Notre Dame Journal of Formal Logic
Volume 33, Number 1, Winter 1992

Two Formal Systems for Situation Semantics

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Abstract We are going to present two formal systems intended to capture some of the basic features of Barwise and Perry's situation semantics. The first one is a multidimensional system which allows formal counterparts of situations (including incoherent ones), the relational theory of meaning, and the strong consequence relation. Our second system is an extension of the former one and considers a set of actual situations, so that the notion of constraint can be expressed in it. Soundness, completeness, and compactness will be proven for both systems.

1 The language $\mathbb{L}$ First of all, we introduce the language $\mathbb{L}$ and its semantics. $\mathbb{L}$ has the following symbols:

Binary connectives: $\wedge$, $\rightarrow$, $|$.

Quantifier: $\forall$.

Unary connectives: $\neg$, $\otimes_i$, for each natural number i.

Identity symbol: $=$.

We also have denumerable sets of predicate letters, constants and variables. All the wffs of $\mathbb{L}$ are closed formulas and are defined in the usual way.

The models for $\mathbb{L}$ are structures $N = \langle W, U, D, [\![\ ]\!], S\rangle$, where W and U are sets, $W \neq \emptyset$, and D is a function defined on W such that for each $v \in W D(v) \subseteq U$. $[\![\ ]\!]$ is a function assigning a value $[\![c]\!] \in U$ to every constant c. S is a function such that for each $v \in W$, $S(v)$ is a set of U-formulas of the form $Pr_1 \ldots r_n$ or $\neg Pr_1 \ldots r_n$, where $r_i \in U$, for $1 \leq i \leq n$. (We are using the concept of U-formula in the same way it is used in Smullyan [9].) We also require that whenever $(\neg)Pr_1 \ldots r_n \in S(u_i)$, $r_1 \ldots r_n \in D(u_i)$. (We use $(\neg)A$ to represent either A or $\neg A$. Obviously, when $(\neg)A$ appears more than once in the same context, all its occurrences should be interpreted in the same way.)

Each element v of W has an associated set $S(v)$ of atomic U-formulas or negations of atomic U-formulas. $S(v)$ can be considered as a set of positive and

Received October 31, 1988; revised April 16, 1990

negative facts, so each $v \in W$ represents a situation. As we have imposed no maximality or coherence restriction on the function S, partial situations and incoherent situations are allowed in our semantics.

By $\vec{u}$ we represent an ω-tuple $\langle u_0, \ldots, u_i \ldots \rangle$ of elements of W. Now we can define the relations $\models$ (support) and $\dashv$ (reject) for ω-tuples, $\vec{u}$, elements v of W and U-wffs A as follows:

Atomic formulas

1. $\vec{u}, v \models Pt_1 \ldots t_n$ iff $P[\![t_1]\!] \ldots [\![t_n]\!] \in S(v)$.
   $\vec{u}, v \dashv Pt_1 \ldots t_n$ iff $\neg P[\![t_1]\!] \ldots [\![t_n]\!] \in S(v)$.
2. $\vec{u},v \models t = t'$ iff $[\![t]\!] \in D(v)$ and $[\![t]\!] = [\![t']\!]$.
   $\vec{u}, v \dashv t = t'$ iff $[\![t]\!], [\![t']\!] \in D(v)$ and $[\![t]\!] \neq [\![t']\!]$.

Complex formulas

3. $\vec{u}, v \models \neg A$ iff $\vec{u}, v \dashv A$.
   $\vec{u}, v \dashv \neg A$ iff $\vec{u}, v \models A$.
4. $\vec{u}, v \models A \wedge B$ iff $\vec{u}, v \models A$ and $\vec{u}, v \models B$.
   $\vec{u}, v \dashv A \wedge B$ iff $\vec{u}, v \dashv A$ or $\vec{u}, v \dashv B$.
5. $\vec{u}, v \models A \rightarrow B$ iff $\vec{u}, v \models A$ only if $\vec{u}, v \models B$.
   $\vec{u}, v \dashv A \rightarrow B$ iff $\vec{u}, v \models A$ and $\vec{u}, v \dashv B$.
6. $\vec{u}, v \models A \mid B$ iff $\vec{u}, v \models A$ and $\vec{u}, v \models B$.
   $\vec{u}, v \dashv A \mid B$ iff $\vec{u}, v \models A$ and $\vec{u}, v \dashv B$.
7. $\vec{u}, v \models \forall x A$ iff for all $r \in U$, $\vec{u}, v \models A[r/x]$.
   $\vec{u}, v \dashv \forall x A$ iff for some $r \in U$, $\vec{u}, v \dashv A[r/x]$.
8. $\vec{u}, v \models \otimes_i A$ iff $\vec{u}, u_i \models A$.
   $\vec{u}, v \dashv \otimes_i A$ iff $\vec{u}, u_i \dashv A$.

Given a model N, an ω-tuple $\vec{u}$ in W and a set Σ of wffs we say that $\vec{u}$ supports Σ in N iff $\vec{u}, u_0 \models A$ for every $A \in \Sigma$, and that $\vec{u}$ rejects Σ (in N) iff $\vec{u}, u_0 \not\models A$, for every $A \in \Sigma$. $\vec{u}, v$ support (reject) a wff A (in M) iff $\vec{u}, v \models A$ ($\vec{u}, v \dashv A$).

Observe that an atomic formula A can be supported and rejected simultaneously by the same $\vec{u}, v$. So, if we interpret $\vec{u}, v \models A$ and $\vec{u}, v \dashv A$ as "A is true relative to $\vec{u}, v$" and "A is false relative to $\vec{u}, v$", we have four truth values: "true" (1), "false" (0), "true and false" (2) and "undefined" (?). The truth tables of our propositional operators are as follows:

$A \wedge B$	1	2	?	0
1	1	2	?	0
2	2	2	0	0
?	?	0	?	0
0	0	0	0	0

$A \rightarrow B$	1	2	?	0
1	1	2	?	0
2	1	2	?	0
?	1	1	1	1
0	1	1	1	1

A/B	1	2	?	0
1	1	2	?	0
2	1	2	?	0
?	?	?	?	?
0	?	?	?	?

A	$\neg A$
1	0
2	2
?	?
0	1

We can define an existential quantifier in the usual way: $\exists x A =_{df} \neg \forall x \neg A$ and then we have:

9. $\vec{u}, v \models \exists x A$ iff there is some $r \in U$ such that $\vec{u}, v \models A[r/x]$.
   $\vec{u}, v \dashv \exists x A$ iff for every $r \in U$, $\vec{u}, v \dashv A[r/x]$.

The quantification domain is U. But we can quantify over the domain of any of the members of $\vec{u}$ using the restricted quantifiers $\forall_i$ and $\exists_i$, which are defined as follows:

$$\forall_i xA =_{df} \forall x(\otimes_i x = x \to A),$$

$$\exists_i xA =_{df} \neg\forall_i x\neg A.$$

(Notice that autoidentity works like an existence predicate.)

With this definition we have that:

10. $\vec{u}, v \models \forall_i A$ iff for every $r \in D(u_i)$, $\vec{u}, v \models A[r/x]$.
 $\vec{u}, v \mathrel{=\!\!|} \forall_i A$ iff for some $r \in D(u_i)$, $\vec{u}, v \mathrel{=\!\!|} A[r/x]$.
11. $\vec{u}, v \models \exists_i xA$ iff for some $r \in D(u_i)$, $\vec{u}, v \models A[r/x]$.
 $\vec{u}, v \mathrel{=\!\!|} \exists_i xA$ iff for every $r \in D(u_i)$, $\vec{u}, v \mathrel{=\!\!|} A[r/x]$.

