

A Note on the "Carving Up Content" Principle in Frege's Theory of Sense

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Abstract In the *Grundlagen* Frege says that "line *a* is parallel to line *b*" differs from "the direction of *a* = the direction of *b*" in that "we carve up the content in a way different from the original way". It seems that such recarving is crucial to Frege's logicist program of defining numbers, but it also seems incompatible with his later theory of sense and reference. I formulate a restriction on recarving, in particular, that no names may be introduced that introduce new possibilities of reference failure, which is observed by Frege's examples. This restriction discriminates between various relatives of the "slingshot" argument which rely on a step of recarving. I offer an argument for the restriction based on Fregean principles, which I formalize in Church's "Logic of Sense and Denotation", and briefly discuss various axioms of his "Alternative (0)" which are incompatible with recarving.

Naive or pre-theoretic semantics holds that names refer to objects and that sentences have a *content*. In developed theories this content becomes an intension, a truth-value, a proposition, truth conditions, and more. Many of these notions, however, have left behind part of the intuition of content as some objective fact or feature of the world which a sentence expresses. In particular, the notion of content differs from the more subjective notions of meaning that often replace it in developed theories. Content seems to depend on what is described, not how it is described, so that replacing co-referential singular terms in a sentence ought to preserve its content. Furthermore, content should be preserved through rephrasings of a sentence, rephrasings that might intuitively alter meaning. It is, after all, some objective feature of the world which constitutes the content, not our way of representing it. For different reasons, developed semantic theories have generally abandoned the notion of content. In this note I wish to consider an aspect of the history of that notion in Frege's thought.

Both the notion of content (*Inhalt*) and the thesis that quite different sen-

tences can express the same content were important for Frege's logicist project. The crucial step in the definition of number, "abstracting" the number of a concept from a class of classes that are equinumerous, depends on claiming that two assertions have the same content. Frege illustrates this with his famous example of the judgments that two lines are parallel and that the direction of one is the same as the direction of the other. The move from the first to the second, which seems to define a new object, a "direction", is justified as just presenting the same content in a different fashion, "recarving" the same content in a new way.¹

64. The judgment "line a is parallel to line b ", or, using symbols, $a \parallel b$, can be taken as an identity. If we do this, we obtain the concept of direction, and say: "the direction of line a is identical with the direction of line b ". Thus we replace the symbol \parallel by the more generic symbol $=$, through removing what is specific in the content of the former and dividing it between a and b . We carve up the content in a way different from the original way, and this yields a new concept. (Frege [8], p. 74)

Presumably, then, the notorious Principle V of the *Grundgesetze der Arithmetik*, which says that coextensive predicates have the same extension, was to be justified as a result of recarving a single content.²

That the same content can be expressed in very different ways also seems consonant with the role of contents as the values of functions. If the content of a sentence is a function of the semantic values of its parts, then it could well be that that very same content could be the value of some other functions applied to other arguments.³ The value of a function need not bear any traces of the argument which yields it and can be the value of many different functions and arguments. The content of the sentence ' a is parallel to b ' is the value of a function for the arguments a and b , but it could also be the value of other functions of other arguments. At least nothing in the notion of a function forbids that.

Thus the notion of "carving up content" seems necessary for Frege's logicist program and natural for his semantic theory. It has been argued that, however, that very flexibility of expression of content makes Frege's notion of content unsuitable as a notion of meaning, for it gives nonsynonymous sentences the same content. It would indeed seem true that to the extent that a content can be expressed with widely varying expressions, that same content cannot serve as meaning. The meaning of a sentence should be close to the words with which it is expressed, or at least close enough that a competent speaker must be able to recognize the different expressions of the same meaning. (See Baker and Hacker [2], pp. 299, 386.)

