or (iii) $(P, \geq)$ contains a chain of cardinality λ. The argument used to prove the preceding theorem can also be used to prove that $T(\kappa)$ implies AC^{κ}. However, in ZFC, $T(\kappa)$ is equivalent to $SH(\kappa)$, and thus it is not possible to prove that in ZF, AC^{κ} implies $T(\kappa)$ (since if κ is regular the existence of a κ-Souslin tree is independent of ZFC).

Using an argument similar to that used to prove that $MA(\aleph_1)$ implies $SH(\aleph_1)$ (Kunen [2], p. 74), it can be shown that in ZF, $MAS(\aleph_1)$ implies $T(\aleph_1)$ (Shannon [3]), and thus it follows from the remark above that in ZF, $MAS(\aleph_1)$ implies $AC^{\aleph_1}$.

REFERENCES

[1] Erdös, P. and A. Tarski, "On families of mutually exclusive sets," *Annals of Mathematics*, vol. 44 (1943), pp. 315–329.

[2] Kunen, K., *Set Theory*, North Holland, Amsterdam, 1980.

[3] Shannon, G.P., "Provable forms of Martin's Axiom," *Notre Dame Journal of Formal Logic*, vol. 31 (1990), pp. 569–573.

Department of Mathematics and Statistics
California State University
Sacramento, CA 95819-6051