Definite descriptions can be contextually defined in the following way:

$$P(\imath xA) =_{df} \exists x(A(x) \wedge \forall y(A(y) \to x = y)) \mid \forall x(A(x) \to P(x)).$$

From this definition it follows that:

12. $\vec{u}, v \models P(\imath xA)$ iff there is exactly one $r \in U$ such that $\vec{u}, v \models A[r/x]$ and for that r, $\vec{u}, v \models P(r)$.
 $\vec{u}, v \mathrel{=\!\!|} P(\imath xA)$ iff there is exactly one $r \in U$ such that $\vec{u}, v \models A[r/x]$ and for that r, $\vec{u}, v \mathrel{=\!\!|} P(r)$.

Substituting restricted quantifiers for the quantifiers in the definition of definite descriptions, we obtain restricted descriptions, which refer to the only element in a domain $D(u_i)$ which satisfies the description condition. Define

$$P(\imath_i xA) =_{df} \exists_i x(A(x) \wedge \forall_i y(A(y) \to x = y)) \mid \forall_i x(A(x) \to P(x)).$$

Then,

13. $\vec{u}, v \models P(\imath_i xA)$ iff there is exactly one $r \in D(u_i)$ such that $\vec{u}, v \models A[r/x]$, and for that r, $\vec{u}, v \models P(r)$.
 $\vec{u}, v \mathrel{=\!\!|} P(\imath_i xA)$ iff there is exactly one $r \in D(u_i)$ such that $\vec{u}, v \models A[r/x]$, and for that r, $\vec{u}, v \mathrel{=\!\!|} P(r)$.

We could have introduced definite descriptions as complex singular terms of $\mathbb{L}$, but, as we have just seen, that would not render us any additional expressive power.

It should be noticed that the propositional fragment of $\mathbb{L}$ containing only the standard operators ($\neg, \wedge, \vee, \to$) and with no nested arrows (i.e., excluding all formulas $A \to B$ where either A or B contains the symbol "$\to$") is just Anderson and Belnap's FDE system (see [1]; the proof can be found in Barba [2]). When we consider formulas with nested arrows, $\mathbb{L}$ differs clearly from relevance logics because formulas like $A \to (B \to A)$ or $A \to (B \to B)$, which are not theorems of any relevance logic, are valid in $\mathbb{L}$.

The language $\mathbb{L}$ and situation semantics As we said above, the elements of the set W in our models represent situations. But each $v \in W$ determines not only a set of facts, but also a set of individuals $D(v)$. This set may be interpreted as the set of individuals existing in v or the set of relevant individuals in situation v. The main purpose of the function D is providing a quantification domain for each situation.

The relations $\models$ and $\dashv$ hold among ω-tuples $\vec{u}$, members v of W, and sentences A. But, for many purposes, the interesting cases are those in which v is u_0 (i.e., the first coordinate of $\vec{u}$), so we shall write $\vec{u} \models A$ instead of $\vec{u}, u_0 \models A$. The meaning of a sentence A may be seen as a relation among situations which holds among the situations of any $\vec{u}$ such that $\vec{u} \models A$. We have a relational account of meaning, as situation semantics requires. Notice that only a finite number of the u_i in $\vec{u}$ are relevant for determining whether $\vec{u} \models A$ (all those u_j such that $\otimes_j$ does not occur in A are irrelevant).

Given any $\vec{u}$, we can consider u_0 as the described situation, some u_i as the discourse situation, and any u_j as a resource situation. It may seem that our semantics would be closer to Situation Semantics if we used ordered triples instead of ω-sequences. Our purpose is, however, to stress the part played by resource situations in natural language semantics. If a resource situation is to be conceived as a set of facts exploited by the speaker along the discourse, there seems to be no finite limit to the number of such situations available to him. We assume that any chunk of discourse may involve any finite number of resource situations, and our use of ω-tuples intends to make explicit such possibility.

Roles can be represented by means of definite descriptions. For instance, the speaker role can be designed using a description of the form $\imath_i xSx$, where u_i is the discourse situation and S is a predicate letter to be interpreted as "speaks". The proposed description refers to the only individual speaking in the discourse situation. That is the interpretation assigned in Barwise and Perry [3] for the word "I". More generally, any expression which purports to refer to an individual may be represented in $\mathbb{L}$ by a (restricted) definite description involving the appropriate situations in $\vec{u}$. This idea can be extended to proper names. We propose to interpret that when someone uses a name "X", he is in fact referring to the only individual named "X" in a contextually given situation. This does not depart strikingly from Barwise and Perry's ideas. They certainly say that proper names are not hidden descriptions, but "names are not unique, and this is reflected in their semantical properties; they serve not only as noun phrases, but also as common nouns. Although my use of 'Aristotle' has no intimate connection with most of the properties I believe Aristotle had, it does have an intimate semantic connection with one he had, being an Aristotle" ([3], pp. 165–166). Moreover, "each name β has an associated property, which we express with the phrase 'being a β'" ([3], p. 167). Our proposal is just to employ that associated property to identify the reference of proper names.

Notice that a definite description is a partial function assigning individuals to tuples $\vec{u}$, so that, implicitly, we have a speaker's connections function c which determines the reference of individual expressions relative to the context (as Barwise and Perry wish): the (partial) function which assigns a reference to each definite description relative to a tuple $\vec{u}$.

Barwise and Perry claim that situation semantics explains the difference between referential and attributive use of definite descriptions: when used attributively, a description must be interpreted in the described situation, whereas when used referentially it must be interpreted in a contextually given situation (a resource situation). Our restricted descriptions capture this distinction, and so attributive and referential uses of definite descriptions can be adequately represented in $\mathbb{L}$.

One of the main purposes of our language $\mathbb{L}$ is providing a rigorous formal account of the notion of strong consequence in situation semantics. First, we define a consequence relation $\models$ between sets of wffs as follows: Let Σ, Ω be any sets of wffs. Then, $\Sigma \models \Omega$ iff for any ω-tuple $\vec{u}$ in any model M, if $\vec{u} \models A$ for every $A \in \Sigma$ then $\vec{u} \models B$, for some $B \in \Omega$. The usual conventions are assumed, so that "$\Gamma, A \models \Delta$" should be read as "$\Gamma \cup \{A\} \models \Gamma$", "$\models\Delta$" as "$\varnothing \models \Delta$", and so on. (We use the symbol "$\models$" to represent both the supporting relation and the consequence relation. Both uses can be found in the literature, and because the context makes clear the intended use in each case we prefer not to introduce a new symbol.)

This is a consequence relation in the standard logical sense of the expression, and it obviously has some of the properties Barwise and Perry demand from a consequence relation (recall the connections between $\mathbb{L}$ and FDE explained above). Also, the relation $\models$ has a syntactic counterpart, as we shall prove below. But $\models$ is a relation between sets of sentences of $\mathbb{L}$, which are intended to be counterparts of natural language sentences, whereas situation semantics' strong consequence is a relation between statements, which are sentences together with an utterance situation (composed of a discourse situation, the speaker's connections, and some resource situations). In our semantics, the resource situation corresponding to a tuple $\vec{u} = \langle u_0 \ldots u_n \ldots \rangle$ would be represented by the tuple $\vec{u}^+ = \langle u_1 \ldots u_n \ldots \rangle$ ($\vec{u}$ minus the described situation u_0). Then the counterpart of a statement is a pair $\langle A, \vec{u}^+ \rangle$. Let $v^\wedge \vec{u}^+ = \langle v, u_1 \ldots u_n \ldots \rangle$. Then the statement $\langle B, \vec{u}^+ \rangle$ is a consequence of the statement $\langle A, \vec{u}'^+ \rangle$ in a model M (of course, we suppose that both $\vec{u}^+$ and $\vec{u}'^+$ have been extracted from M) iff for every $v \in W$, if $v^\wedge \vec{u}'^+ \models A$ then $v^\wedge \vec{u}^+ \models B$. (Notice once more that only a finite number of the coordinates in $\vec{u}^+$ and $\vec{u}'^+$ are relevant.)

Observe that this consequence relation is unavoidablely relative to models and that it is not a consequence relation in the standard logical sense.