A more radical consequence that has been drawn from the carving up content principle is that the only possible content for a sentence is its truth value, and thus the intuitive notion of content as what a sentence is "about" must be abandoned. Olson ([11], ch. 4) has argued that it was the recognition of this that led Frege to abandon his original notion of "judgable content", *beurteilbarer Inhalt*, and to develop his theory of sense, *Sinn*, and reference, *Bedeutung*, essentially to bifurcate his semantic theory with reference on one side and notions of synonymy and content on the other side with sense, abandoning some of the pretheoretic connections among them. The content (*Inhalt*) of a sentence in Frege's earlier philosophy is replaced by two notions, truth value and thought,

and that the way in which a thought is “about” the world is not just a matter of the references of its constituents.⁴

Although Frege does not give a clear-cut argument for the collapse of content into truth-value, an argument has been proposed on his behalf by Church, Gödel, and others: the notorious “slingshot”.⁵ The slingshot presents a series of sentences each derived from the last by steps of “recarving” content or substitution of co-referential terms, both of which seem to preserve content. While the co-referential terms have the same content in terms of reference, one smuggles in content with that substitution which is then exposed in the recarving step. The first and last members of this series share nothing but a truth-value, which violates the assumption that there are many different contents. The distinction of sense and reference allows one to counter the slingshot. Substitution of co-referential terms may well not preserve sense, and so the argument is blocked for sense, while still allowed for reference. A new semantic notion of “mode of presentation” relates constituents of thoughts to objects in the world. The crucial principle about content that seems to lead Frege into these troubles is the notion that the same content can be expressed in different ways carved up with references to quite distinct objects.

The question now arises, to what extent is the carving-up-content principle compatible with Frege’s new theory of sense and reference? It would seem that it must be if the uses of recarving in the philosophy of mathematics are now justified as sense preserving. How else could Principle V be seen as *analytic* if it does not recarve the same sense in two different ways? There is also the question of how well senses will serve as meanings. How close is their structure to that of sentences? Furthermore, if the principles about content that lead to the slingshot do not hold for sense, then the intuitive connection of thoughts and “aboutness” seems to be severed. Fregean thoughts do not represent what a sentence is about, but seem to be mere shadows of the strings of words with which they are expressed, and so lose the feature of being about the world, which is characteristic of contents.

The historical question of the continuity of Frege’s thought is an important one. My interest is in just a portion of this problem, namely, to what extent thoughts can be carved up, compatible with the logical principles governing the relations of sense and reference. Still, at least one tantalizing quotation exists which seems to explicitly make just such a claim. It occurs in the paper “Function and Concept” from 1891 which contains a brief summary of “On Sense and Reference” and the following passage.

Accordingly, e.g., $\acute{\epsilon}(\epsilon^2 - 4\epsilon)$ is the value-range of the function $x^2 - 4x$ and $\acute{\alpha}(\alpha.(\alpha - 4))$ is the value range of the function $x(x - 4)$, so that in ‘ $\acute{\epsilon}(\epsilon^2 - 4\epsilon) = \acute{\alpha}(\alpha.(\alpha - 4))$ ’ we have the expression for: the first range of values is the same as the second. . . . If we understand ‘ $x^2 - 4x = x(x - 4)$ ’ in the same sense as before [as a universal generalization], this expresses the same sense, but in a different way. It presents the sense as an equality holding generally; whereas the newly-introduced expression is simply an equation, and its right side, and equally its left side, stands for something complete in itself. ([9], p. 27)⁶

Here it seems to be explicitly claimed that a universal generalization of an equivalence relation and a proposition asserting the identity of objects “ab-

stracted” from that relation express the same *sense*. This looks like an application of the “carving up content” principle applied to senses.

In this paper I wish to suggest that other doctrines manifestly accepted by Frege place a limitation on the extent to which any carving could be allowed. There are quite formal principles that Frege adopts which lead to a precisely specifiable limit to recarving. Recarving cannot introduce any new terms which are “independent”, in a sense to be specified below.

