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# Coherence in Category Theory and the Church–Rosser Property

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**Abstract** Szabo's derivation systems on sequent calculi with exchange and product are not Church–Rosser. Thus, his coherence results for categories having a symmetric product (either monoidal or cartesian) are false.

***Introduction*** Gentzen's sequent calculi (Szabo [9]) have been applied extensively in category theory (e.g., Kelly and MacLane [2], Lambek [3] and [4], MacLane [6], Minc [7] and [8]). Sequents correspond to morphisms of a category, and the rules of the calculus correspond to categorical structures (e.g., having an associative tensor product). Cut-elimination was then used to put bounds on the complexity of these structures, e.g. to produce exhaustive lists (perhaps with duplications) of the canonical natural transformations between given functors. For symmetric, monoidal closed categories it was shown in Voreadou [12] how to decide in principle whether two such transformations are equal, whereas an effective, linear-time decision procedure was given in Jay [1].

Derivation systems (reduction rules) can be used to eliminate some duplicates in the list of cut-free proofs (e.g. [8]). However, in *Algebra of Proofs* [11] and its forerunner [10], Szabo claims to have produced derivation systems in which *all* duplicates have been eliminated, so that every proof has a unique normal form. In fact, none of his systems which include a symmetry for the product are Church–Rosser (confluent).

The two major applications of his work were complete coherence theorems for symmetric, monoidal closed categories (mentioned above) and for cartesian closed categories. The latter problem is solved in Lambek and Scott [5] using similar methods, but they avoid adopting symmetry as a primitive by exploiting the universal property of the cartesian product.

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***The counterexample*** Here is Szabo's sequent calculus for symmetric, monoidal categories [11]

$$\frac{\Gamma \to \gamma \quad \Delta\gamma\Lambda \to \phi}{\Delta\Gamma\Lambda \to \phi}\ (Cut) \qquad \frac{\Gamma\Delta \to \phi}{\Gamma I \Delta \to \phi}\ (LI) \qquad \frac{\Gamma\alpha\beta\Delta \to \phi}{\Gamma\beta\alpha\Delta \to \phi}\ (Ex)$$

$$\frac{\Gamma \to \alpha \quad \Delta \to \beta}{\Gamma\Delta \to \alpha \otimes \beta}\ (R\otimes) \qquad \frac{\Gamma\alpha\beta\Delta \to \phi}{\Gamma\alpha\otimes\beta\Delta \to \phi}\ (L\otimes)$$

which are there labelled (R1), (R2), (R4), (R8), and (R9), respectively. Here is an example of a proof in the calculus:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\alpha \to \alpha \quad \beta \to \beta}{\alpha\beta \to \alpha \otimes \beta}\ (R\otimes) \qquad \gamma \to \gamma}{\alpha\beta\gamma \to (\alpha \otimes \beta) \otimes \gamma (=\phi)}\ (R\otimes)}{\alpha\gamma\beta \to \phi}\ (Ex)}{\alpha\beta\gamma \to \phi}\ (Ex)}{\beta\alpha\gamma \to \phi}\ (Ex)}{\beta(\alpha \otimes \gamma) \to \phi}\ (L\otimes)}{(\alpha \otimes \gamma)\beta \to \phi}\ (Ex)}{(\alpha \otimes \gamma) \otimes \beta \to \phi}\ (L\otimes)$$

Two derivations (reductions) applicable to this proof are:

(D.24.1) For any permutation $\pi$ of the integers $1, \ldots, n$, and any successions $(\sigma)$ and $(\tau)$ of instances of (R4),

$$\frac{\Gamma \to \Phi}{\Delta \to \Phi}\ (\sigma) \geq \frac{\Gamma \to \Phi}{\Delta \to \Phi}\ (\tau)$$

provided that $\Gamma = \alpha_1 \ldots \alpha_n$, $\Delta = \beta_1 \ldots \beta_n$, $\beta_i = \alpha_{\pi(i)}$ for $1 \leq i \leq n$, and $(\tau)$ is the unique string of interchanges which first moves $\alpha_{\pi(1)}$ to $\beta_1$, then $\alpha_{\pi(2)}$ to $\beta_2$, etc.

If $\pi$ is the identity permutation, the right-hand side denotes [the null proof].

(D.27.3)
$$\cfrac{\cfrac{\cfrac{\Gamma\alpha\beta\gamma\Delta \to \Phi}{\Gamma\alpha\gamma\beta\Delta \to \Phi}\ (Ex)}{\Gamma\gamma\alpha\beta\Delta \to \Phi}\ (Ex)}{\Gamma\gamma(\alpha \otimes \beta)\Delta \to \Phi}\ (L\otimes) \quad \geq \quad \cfrac{\cfrac{\Gamma\alpha\beta\gamma\Delta \to \Phi}{\Gamma(\alpha \otimes \beta)\gamma\Delta \to \Phi}\ (L\otimes)}{\Gamma\gamma(\alpha \otimes \beta)\Delta \to \Phi}\ (Ex)$$

The first two applications of $(Ex)$ in this proof can be deleted by (D.24.1) to yield an irreducible proof. However, we can also apply (D.27.3) followed by (D.24.1) to obtain a distinct irreducible proof. Thus the system is not Church–Rosser. Categorically, the two irreducible proofs correspond to the morphisms

$$ca(c \otimes 1) = a^{-1}(1 \otimes c)a : (\alpha \otimes \gamma) \otimes \beta \to (\alpha \otimes \beta) \otimes \gamma$$

whose proof of equality requires the expansion of an identity to $c^2$.

The loss of confluence leads to incorrect combinatorial calculations of the size of homsets. Let $\mathcal{V}$ be a free symmetric monoidal closed category on a set of objects. For objects $A$ and $B$ of $\mathcal{V}$ let $\|A,B\|$ denote the cardinality of $\mathcal{V}(A,B)$. Further, let $A^{(1)} = [A,I]$ denote the *dual* of $A$ and $A^{(n)}$ its $n$th dual. Claim (2) of Corollary 8.6.13 is that if $A$ is not isomorphic to $I$ we have

$$\|A^{(2n)}, A^{(2m)}\| = \binom{n+m-1}{n} \|A,A\|.$$

This formula is self-contradictory since it implies that

$$\|A^{(4)}, A^{(4)}\| = \binom{3}{2} \|A,A\| = 3\|A,A\|,$$

while applying it twice shows that

$$\|A^{(4)}, A^{(4)}\| = \|A^{(2)}, A^{(2)}\| = \|A,A\|.$$

Similar calculations show that claims (3), (4), and (5) are also false.

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