These are not the only implication relations which could be considered in our semantics. An interesting one could be, for instance, "A implies B independently of the contexts in which A and B are uttered (i.e., independently of whether they are uttered in the same context or in different ones)". The formal counterpart of such relation would be: For any wffs A and B, A implies B independently of the context iff for any $M, v, \vec{u}, \vec{u}'$, if $v^\wedge \vec{u} \models A$ then $v^\wedge \vec{u}' \models B$.

A sequent system for $\mathbb{L}$: $S_{\mathbb{L}}$ A sequent is a pair $\langle \Gamma, \Delta \rangle$ where Γ, Δ are finite sets of wffs of $\mathbb{L}$. We write $\Gamma \vdash \Delta$ to denote that $\langle \Gamma, \Delta \rangle$ is provable. According to the usual conventions, we write A, A, B, and Γ, A instead of $\{A\}$, $\{A, B\}$, and $\Gamma \cup \{A\}$, respectively, and $\Gamma\vdash$, $\vdash\Delta$ instead of $\Gamma \vdash \varnothing$ and $\varnothing \vdash \Delta$. $S_{\mathbb{L}}$ has the following axiom schemes and rules:

Axiom schemes

1. $A \vdash A$
2. $\otimes_i t = t'$, $\neg \otimes_i t = t' \vdash$
3. $\otimes_i t_1 = s_1 \ldots \otimes_i t_n = s_n, (\neg) \otimes_i P t_1 \ldots t_n \vdash (\neg) \otimes_i P s_1 \ldots s_n$.

Rules I

W $\dfrac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$

1a $\dfrac{\Gamma, A \vdash \Delta}{\Gamma, \neg\neg A \vdash \Delta}$ **1b** $\dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash \neg\neg A, \Delta}$

2 $\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}$ **3** $\dfrac{\Gamma \vdash A, \Delta;\ \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}$

4 $\dfrac{\Gamma, \neg A \vdash \Delta;\ \Gamma \neg B \vdash \Delta}{\Gamma, \neg (A \wedge B) \vdash \Delta}$ **5** $\dfrac{\Gamma \vdash \neg A, \neg B, \Delta}{\Gamma \vdash \neg (A \wedge B), \Delta}$

6 $\dfrac{\Gamma \vdash A, \Delta;\ \Gamma, B \vdash \Delta}{\Gamma, A \rightarrow B \vdash \Delta}$ **7** $\dfrac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \rightarrow B, \Delta}$

8 $\dfrac{\Gamma, A, \neg B \vdash \Delta}{\Gamma, \neg (A \rightarrow B) \vdash \Delta}$ **9** $\dfrac{\Gamma \vdash A, \Delta;\ \Gamma \vdash \neg B, \Delta}{\Gamma \vdash \neg (A \rightarrow B), \Delta}$

10 $\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A | B \vdash \Delta}$ **11** $\dfrac{\Gamma \vdash A, \Delta;\ \Gamma \vdash B, \Delta}{\Gamma \vdash A | B, \Delta}$

12a $\dfrac{\Gamma, A | \neg B \vdash \Delta}{\Gamma, \neg (A | B) \vdash \Delta}$ **12b** $\dfrac{\Gamma \vdash A | \neg B, \Delta}{\Gamma \vdash \neg (A | B), \Delta}$

13 $\dfrac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall x A(x) \vdash \Delta}$ **14** $\dfrac{\Gamma \vdash A(t), \Delta}{\Gamma \vdash \Delta, \forall x A(x)}$ (*)

15 $\dfrac{\Gamma, \neg A(t) \vdash \Delta}{\Gamma, \neg \forall x A(x) \vdash \Delta}$ (*) **16** $\dfrac{\Gamma \vdash \neg A(t), \Delta}{\Gamma \vdash \neg \forall x A(x), \Delta}$

(*) provided that t does not occur in Γ, Δ, $\forall x A(x)$.

17a $\dfrac{\Gamma, (\neg) \otimes_0 A \vdash \Delta}{\Gamma, (\neg) A \vdash \Delta}$ (*) **17b** $\dfrac{\Gamma \vdash (\neg) \otimes_0 A, \Delta}{\Gamma \vdash (\neg) A, \Delta}$ (*)

(*) A is an atomic formula.

18 $\dfrac{\Gamma, \otimes_i t = t \vdash \Delta}{\Gamma, (\neg) \otimes_i P(t) \vdash \Delta}$ **19** $\dfrac{\Gamma, \otimes_i t = t \vdash \Delta}{\Gamma, (\neg) \otimes_i t = t' \vdash \Delta}$

20 $\dfrac{\Gamma, \otimes_i t_2 = t_0 \vdash \Delta}{\Gamma, \otimes_i t_0 = t_1,\ \otimes_k t_1 = t_2 \vdash \Delta}$

21 $\dfrac{\Gamma, \otimes_i t_0 = t_1 \vdash \Delta}{\Gamma, \otimes_i t_0 = t_0,\ \otimes_i t_1 = t_1 \vdash \neg \otimes_i t_0 = t_1, \Delta}.$

The most salient feature of this set of rules is the inexistence of independent rules for the negation symbol $\neg$, having instead many rules for different kinds of negated formulas. It would be desirable to have a "classical" rule for negation as, for instance, from $\Gamma, A \vdash \Delta$ infer $\Gamma \vdash \neg A, \Delta$. But it would not be sound: take the axiom $A \vdash A$ and apply the rule to obtain $\vdash A, \neg A$. As $A \models A$, soundness would require that $\models A, \neg A$. But this is not true: if A is an atomic formula, we would need that for any u in any model, either $A \in S(u)$ or $\neg A \in S(u)$, contradicting the partiality of situations.

Notice that all the rules above are introduction rules except 17a and 17b. In both cases we could have used introduction rules (a simple inversion of the actual rules would do), but then we would have to modify Rules 18–21 in such a way that any of the atomic formulas appearing in those rules preceded by $\otimes_i$ should be allowed to appear with no prefix. That would complicate not only the sequent system but the completeness proof as well. We have chosen the simplest way, despite the loss of elegance.

Rules II We include here some rules affecting the operators $\otimes_i$. By $C(B)$ we mean any wff containing B. We have the following rules:

1′a $$\frac{\Gamma, C(\otimes_i \neg A) \vdash \Delta}{\Gamma, C(\neg \otimes_i A) \vdash \Delta}$$ **1′b** $$\frac{\Gamma \vdash C(\otimes_i \neg A), \Delta}{\Gamma \vdash C(\neg \otimes_i A), \Delta}$$

2′a $$\frac{\Gamma, C(\otimes_i (A \circ B)) \vdash \Delta}{\Gamma, C(\otimes_i A \circ \otimes_i B) \vdash \Delta}$$ **2′b** $$\frac{\Gamma \vdash C(\otimes_i (A \circ B)), \Delta}{\Gamma \vdash C(\otimes_i A \circ \otimes_i B), \Delta}$$

where $\circ$ is any binary operator.

3′a $$\frac{\Gamma, C(\otimes_i \forall x A) \vdash \Delta}{\Gamma, C(\forall x \otimes_i A) \vdash \Delta}$$ **3′b** $$\frac{\Gamma \vdash C(\otimes_i \forall x A), \Delta}{\Gamma \vdash C(\forall x \otimes_i A), \Delta}$$

4′a $$\frac{\Gamma, C(\otimes_i \otimes_j A) \vdash \Delta}{\Gamma, C(\otimes_j A) \vdash \Delta}$$ **4′b** $$\frac{\Gamma \vdash C(\otimes_i \otimes_j A), \Delta}{\Gamma \vdash C(\otimes_j A), \Delta}.$$

Theorem 1 (Soundness theorem) $\Gamma \vdash \Delta$ *only if* $\Gamma \models \Delta$.

Proof: It can be easily checked that: (i) if $\Gamma \vdash \Delta$ is an axiom, then $\Gamma \models \Delta$, and (ii) the rules preserve the $\models$ relation.