Suppose that the carving-up-content principle has the consequence that there could be: (1) two sentences ‘ $\phi(a)$ ’ and ‘ ψ ’, where ‘ a ’ does not appear in ‘ ψ ’, yet both sentences have the same sense. Suppose further that: (2) the name ‘ a ’ is *independent* of ‘ ψ ’; that is, it is possible for ‘ ψ ’ to have a reference when ‘ a ’ has none. Suppose that the general doctrine that sense determines reference has not only the consequence that if the sense of ‘ t_1 ’ = the sense of ‘ t_2 ’ then the truth value of ‘ $\phi(t_1)$ ’ = the truth value of ‘ $\phi(t_2)$ ’ but, further: (3) if ‘ t_1 ’ *lacks* a reference then the thought which is the sense of ‘ $\phi(t_1)$ ’ lacks a truth value. One needs also to assume that another consequence of the doctrine that sense determines reference is that: (4) if it is possible that thought p has a truth value while thought q does not, then $p \neq q$.

The argument then is very fast. Suppose that ‘ $\phi(a)$ ’ and ‘ ψ ’ are the result of an application of the carving-up-content principle, where ‘ a ’ is independent of ‘ ψ ’ as described above. Consider a possible world in which ‘ a ’ does not have a referent, while ‘ ψ ’ does. Then ‘ ψ ’ has a truth value but ‘ $\phi(a)$ ’ does not, so they express different senses. A recarving thus cannot introduce an independent term if senses satisfy (3) and (4) above, that is, if reference failure produces truth value gaps, and if a necessary condition for two sentences expressing the same thought is that they must have truth values at all the same possible worlds.

Is Frege committed to the assumptions and logic of the above argument? That reference failure of a name leads to reference failure of a containing sentence seems explicit in Frege’s discussion in “On Sense and Reference” ([9], p. 62), “The sentence ‘Odysseus was set ashore at Ithaca while sound asleep’ obviously has a sense. But since it is doubtful whether the name ‘Odysseus’, occurring therein, has a reference, it is also doubtful whether the whole sentence has one.” Discovering that ‘Odysseus’ has no referent would, presumably, resolve that doubt, showing that the sentence lacks a truth value. Below I consider how Frege could express the inference in (4) above. It seems, then, that Fregean principles justify the “independence” restriction on carving up content.⁷

The question now is just how much of a restriction is this? Uses of the carving-up-content principle including Frege’s example of ‘line a is parallel to line b and ‘the direction of a = the direction of b ’ satisfy the condition above. Neither ‘the direction of a ’ nor ‘ a ’ are independent in the relevant sense, if ‘ a ’ lacks a reference then so does ‘the direction of a ’ (and sentences containing it), and vice versa. Let us consider this carefully. Notice that it is possible for the direction of a to exist but not a . Suppose that a is a street and the direction is north-south which exists even if a does not. The point is that the lack of reference for the name ‘ a ’ will infect the *name* ‘the direction of a ’. Now what about the converse? How might ‘the direction of a ’ lack a reference? Only if ‘ a ’ does or terms with the expression ‘direction of’ lack a reference because there are no directions. If directions, numbers, and other entities involved in the entities defined by def-

initions by abstracting from equivalence classes or other uses of the carving-up-content principle must exist, then the use will not run afoul of the above restrictions. Frege seems to use the carving-up-content principle to *introduce* the notion of direction, in the example under consideration, and, with Principle V, the notion of extension, or “course of values”. If the carving principle is taken to *justify* belief in the introduced entities, then of course determining whether the independence condition has been met just begs the question. ‘The direction of *a*’ will lack a referent if there are no directions. But if directions do exist then the expression ‘the direction of . . .’ can never introduce new reference failure. What are introduced by these definitions by “abstraction” are mathematical objects which will exist necessarily, or at least whenever the objects from which they are abstracted exist. If there are parallel lines then necessarily there are directions. The skeptic about numbers or directions will simply block the recarvings on the grounds that they introduce nondenoting terms, not that they introduce *independent* terms. It is not clear how Frege could convince the skeptic that the recarving is legitimate.⁸