The set of rules II implies that, for any wff A of $\mathbb{L}$, there is a formula A' such that every operator $\otimes_i$ in A has in its scope only expressions of the form $t = t'$ or $Pt_1 \dots t_n$, where $t_1 \dots t_n$ are singular terms (parameters or variables) and such that A and A' are equivalent in the following sense: $\vec{u}, v \models A$ iff $\vec{u}, v \models A'$ and $\vec{u}, v \dashv A$ iff $\vec{u}, v \dashv A'$. Accordingly, we can ignore any formula which contains propositional operators or quantifiers within the scope of any $\otimes_i$ without loss of generality. That is what we shall do in the sequel. Thus, the set of rules II will not be needed at all in our further work. This explains why we have divided our rules into two different sets.

Completeness of $S_{\mathbb{L}}$ We shall prove that, for any two sets of wffs Σ and Ω, $\Sigma \models \Omega$ only if there are finite sets Γ, Δ, $\Gamma \subseteq \Sigma$, and $\Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$.

Extend the language $\mathbb{L}$ with a denumerable set *Cons* of new constants, and fix an enumeration of *Cons*.

Let $H = \{\langle A, 1\rangle : A \in \Sigma\} \cup \{\langle B, 0\rangle : B \in \Omega\}$, and let $h_1 \dots h_n \dots$ be any enumeration of H. We shall show that either there are sets $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$ or there are a model M and an ω-tuple $\vec{u}$ which supports Σ and rejects Ω. In order to do so, we construct a tree from which we can obtain either a proof of $\Gamma \vdash \Delta$ or a model M and an ω-tuple $\vec{u}$ as required. Each node of the tree contains a pair $\langle A, i \rangle$, $i = 1,0$, where i indicates whether or not A is a formula which should be supported by the desired $\vec{u}$ and M.

We construct a tree for $\langle \Sigma, \Omega \rangle$ step by step, according to the following instructions:

Step 0: Put h_0 in the topmost node of the tree, and go to the next step.

Step n + 1: Consider the nth node of each open branch, and proceed according to the form of the pair $\langle C, i \rangle$ contained in it. We shall not consider every case, as they can be easily deduced from corresponding sequent rules. See Smullyan [9] for the general principles of tableaux construction. However, it should be noticed that there is an important difference: in classical logic, the truth of a formula A is equivalent to nontruth of $\neg A$. In our semantics, $\vec{u} \vDash A$ does not imply $\vec{u} \nvDash \neg A$, and this causes the necessity of marking each formula in the tableau with "1" or "0", as explained above. We detail only the cases concerning the most novel operators, $|$ and $\otimes$. (The number(s) preceding each case refer to the corresponding sequent rule(s). A fully detailed proof can be found in Barba [2].) If $\langle C, i \rangle$ is:

10. $\langle A | B, 1 \rangle$: Add the nodes $\langle A, 1 \rangle$ and $\langle B, 1 \rangle$.
11. $\langle A | B, 0 \rangle$: Divide the branch putting $\langle A, 0 \rangle$ in a new branch and $\langle B, 0 \rangle$ in the other.
12. $\langle \neg (A | B), i \rangle$ $(i = 1,0)$: Add the node $\langle A | \neg B, i \rangle$.
17. $\langle (\neg) A, i \rangle$, where A is an atomic formula and $i = 1,0$: Add the node $\langle (\neg) \otimes_0 A, i \rangle$.
18. $\langle (\neg) \otimes_i Pc_1 \ldots c_n, 1 \rangle$: Add successively the nodes $\langle \otimes_i c_1 = c_1, i \rangle \ldots \langle \otimes_i c_n = c_n, 1 \rangle$.
19. $\langle \neg \otimes_i c = c', 1 \rangle$: Add $\langle \otimes_i c = c, 1 \rangle$ and $\langle \otimes_i c' = c', 1 \rangle$.

19,20. $\langle \otimes_i c_0 = c_1, 1 \rangle$: First of all, add the nodes $\langle \otimes_i c_0 = c_0, 1 \rangle$ and $\langle \otimes_i c_1 = c_1, 1 \rangle$, if they are not in the branch. Then, for each node $\langle \otimes_j c_1 = c_2, 1 \rangle$ in the branch add $\langle \otimes_i c_2 = c_0, 1 \rangle$ (if it does not appear previously in the branch). Finally, repeat $\langle \otimes_i c_0 = c_1, 1 \rangle$.

21. $\langle \neg \otimes_i c = c', 0 \rangle$: Check whether $\langle \otimes_i c = c, 1 \rangle$ and $\langle \otimes_i c' = c', 1 \rangle$ both appear in the branch. If so, add $\langle \otimes_i c = c', 1 \rangle$. Otherwise, repeat $\langle \neg \otimes_i c = c', 0 \rangle$.

Then add h_{n+1} to the branch. This finishes the n + 1-th step.

A branch closes iff it contains a group of nodes of one of the following forms:

1. $\langle A, 1 \rangle$, $\langle A, 0 \rangle$
2. $\langle \otimes_i c = c', 1 \rangle$, $\langle \neg \otimes_i c = c', 1 \rangle$
3. $\langle \otimes_i t_1 = s_1, 1 \rangle \ldots \langle \otimes_i t_n = s_n, 1 \rangle$, $\langle (\neg) Pt_1 \ldots t_n, 1 \rangle$, $\langle (\neg) \otimes_i Ps_1 \ldots s_n, 0 \rangle$.

(Notice that the described tree has the finite branching property.)

Lemma 2 *If all the branches of a tree for $\langle \Sigma, \Omega \rangle$ close, then there are finite sets $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$ ($\langle \Gamma, \Delta \rangle$ is provable).*

Proof: For each branch b there is a set of closing nodes $\langle A, i\rangle$ which determine it to be closed. Let $C_{b,1}$ be the set of closing nodes with $i = 1$, and $C_{b,0}$ the set of closing nodes with $i = 0$. Obviously, $C_{b,1} \vdash C_{b,0}$ is an axiom. Take now the sets

$$\Sigma_b = \{A : \langle A,1\rangle \text{ appears in the branch } b\} \text{ and}$$

$$\Omega_b = \{A : \langle A,0\rangle \text{ appears in the branch } b\}.$$

$\Sigma_b \vdash \Omega_b$ can be proven from the axiom above by Rule W. So for each branch b we have a provable sequent $\langle \Sigma_b, \Omega_b\rangle$. Every wff in $\Sigma_b(\Omega_b)$ is either: (a) a formula in $\Gamma(\Delta)$ or (b) a formula B such that $\langle B,1\rangle$ ($\langle B,0\rangle$) was introduced in the tree in the i-th step, according to the form of the i-th node in the branch (for some i). Our task is to show that a sequent $\langle \Sigma, \Omega\rangle$ in which every formula of type (b) has been eliminated can be proven. The proof of such sequent is the inverse image of the tree: the latest formulas introduced in the tree are eliminated first. Suppose the largest branch in the tree has n steps. We can prove that for each branch b and each $j \leq n$, there is a provable sequent $\langle \Sigma_{b,j}, \Omega_{b,j}\rangle$ in which every formula of type (b) in $\Sigma_{b,j}$ or in $\Omega_{b,j}$ has been introduced in the tree before the i-th step. The proof is by induction on $n - i$. If $n - i = 0$, the desired sequent is just the above-mentioned $\langle \Sigma_b, \Omega_b\rangle$. For $n - i > 0$, if the length of b is less than $n - i$, there is nothing to prove. Otherwise, by induction hypothesis, we assume that we have a provable sequent in which every formula of type (b) has been introduced before the $i + 1$-th step, so we must only show how to obtain a provable sequent in which the formulas introduced in the i-th step have disappeared. We shall consider (as examples) only the cases corresponding to some of the instructions for tableaux construction detailed above, namely those numbered 11, 19,20, and 21. The remaining cases are similar and left to the reader.

Suppose the i-th node in the tree was $\langle A \mid B, 0\rangle$. According to Rule 11, the tree branches, and we can assume that we have two provable sequents $\langle \Sigma_1, \Omega_1\rangle$ and $\langle \Sigma_2, \Omega_2\rangle$. The new pairs introduced in the tree are $\langle A,0\rangle$ and $\langle B,0\rangle$, one in each new branch, so suppose that $A \in \Omega_1$ and $B \in \Omega_2$ (otherwise, apply Rule W). By induction hypothesis, A and B are the only formulas of type (b) in these sequents, as they are the only ones introduced in the i-th step. Let $\Omega_1' = \Omega_1 - \{A\}$ and $\Omega_2' = \Omega_2 - \{B\}$. We can assume that $\Omega_1' = \Omega_2'$, and $\Sigma_1 = \Sigma_2$ (otherwise, apply Rule W to obtain sequents with this property). Thus we have $\Sigma_1 \vdash \Omega_1', A$ and $\Sigma_1 \vdash \Omega_1', B$, and by Rule 11 we obtain $\Sigma_1 \vdash \Omega_1', A \mid B$. $A \mid B$ was introduced in the tree before the i-th step, so the sequent $\langle \Sigma_1, \Omega_1' \cup \{A \mid B\}\rangle$ is provable and contains no formula of type (b) introduced in the i-th step or later, as wished.

Suppose now that the i-th node in the tree was $\langle \otimes_i c_0 = c_1, 1\rangle$. By induction hypothesis, we have a provable sequent $\langle \Sigma, \Omega\rangle$. According to instruction 19,20, both $\langle \otimes_i c_0 = c_0, 1\rangle$ and $\langle \otimes_i c_1 = c_1, 1\rangle$ have been introduced in the branch, so let $\Sigma' = \Sigma - \{\otimes_i c_0 = c_0, \otimes_i c_1 = c_1\}$. From $\Sigma \vdash \Omega$ we can easily prove $\Sigma', \otimes_i c_0 = c_1 \vdash \Omega$ by two applications of Rule 19. Σ' may still contain some formula $\otimes_i c_2 = c_0$. If so, we know that there was a node $\langle \otimes_j c_i = c_2, 1\rangle$ in the branch introduced before the i-th step. Thus, let $\Sigma'' = \Sigma' - \{\otimes_i c_2 = c_0\}$. From $\Sigma' \vdash \Omega$, applying Rule 20, we prove Σ'', $\otimes_i c_0 = c_1$, $\otimes_j c_1 = c_2 \vdash \Omega$. In this sequent, the formula introduced in the i-th step ($\otimes_i c_2 = c_0$) has been replaced by two formulas which were introduced in the tree before the i-th step. Iterating this procedure, we prove a sequent in which every formula of type (b) was introduced in the tree before the i-th step, as desired.

Finally, consider the case in which the i-th node is $\langle \neg\otimes_i c = c', 0\rangle$. If some new formula has been added, it has been in a node $\langle \otimes_i c = c', 1\rangle$, and provided that there were nodes $\langle \otimes_i c = c, 1\rangle$ and $\langle \otimes_i c' = c', 1\rangle$ previously introduced in the branch. Thus we can suppose to have a proof of $\Sigma, \otimes_i c = c' \vdash \Omega$, from which we prove $\Sigma, \otimes_i c = c, \otimes_i c' = c' \vdash \neg\otimes_i c = c', \Omega$ by Rule 20. Iterating this procedure, every formula introduced in the i-th step has been replaced by formulas introduced earlier in the tree.

It should be noticed that in every sequent above, each formula A appears on the left or right part of the sequent depending on whether the corresponding node in the tree is $\langle A, 1\rangle$ or $\langle A, 0\rangle$. This remark finishes the induction step and our summary of the proof of Lemma 2. The remaining cases and details are routine and tedious.

Lemma 3 *If a tree for $\langle \Sigma, \Omega\rangle$ has an open branch, there is a model M and an ω-tuple $\vec{u}$ in M such that $\vec{u}, u_0 \models A$ and $\vec{u}, u_0 \not\models B$ for each $A \in \Sigma$ and each $B \in \Omega$. I.e., if the tree is not closed, $\Sigma \not\models \Omega$.*

Proof: Consider an open branch. Let *Ctes* be the set of all constants c occurring in the branch. Define the relation $\approx$ in *Ctes* as follows:

a. $c \approx c$, for every $c \in Ctes$.
b. if $c \neq c'$, $c \approx c'$ iff $\langle \otimes_i c = c', 1\rangle$ appears in the branch for some i.

It can be checked that $\approx$ is an equivalence relation. For each $c \in Ctes$, let $|c|$ be the equivalence class to which c belongs.

We can now define a model $M = \langle W, U, D, [\![\ ]\!], S\rangle$ in the following way:

$W = \{u_i : \otimes_i$ occurs in the branch$\}$

$U = \{|c| : c \in Ctes\}$

$D(u_i) = \{|c| : \langle \otimes_i c = c, 1\rangle$ appears in the branch$\}$

$[\![c]\!] = |c|$ for each $c \in Ctes$. $[\![c]\!]$ takes any arbitrary value in U whenever $c \notin Ctes$.

$S(u_i) = \{P|c_1| \ldots |c_n| : \langle \otimes_i Pc_1 \ldots c_n, 1\rangle$ appears in the branch$\} \cup \{\neg P|c_1| \ldots |c_n| : \langle \neg\otimes_i Pc_1 \ldots c_n, 1\rangle$ appears in the branch$\}$.

Instruction 18 ensures that whenever $\langle(\neg)\otimes_i Pc_1 \ldots c_n, 1\rangle$ appears in the branch, so do $\langle \otimes_i c_j = c_j, 1\rangle$ $(1 \leq j \leq n)$, so that if $P|c_1| \ldots |c_n| \in S(u_i)$, then $|c_1| \ldots |c_n| \in D(u_i)$.

Let $\vec{u} = \langle u_0 \ldots u_n \ldots\rangle$. For those i such that $\otimes_i$ does not occur in the branch, let u_i be any member of W. Then, $M, \vec{u}$ support Σ and reject Ω. This can be established using the following claim:

Claim *For every node $\langle A, k\rangle$ in the branch,*
a. *if $k = 1$, $\vec{u}, u_0 \models A$*
b. *if $k = 0$, $\vec{u}, u_0 \not\models A$.*

Proof of the claim: Induction on the complexity of A. However, the induction is slightly more complicated than usual: first of all, the claim must be proved for atomic formulas and its negations and for atomic formulas preceded by $\otimes_i$ and its negations. Then, in the induction step, it can be proven for more complex formulas. We omit details here.

Now, as the branch is open, every $h_i \in H$ has been introduced in it (in the i-th step). So, for every $A \in \Sigma$, $\langle A, 1\rangle$ appears in the branch and, by our claim above, $\vec{u}, u_0 \models A$. Similarly, $\langle B, 0\rangle$ appears in the branch for each $B \in \Omega$, and $\vec{u}, u_0 \not\models B$. Then $M, \vec{u}$ support Σ and reject Ω, and the lemma is proven.

We can now establish our completeness theorem:

Theorem 4 (Completeness of $S_{\mathbb{L}}$) *For any two sets of wffs* Σ, Ω, $\Sigma \models \Omega$ *only if there are finite sets* $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ *such that* $\Gamma \vdash \Delta$.

Proof: Consider a tree for $\langle \Sigma, \Omega\rangle$. The tree closes, because otherwise $\Sigma \not\models \Omega$, according to Lemma 3. But then, by Lemma 2, there are finite sets $\Gamma \subseteq \Sigma, \Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$.

Corollary 5 $\Sigma \models \Omega$ *iff* $\Gamma \vdash \Delta$, *for some* $\Gamma \subseteq \Sigma$ *and* $\Delta \subseteq \Omega$.

Proof: Follows immediately from soundness and completeness theorems.

Theorem 6 (Compactness theorem) *Let* Σ, Ω *be two sets of wffs.* $\Sigma \models \Omega$ *iff there are finite sets* $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ *such that* $\Gamma \models \Delta$.