The “slingshot” argument shows that any two true sentences must have the same “content” if one assumes only that “content” is preserved by substitution of co-referential singular terms and by a fair amount of alternative carvings. After accepting the sense–reference distinction, Frege could accept the consequence of the argument for reference but deny it for sense, because substitution of co-referential terms does not preserve sense. It is interesting to ask, however, whether the “carving-up-content” moves in the argument are justified for the theory of sense. In other words, does Frege’s theory block the argument in two places or in just one by objecting to the replacement of coreferential singular terms?

Let us look at some of the recarving moves in some versions of the slingshot (taken from [11]) and see if they satisfy the above restriction. First, Church’s original version:

- (A) Sir Walter Scott is the author of *Waverly*.
- (B) Sir Walter Scott is the man who wrote twenty-nine *Waverly* Novels altogether.
- (C) The number such that Sir Walter Scott is the man who wrote that many *Waverly* novels altogether is twenty-nine.
- (D) The number of counties in Utah is twenty-nine.

The steps from (A) to (B) and from (C) to (D) result from replacing co-referential terms. The recarving occurs in the move from (B) to (C). Let ‘ ψ ’ be the sentence (C), ‘*a*’ be ‘Sir Walter Scott’, ‘ ϕ ’ be the rest of (B), and we find that ‘*a*’ is *not* independent of ‘ ψ ’, for if Sir Walter Scott did not exist then (C) would also lack a reference, and similarly for the other ways of viewing (B) as a recarving of (C), or vice versa. So Church’s slingshot satisfies the “independence” condition for carving up content.⁹

On the other hand it is clear that the move from (A) to (B) and again from (C) to (D) introduces new content into the sentences. A completely adequate account of content would have to investigate the way in which substitution of co-referential terms affects content. Barwise and Perry ([5], p. 150) want to distinguish two notions of interpretation, one “value free” and the other “value

loaded”. Substitutions of co-referential terms would automatically preserve content only in the “value loaded” sense, where the *value* of the expression is what it contributes to content. Recarving requires that the content be available in the “value free” form in which the way an object is denoted makes a contribution to content. The “independence” condition, or others, could then limit the replacement of terms in the value free form. In the Church version of the argument, for example, mention of “Utah” in (D) introduces an independent singular term as the result of substituting coreferential terms, not by recarving. In “value free” sense, at least, the step does not seem to preserve content. However the development of a theory of content might go, it is clear that the “independence” condition proposed here will not by itself be adequate to block slingshot arguments.

Donald Davidson uses the slingshot against facts, arguing that all true sentences “correspond” with the same fact. He justifies the recarving step by citing the *logical equivalence* of:

(E) “S” corresponds to the fact that *S*.

and

(F) “S” corresponds to the fact that

$$\iota x(x = \textit{Diogenes} \ \& \ S) = \iota x(x = \textit{Diogenes}).$$

Here ‘Diogenes’ will serve as ‘*a*’ and *S* as ‘ ψ ’, as long as ‘Diogenes’ does not appear in ‘*S*’. The term ‘Diogenes’ is independent of *S*, so the move from (E) to (F) does not qualify as a legitimate recarving of *sense*, even though this is a pair of *logically equivalent* sentences. Thus not every replacement of one sentence by another which is logically equivalent counts as a recarving of its sense.¹⁰

Another slingshot which Olson presents is one he extracts from Gödel. This version assumes the following case of recarving;

(G) $\phi(a)$

(H) $a = \iota x(\phi(x) \ \& \ x = a)$.

Here there do not seem to be any candidates for independent singular terms, for if the ‘*a*’ in (G) fails to have a reference then both terms in (H) do.