Proof: One direction is trivial. For the converse, suppose that $\Sigma \models \Omega$. By the completeness theorem, there are sets $\Gamma \subseteq \Sigma$ and $\Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$. Applying the soundness theorem, $\Gamma \models \Delta$.

2 *A system for actual situations*

Semantics and language for actual situations In Barwise and Perry [3], the authors present structures called "situation structures". From their point of view, reality must not be conceived as a completely determined whole (a world), but as a collection of situations, each of which represents a part of reality. These structures are not models for situation semantics (because situation semantics needs nonactual situations). They are a way of representing reality.

Now we are going to extend our system in such a way that we can deal with actuality. First, we must introduce actual situations in our models. Our new models are similar to the old ones but contain a new element: a subset $\mathbb{A}$ of W, to be considered as the set of actual situations in the model. Models have the following form: $M = \langle W, U, D, [\![\ ]\!], S, \mathbb{A}\rangle$. W, U, D, $[\![\ ]\!]$, and S are as before, while $\mathbb{A}$ satisfies the following conditions:

1. $\mathbb{A} \subseteq W$.
2. If $\mathbb{a} \in \mathbb{A}$, then there is no U-formula A such that $A, \neg A \in S(\mathbb{a})$.
3. If $\mathbb{a}_0, \mathbb{a}_1 \in \mathbb{A}$, then there is some $\mathbb{a}_2 \in \mathbb{A}$ such that $\mathbb{a}_0 \subseteq \mathbb{a}_2$ and $\mathbb{a}_1 \subseteq \mathbb{a}_2$.

The relation $\sqsubseteq$ between elements of W appearing in Clause 3 is defined as follows: For any $u, v \in W$, $u \sqsubseteq v$ iff $D(u) \subseteq D(v)$ and $S(u) \subseteq S(v)$. Clause 2 says simply that every actual situation is coherent, and Clause 3 says that any two actual situations are included in a greater one. This agrees with the conditions imposed by Barwise and Perry on their situation structures. Barwise and Perry include in their structures a collection of factual situations. We have not included such a set because it is obviously superfluous: the collection of factual situations, as defined in Barwise and Perry [3], contains exactly those situations which are included in some actual situation. So once we have fixed a collection of actual situations, the collection of factual ones is completely determined. Notice that $\mathbb{A}$ is a *set*, not a *collection*. Our models include also sets (not collections) of individuals and situations. At this point, we are following the usual practice in model-theoretic semantics, despite Barwise and Perry's opinions.

Observe that $\mathbb{A}$ may not contain any maximal element (that is, any $\mathbb{a}^*$ such that $\mathbb{a} \sqsubseteq \mathbb{a}^*$ for every $\mathbb{a} \in \mathbb{A}$).

We extend our language $\mathbb{L}$ with a new operator @, obtaining a new language $\mathbb{L}(@)$. The set of wffs is defined as that for $\mathbb{L}$, adding a new clause: if A is a wff, so is $@A$.

$\models$ and $\Dashv$ are defined according to the clauses given in Section 1 plus the following ones:

$\vec{u}, u \models @A$ iff $\vec{u}(\mathbb{a}), \mathbb{a} \models A$, for some $\mathbb{a} \in \mathbb{A}$.

$\vec{u}, u \Dashv @A$ iff $\vec{u}(\mathbb{a}), \mathbb{a} \Dashv A$, for every $\mathbb{a} \in \mathbb{A}$.

(Given an ω-tuple $\vec{u}$ in W and $v \in W$, by $\vec{u}(v)$ we mean $\langle v, u_1 \dots u_n \dots \rangle$; i.e., u_0 has been dropped and replaced by v.)

Our operator @ behaves, relative to $\mathbb{A}$, like an $S5$ modal possibility operator, and its dual $\neg @ \neg$, like the corresponding necessity operator. It can be easily checked that

$\vec{u}, u \models \neg @ \neg A$ iff for every $\mathbb{a} \in \mathbb{A}$, $\vec{u}(\mathbb{a}), \mathbb{a} \models A$.

$\vec{u}, u \Dashv \neg @ \neg A$ iff for some $\mathbb{a} \in \mathbb{A}$, $\vec{u}(\mathbb{a}), \mathbb{a} \Dashv A$.

However, the intuitive meaning of @ is more akin to that of the "Actually" operator considered in Davies and Humberstone [4] and Hodes [6] (from which the symbol "@" has been borrowed).

The notion of constraint employed by Barwise and Perry [3] can be captured in $\mathbb{L}(@)$ using formulas of the form

$$\forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m (@A \to @B),$$

where $x_1 \dots x_n$ are all the variables free in A, and $y_1 \dots y_m$ are all the variables free in B but not free in A.

We define $\Sigma \models \Omega$, supporting and rejecting just as we did for $\mathbb{L}$.

We shall now consider a set of indexed operators of the form $@_e$, where $e \in E$ and E is the set of nonempty finite subsets of natural numbers. By E' we mean the subset of E which contains all and only the indices $\{n\}$, for every natural number n. Indices are introduced only for auxiliary purposes, in order to present a deductive system and a completeness proof. Our use of indices is an adaptation of the ideas contained in Fitting [5], Smullyan [10] and Marraud [7,8].

Substituting indexed operators for some occurrences of @ in wffs, we obtain indexed wffs.

Given a model M, an index reading in M is a function f from E to $\mathbb{A}$ such that:

For any $d, e \in E$, if $d \subseteq e$ then $f(d) \subseteq f(e)$.

When dealing with indexed wffs, we shall use the relations $\models_f$ and $=\!\!|_f$ (relative to an index reading f), which are defined by the same clauses as $\models$ and $=\!\!|$ (with the obvious modifications) plus two new ones:

$\vec{u}, u \models_f @_e A$ iff $\vec{u}(f(e)), f(e) \models_f A$.

$\vec{u}, u =\!\!|_f @_e A$ iff $\vec{u}(f(e)), f(e) =\!\!|_f A$.

Let Γ, Δ be two sets of indexed wffs. $\langle \Gamma, \Delta \rangle$ is supported by $\vec{u}$ in M iff for every index reading f,

if $\vec{u}, u_0 \models_f A$ for every $A \in \Gamma$, then $\vec{u}, u_0 \models_f B$, for some $B \in \Delta$.

Otherwise, $\langle \Gamma, \Delta \rangle$ is rejected by $\vec{u}$ in M. Observe that if Γ, Δ contain no indexed operator $@_e$, f is superfluous, and this definition is just the standard one for $\mathbb{L}(@)$. $\langle \Gamma, \Delta \rangle$ is valid in M iff it is supported by any $\vec{u}$ in M, and it is valid iff it is valid in any M. Thus, if $\langle \Gamma, \Delta \rangle$ is not valid, there are M, $\vec{u}$, and f such that for every $A \in \Gamma$, $\vec{u} \models_f A$, while for every $B \in \Delta$, $\vec{u} \not\models_f B$. Then we say that $M, \vec{u}, f$ refute $\langle \Gamma, \Delta \rangle$.

The sequent system $S_{\mathbb{L}}(@)$ $S_{\mathbb{L}}(@)$ is the result of adding to $S_{\mathbb{L}}$ the following axiom schemes and rules:

Axiom schemes

4. $@_e A,\ @_e \neg A \vdash$, for any atomic wff A.
5. $@_e t_1 = s_1 \ldots @_e t_n = s_n,\ @_e(\neg)Pt_1 \ldots t_n \vdash @_e(\neg)Ps_1 \ldots s_n$.

Rules:

22 $\dfrac{\Gamma, @_e A \vdash \Delta}{\Gamma, @A \vdash \Delta}$ (*) **23** $\dfrac{\Gamma \vdash @_e A, \Delta}{\Gamma \vdash @A, \Delta}$

(*) $e \cap d = \varnothing$, for any d in Γ, Δ, A.

24 $\dfrac{\Gamma, @_e \neg A \vdash \Delta}{\Gamma, \neg @A \vdash \Delta}$ **25** $\dfrac{\Gamma \vdash @_e \neg A, \Delta}{\Gamma \vdash \neg @A, \Delta}$ (*)

(*) $e \cap d = \varnothing$, for any d in Γ, Δ, A.

26a $\dfrac{\Gamma, @_e A \vdash \Delta}{\Gamma, @_e \neg\neg A \vdash \Delta}$ **26b** $\dfrac{\Gamma \vdash @_e A, \Delta}{\Gamma \vdash @_e \neg\neg A, \Delta}$

27 $\dfrac{\Gamma, @_e \neg A \vdash \Delta}{\Gamma, \neg @_e A \vdash \Delta}$ **28** $\dfrac{\Gamma \vdash @_e \neg A, \Delta}{\Gamma \vdash \neg @_e A, \Delta}$

29a $\dfrac{\Gamma, (\neg)(@_e A \circ @_e B) \vdash \Delta}{\Gamma, @_e(\neg)(A \circ B) \vdash \Delta}$ **29b** $\dfrac{\Gamma \vdash (\neg)(@_e A \circ @_e B), \Delta}{\Gamma \vdash @_e(\neg)(A \circ B), \Delta}$

where $\circ$ is any binary operator.

30 $\dfrac{\Gamma, @_e A(c) \vdash \Delta}{\Gamma, @_e \forall x A(x) \vdash \Delta}$ **31** $\dfrac{\Gamma \vdash @_e A(c), \Delta}{\Gamma \vdash @_e \forall x A(x), \Delta}$ (**)

32 $\dfrac{\Gamma, @_e \neg A(c) \vdash \Delta}{\Gamma, @_e \neg \forall x A(x) \vdash \Delta}$ (**) **33** $\dfrac{\Gamma \vdash @_e \neg A(c), \Delta}{\Gamma \vdash @_e \neg \forall x A(x), \Delta}$

(**) c is a constant not occurring in Γ, Δ, or $\forall_i A(x)$.

34a $\dfrac{\Gamma, (\neg) @_e \vdash \Delta}{\Gamma, @_d(\neg) @_e A \vdash \Delta}$ **34b** $\dfrac{\Gamma \vdash (\neg) @_e, \Delta}{\Gamma \vdash @_d(\neg) @_e A, \Delta}$

35a $\dfrac{\Gamma, (\neg) \otimes_i A \vdash \Delta}{\Gamma, @_e(\neg) \otimes_i A \vdash \Delta}$ (*) **35b** $\dfrac{\Gamma \vdash (\neg) \otimes_i A, \Delta}{\Gamma \vdash @_e(\neg) \otimes_i A, \Delta}$ (*)

(*) A is an atomic formula and $i \neq 0$.

36a $\dfrac{\Gamma, @_e(\neg) A \vdash \Delta}{\Gamma, @_e(\neg) \otimes_0 A \vdash \Delta}$ (**) **36b** $\dfrac{\Gamma \vdash @_e(\neg) A, \Delta}{\Gamma \vdash\partial\ @_e(\neg) \otimes_0 A, \Delta}$ (**)

(**) A is an atomic formula.

37 $\dfrac{\Gamma, @_{e \cup d} A \vdash \Delta}{\Gamma, @_e A \vdash \Delta}$ (*) **38** $\dfrac{\Gamma \vdash @_e A, \Delta}{\Gamma \vdash @_{e \cup d} A, \Delta}$ (*)

(*) A is an atomic formula or its negation.

39 $\dfrac{\Gamma, @_e t = t \vdash \Delta}{\Gamma, @_e(\neg) P \ldots t \ldots \vdash \Delta}$ **40** $\dfrac{\Gamma, @_e t = t \vdash \Delta}{\Gamma, @_e(\neg) t = t' \vdash \Delta}$

41 $\dfrac{\Gamma, op_1 t_2 = t_0 \vdash \Delta}{\Gamma, op_1 t_0 = t_1,\ op_2 t_1 = t_2 \vdash \Delta}$

where op_j is either $@_e$ or $\otimes_i$, for $j = 1,2$.

42 $\dfrac{\Gamma, @_e t_0 = t_1 \vdash \Delta}{\Gamma @_e t_0 = t_0,\ @_e t_1 = t_1 \vdash @_e \neg t_0 = t_1, \Delta}.$

Notice that $\otimes_i @_e A$ is equivalent to $@_e A$ (and $\otimes_i @A$ is equivalent to $@A$), so we can add to our set of rules II the following ones:

5′a $\dfrac{\Gamma, C(@_e A) \vdash \Delta}{\Gamma, C(\otimes_i @_e A) \vdash \Delta}$ **5′b** $\dfrac{\Gamma \vdash C(@_e A), \Delta}{\Gamma \vdash C(\otimes_i @_e A), \Delta}$

5′c $\dfrac{\Gamma, C(@A) \vdash \Delta}{\Sigma, C(\otimes_i @A) \vdash \Delta}$ **5′d** $\dfrac{\Gamma \vdash C(@A), \Delta}{\Gamma \vdash C(\otimes_i @A), \Delta}.$

According to these rules and to our former observation about the set of rules II, there is no loss of generality in considering only wffs in which each operator $\otimes_i$ has under its scope only atomic formulas. We shall do so in the sequel.

The following theorem can be easily proven:

Theorem 7 (Soundness theorem for $S_{\mathbb{IL}}(@)$) $\Gamma \vdash \Delta$ *only if* $\Gamma \vDash \Delta$.

Proof: Soundness can be easily checked, keeping in mind the definition of index reading and the provisos added to the $S_{\mathbb{IL}}(@)$ rules. We shall give the details for Rules 22 and 23 only.

For Rule 22: Suppose the lower sequent is not valid, i.e., there are $M, \vec{u}, f$ refuting it. Then we have $\vec{u}, u_0 \vDash_f @A$, so that there is some $\mathbb{a} \in A$ such that $u(\mathbb{a}), \mathbb{a} \vDash_f A$. Define a new index reading g satisfying $g(d) = f(d)$ for every d such that there is some index d' in Γ, Δ, A such that $d \cap d' \neq \varnothing$, and $g(e) = \mathbb{a}$, while for any other index g takes any suitable value. As the rule establishes that $e \cap d' = \varnothing$, for every d' in Γ, Δ, A, there is an index reading g satisfying the required conditions. Thus, $\vec{u}(\mathbb{a}), \mathbb{a} \vDash_g A$, so that $\vec{u}, u_0 \vDash_g @_e A$. As g takes the same values as f relative to indices in Γ and Δ, M, $\vec{u}$, and g refute the upper sequent.

For Rule 23: Suppose again that the lower sequent is refuted by $M, \vec{u}, f$. Then $\vec{u}, u_0 \nvDash_f @A$, and $\vec{u}(\mathbb{a}), \mathbb{a} \nvDash_f A$, for every $\mathbb{a} \in \mathbb{A}$, and, particularly, for $\mathbb{a} = f(e)$. Thus, $\vec{u}, u_0 \nvDash_f @_e A$.

Completeness of $S_{\mathbb{IL}}(@)$ Completeness of $S_{\mathbb{IL}}(@)$ is proved exactly in the same way that completeness of $S_{\mathbb{IL}}$ has been proved. We add to the instructions for the construction of a tree some new ones relative to the new forms that formulas can have. The new instructions are these (we omit instruction corresponding to Rules 26 to 36, as they are straightforward):

22. $\langle @A, 1\rangle$: Add the node $\langle @_e A, 1\rangle$, for the first $e \in E'$ not occurring in the branch. (We suppose a fixed enumeration of E.)
23. $\langle @A, 0\rangle$: Add all nodes $\langle @_e A, 0\rangle$ not appearing in the branch before such that $e \in E$ occurs in the branch, and then repeat $\langle @A, 0\rangle$.
24. $\langle \neg @A, 1\rangle$: For each $e \in E$ in the branch, add $\langle @_e \neg A, 1\rangle$ (if it does not appear in the branch) and then repeat $\langle \neg @A, 1\rangle$.
25. $\langle \neg @A, 0\rangle$: Add $\langle \neg @_e A, 0\rangle$, for the first $e \in E'$ not appearing in the branch.