Consider another case of replacement of logical equivalents, this in Barwise [3]. It is not presented as a use of recarving in a slingshot, but is a good illustration of a use of substitution of logical equivalents which leads to some trouble. He goes from;

(I) *f* saw $F(m)$

to

(J) *f* saw $((F(m) \wedge B(b)) \vee (F(m) \wedge \sim B(b)))$.

This interchange of logical equivalents clearly introduces a term ‘*b*’ which is independent of (I).

Take any sentence *S* and term denoting term ‘*a*’ independent of it. Then ‘*S*’ and ‘*S* & (*a* = *a*)’ will be logically equivalent, but will express different senses, for if ‘*a*’ were to lack a referent then the thought that *S* would still have a truth value while the thought that *S* & (*a* = *a*) would not.

How much of this would Frege have accepted? He certainly made little use

of modal notions; in particular, he never spoke about what the reference of an expression *would be* in certain counterfactual circumstances. Such talk is more in place in the context of a Carnapian reconstruction of Fregean senses as intensions, or functions from possible worlds to referents. Can the restrictions on the carving-up-content principle be put in terms more congenial to Frege? One way to put the dependence of truth value upon reference is in terms of logical consequences. If it follows as a matter of *logic* that nondenoting terms lead to reference failure of sentences, then that fact can express the *modal dependency* of truth value on reference of names. If p logically entails q , then if p were true, then so *would q be*. How can we represent this logical dependence of ‘ $\phi(a)$ ’ having a truth value on ‘ a ’ having a reference? Frege never formulated a new logic to incorporate the new concepts of sense and reference. As well, in “On Sense and Reference”, there is the remark in a footnote that nonreferring expressions such as ‘the negative square root of 4’ “. . . must actually always be assured of reference, by means of a special stipulation, e.g. by the convention that 0 shall count as its reference, when the concept applies to no object or to more than one” ([9], p. 71). This is often taken to indicate that Frege did not allow that formalized languages could tolerate reference failure. I take this remark to suggest that Frege intended his logic to apply to possibly nondenoting terms and that, while the logic must itself assign some referent to every term, the phenomena of reference failure must still be expressible.¹¹

Claims about the relation of sense to reference seem to have the formal character of other statements that are clearly logical for Frege. Church has developed a “logic” which can be interpreted as a development of Frege’s original logic to incorporate the new notion of sense and to handle inferences involving the relation of sense to reference. In particular, it allows one to formalize the argument that expressions with independent terms cannot have the same sense. It does so without itself being a free logic. No terms *in* the logic lack a reference. The logic does, however, supply names for the *senses* of ordinary language expressions like “Odysseus” which lack a referent. With an expression for the denotation relation, it is then possible to express the fact that such terms are nondenoting: simply, there does not exist an object denoted by that sense. It is instructive, then, to formulate the independence condition and the argument that violating it will not preserve sense in Church’s “Logic of Sense and Denotation” (Alternative 0) as presented by Anderson [1]. We are to suppose that there could be two sentences ‘ $F_{o_i} a_i$ ’ and ‘ $G_{o_i} b_i$ ’ where $a_i \neq b_i$ yet the two sentences express the same thought, i.e. $F_{o_i \iota} a_i = G_{o_i \iota} b_i$. (ι is the type of objects, o the type of truth values, and the subscript 1 indicates a *first ascendant*, the *sense* of an expression. Thus a sentence will be of type o and its sense, a thought, of type o_1 . A predicate which is a function from objects to truth values will be of type o_i . ‘ Δ ’ expresses the denotation relation which is typed, although the lowest order 0 is sufficient for the example below and so the superscript may be suppressed. ‘*Con*’ is a predicate true of senses.) The crucial axiom, which has the consequence that if a sentence has a nondenoting term then it will not have a truth value, is $A(0)3^{m\alpha\beta}$:

$$\begin{aligned} & (f_{\alpha\beta})(f_{\alpha_1\beta_1})(y_{\alpha})(x_{\beta_1}) \cdot \Delta^m f_{\alpha\beta} f_{\alpha_1\beta_1} \\ & \rightarrow \Delta^m y_{\alpha} (f_{\alpha_1\beta_1} x_{\beta_1}) \rightarrow \cdot \text{Con} x_{\beta_1} \rightarrow \cdot (\exists x_{\beta}) \Delta^m x_{\beta} x_{\beta_1}. \end{aligned}$$

Assume that ‘ $F_{o_i}a_i$ ’ has a truth value, in particular that it is true, i.e., that $F_{o_i}a_i$ denotes T_o . (The other case is identical). An instance of $A(0)3^{0o_i}$ with respect to G will be:

$$\Delta G_{o_i}G_{o_i t_i} \rightarrow \Delta T_0(G_{o_i t_i} b_{t_i}) \rightarrow .Conb_{t_i} \rightarrow .(\exists x_i)\Delta x_i b_{t_i}.$$

The three antecedents of this conditional will hold if ‘ G ’ has a denotation and ‘ b ’ has a sense. Since the sense of ‘ Fa ’ denotes the True, and we know that sense is the same as the sense of ‘ Gb ’, we can get the second antecedent by substitution. (From ‘ $\Delta T_0(F_{o_i t_i} a_{t_i})$ ’ and ‘ $F_{o_i t_i} a_{t_i} = G_{o_i t_i} b_{t_i}$ ’, we get ‘ $\Delta T_0(G_{o_i t_i} b_{t_i})$ ’.) So, from the hypothesis that ‘ Fa ’ is true we get that ‘ b ’ has a denotation, the desired consequence.

It should be noted that this result uses very little of the logic of $LSD(0)$. It does not, in particular, use axioms or notions that are too heavily committed to the concept of “synonymous isomorphism” as the identity criterion for senses, which Anderson himself sees as the underlying intuition behind Alternative (0). That intuition does prejudge the question of to what extent thoughts can be recarved, namely, never beyond the limits of that isomorphism. The argument does seem to assume that the only ‘ ψ ’ which is a candidate for being synonymous with the atomic subject–predicate sentence ‘ Gb ’ will be another sentence with the same structure, ‘ Fa ’. Another axiom ($A(0)8^{\alpha\beta\gamma}$) requires that atomic sentences with different numbers of singular terms cannot be synonymous.¹²

Other axioms place even stricter limits on recarving. For example, axiom $A(0)6^{\alpha\beta}$ prohibits just the sort of recarving that Frege himself cited in his early examples:

$$(f_{\alpha_1\beta_1})(g_{\alpha_1\beta_1})(x_{\beta_1})(y_{\beta_1}).Conf \rightarrow .Cong \rightarrow .Conx \rightarrow .Cony \rightarrow .fx = gy \rightarrow f = g.$$

This would seem to rule out even the “direction” example because ‘ a is parallel to b ’ will involve two singular terms and a two-place relation, while ‘the direction of $a =$ the direction of b ’ involves singular terms, a two-place relation (identity) but also the function ‘direction of’. Formalized with a predicate ‘ P_{ou} ’ for the relation ‘is parallel to’ and ‘ d_u ’ for the direction of function we get as the two sentences ‘ $d_u a_i = d_u b_i$ ’ and ‘ $P_{ou} a_i b_i$ ’, the first ascendants of which, it would seem, are not identical by $A(0)6$. The identity symbol is defined so that the first is ‘ $Q_{o_i t_i t_i} d_{t_i t_i} b_{t_i} d_{t_i t_i} a_{t_i}$ ’, where ‘ Q ’ is in turn defined to express indiscernibility, the second, ‘ $P_{o_i t_i t_i} a_{t_i} b_{t_i}$ ’. The hypothesis that the original sentences are synonymous leads to the consequent of an instance of axiom $A(0)6$ which is absurd: $P_{o_i t_i t_i} = Q_{o_i t_i t_i}$.¹³ $LSD(0)$ thus has very strict limitations on what can count as synonymous expressions. One would think that it should be possible to *define* ‘ x is parallel to y ’ with the more complex formula ‘the direction of $x =$ the direction of y ’, using the notions of identity and direction, even though that reverses the order of conceptual priority intended by Frege’s examples (see [7], p. 9). *Some* definitions of primitive terms by complex formulas ought to be allowed as preserving sense, and, as Anderson points out, this is an unresolved problem for $LSD(0)$ (see [1], p. 383). My hope is to have captured the essential logic of the argument without having imported any assumptions about sense that would make it trivial.

Despite the intricate interconnections between various aspects of Frege’s thought, I hope to have isolated a somewhat independent connection between

plausible properties of sense and the “carving-up-content” principle, a connection that should be of use both to those interpreting Frege and to those who are attempting to construct alternative notions of “content”.

NOTES

1. This name is given to the principle by Leora Weitzman in her dissertation [13]. Harold Hodes speaks of the “polymorphous composition” of thoughts in [10]. Hodes provides numerous examples of recarvings from Frege’s writings and argues for the necessity of the principle in Frege’s logicist program. See, however, Dummett [7], who denies Frege held that recarvings preserve sense.
2. Although this is also denied by Dummett.
3. Frege says in the *Begriffsschrift* ([9], pp. 12–13) that “Hydrogen is lighter than carbon dioxide” and “Carbon dioxide is heavier than hydrogen” express the application of different functions to hydrogen and carbon dioxide, respectively. In the later “Concept and Object” ([9], p. 49) he analyzes “There is at least one square root of 4” as, equivalently, “The number 4 has the property that there is something of which it is the square” and “The concept square root of 4 is not empty”.
4. Frege says in “On Concept and Object”, published in 1892, “When I wrote my *Grundlagen der Arithmetik*, I had not yet made the distinction between sense and reference; and so, under the expression ‘a content of possible judgement,’ I was combining what I now designate by the distinctive words ‘thought’ and ‘truth-value’” ([9], p. 47).
5. This name comes from Barwise and Perry [4] who also present an analysis of the argument intended to save a notion of semantic value for sentences which is much more fine-grained than truth-value and somehow “in between” the levels of sense and reference.
6. Dummett [7] claims that this is a remnant of his earlier views and is never asserted again and that principle V is never claimed to be analytic.
7. I ignore the other possible source of reference failure for complex terms, *category mistakes* and partially defined predicates. ‘The direction of the number 4’ may well lack a reference because ‘direction of’ is not defined for numbers. I assume that Frege considered such reference failures to indicate defects in natural language, while the reference failure of ‘Odysseus’ needed to be representable in a formal language of sense and reference, even if that technical language itself contained no non-referring terms. Church’s logic, used below, incorporates these assumptions as well.
8. I owe awareness of this issue to Peter Morton.
9. Church ([6], p. 25) remarks that “. . . it is plausible to suppose [that] if [B] is not synonymous with [C], it is at least so nearly so as to ensure its having the same denotation”. This paper is an attempt to clarify the extent of that synonymy.
10. This, despite Frege’s remarks that logical equivalence is sufficient. Weitzman [13] resolves this by arguing that it is *thoughts* which are logically equivalent and that it is a serious problem to determine which thought a sentence expresses—in fact, one which renders syntactic determination impossible.

11. It is also clear, then, that the logic must allow ascriptions of logical equivalence to sentences that include terms that may only contingently refer, *pace* Wagner ([12], pp. 439ff.)
12. $(f_{\alpha_1\beta_1})(g_{\alpha_1\beta_1})(x_{\beta_1})(y_{\gamma_1}).Conf \rightarrow .Cong \rightarrow .Conx \rightarrow .Cony \rightarrow .fx \neq gy. (When \beta \neq \gamma.)$
13. The absurdity is not immediate as 'Q' is a defined symbol, but Axiom 6 suffices to deduce it.

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