37–39. $\langle @_e(\neg)Pc_1 \ldots c_n, 1\rangle$: Add the following nodes: (a) $\langle @_e c_i = c_i, 1\rangle$, for each i, $1 \leq i \leq n$, if it does not appear in the branch before. (b) $\langle @_{e\cup d}(\neg)Pc_1 \ldots c_n, 1\rangle$ for every $d \in E$ in the branch, if it does not appear in the branch before. (c) Repeat $\langle @_e(\neg)Pc_1 \ldots c_n, 1\rangle$.

38. $\langle @_e A, 0\rangle, A$ is $(\neg)Pc_1 \ldots c_n$ or $c = c'$: Add every node $\langle @_d, A, 0\rangle$ not previously in the branch such that $d \subseteq e$.

38–42. $\langle @_e \neg c = c', 0\rangle$: (a) Add every node $\langle @_d \neg c = c', 0\rangle$ not previously in the branch such that $d \subseteq e$, and (b) if both $\langle @_e c = c, 1\rangle$ and $\langle @_e c' = c', 1\rangle$ appear in the branch, add $\langle @_e c = c', 1\rangle$, and otherwise repeat $\langle @_e \neg c = c', 0\rangle$.

40. $\langle @_e \neg c = c', 1\rangle$: Add $\langle @_e c = c, 1\rangle$ and $\langle @_e c' = c', 1\rangle$.

40–41–37. $\langle @_e c_0 = c_1, 1\rangle$: (a) Add $\langle @_e c_0 = c_0, 1\rangle$ and $\langle @_e c_1 = c_1, 1\rangle$. (b) For each node $\langle opc_1 = c_2, 1\rangle$ in the branch (where op is $@_d$ or $\otimes_i$) add $\langle @_e c_2 = c_0, 1\rangle$, if it does not appear in the branch. (c)

Add every pair $\langle @_{e\cup d} c_0 = c_1, 1\rangle$ not previously in the branch such that $d \in E$ appears in the branch. (d) Repeat $@_e c_0 = c_1, 1\rangle$.

19–41. $\langle \otimes_i c_0 = c_1, 1\rangle$: (a) Add $\langle \otimes_i c_0 = c_0, 1\rangle$ and $\langle \otimes_i c_1 = c_1, 1\rangle$, if they do not appear previously in the branch. (b) For each node $\langle op c_1 = c_2, 1\rangle$ (op is either $@_e$ or $\otimes_i$) in the branch, add $\langle \otimes_i c_2 = c_0, 1\rangle$ (if it does not appear in the branch before). (c) Repeat $\langle \otimes_i c_0 = c_1, 1\rangle$. (This clause must be substituted for the former 19,20.)

A branch closes iff it contains a set of nodes of one of the Forms 1–3 considered for $S_{\mathbb{L}}$ or of one of the following forms:

4. $\langle @_e A, 1\rangle$, $\langle @_e \neg A, 1\rangle$.
5. $\langle @_e t_1 = s_1, 1\rangle \ldots \langle @_e t_n = s_n, 1\rangle$, $\langle @_e(\neg)Pt_1 \ldots t_n, 1\rangle$, $\langle @_e(\neg)Ps_1 \ldots s_n, 1\rangle$.

Lemma 8 *If a tree constructed for* $\langle \Sigma, \Omega\rangle$ *closes, then there are finite sets* $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ *such that* $\Gamma \vdash \Delta$.

Proof: The structure of the proof is exactly the same as that of Lemma 2. It is enough to consider the cases corresponding to the new rules and new instructions for tableaux construction. In the most complex cases (those concerning Rules 37 to 42) indexed operators $@_e$ are handled in almost the same way as operators $\otimes_i$ in the proof of Lemma 2 (notice that Rule 41 does not distinguish between these operators), and the proofs are similar to those concerning Rules 19–20 and 21 in the proof of Lemma 2. It is not difficult (but boring) to reconstruct the whole proof from the tableaux constructing instructions above, keeping in mind that the numbers preceding each instruction indicate the rules needed to prove the corresponding sequent, as explained in the proof of Lemma 2. The conditions imposed on indices ensure that the provisos of the rules are satisfied in every case.

Suppose we have constructed a tree for $\langle \Sigma, \Omega\rangle$ which contains an open branch. From that open branch we can construct the following model $M = \langle W, U, D, [\![\ ]\!], S, \mathbb{A}\rangle$:

$$\mathbb{A} = \{ a_e : @_e \text{ occurs in the branch}\}$$
$$W = \{ u_i : \otimes_i \text{ occurs in the branch}\} \cup \mathbb{A}.$$

Define *Ctes* and $\approx$ as before, substituting op_j for $\otimes_i$ in the definition of $\approx$ (op_j stands for $\otimes_i$ or $@_e$). As before, $\approx$ is an equivalence relation. Define $|c|$, U, $D(u_i)$, $[\![c]\!]$, and $S(u_i)$ just as in the former case, but

$$D(a_e) = \{|c| : \langle @_e c = c, 1\rangle \text{ appears in the branch}\}.$$

and

$$S(a_e) = \{(\neg)P|c_1| \ldots |c_n| : \langle (\neg)@_e Pc_1 \ldots c_n, 1\rangle \text{ appears in the branch}\}.$$

We can check that the conditions imposed on $\mathbb{A}$ in our definition of model are satisfied by M:

1. Obviously, $\mathbb{A} \subseteq W$.
2. It is not the case that $A, \neg A \in S(\mathbb{a}_e)$, for any A. If this were the case, both $\langle @_e A, 1\rangle$ and $\langle @_e \neg A, 1\rangle$ should appear in the branch, which would close.
3. If $\mathbb{a}_e, \mathbb{a}_d \in \mathbb{A}$, there is some $\mathbb{a}$ in $\mathbb{A}$ such that $\mathbb{a}_e \sqsubseteq \mathbb{a}$, $\mathbb{a}_d \sqsubseteq \mathbb{a}$. This is the case for $\mathbb{a} = \mathbb{a}_{d \cup e}$:
   a. $D(\mathbb{a}_e) \subseteq D(\mathbb{a}_{d\cup e})$: Let $r \in D(\mathbb{a}_e)$. Then $r = |c|$, and $\langle @_e c = c, 1\rangle$ appears in the branch. Then, by Instruction 40–41–37, so does $\langle @_{d\cup e} c = c, 1\rangle$ (as d must occur in the branch), and so $|c| = r \in D(\mathbb{a}_{d\cup e})$. Similarly, $D(\mathbb{a}_d) \subseteq D(\mathbb{a}_{d\cup e})$.
   b. $S(\mathbb{a}_e) \subseteq S(\mathbb{a}_{d\cup e})$: Suppose that $Pr_1 \dots r_n \in S(\mathbb{a}_e)$. Then $\langle @_e Pc_1 \dots c_n, 1\rangle$ belongs to the branch, for some $c_1 \dots c_n$ such that $r_i = |c_i|$ $(1 \leq i \leq n)$. Then, by 37–39, $\langle @_{d\cup e} Pc_1 \dots c_n, 1\rangle$ appears in the branch, and so $Pr_1 \dots r_n \in S(\mathbb{a}_{d\cup e})$. The same can be proved for $\neg Pr_1 \dots r_n$.

Define an *index reading* f as follows: for each index e in the branch, $f(e) = \mathbb{a}_e$, and for any other e, $f(e)$ takes any adequate value in $\mathbb{A}$. It can be easily checked that this definition is sound.

Finally, define the ω-tuple $\vec{u}$ exactly as we did in the completeness proof for $S_{\mathbb{L}}$. We can prove the following lemma:

Lemma 9 *Suppose a tree constructed for $\langle \Sigma, \Omega\rangle$ which contains an open branch, and let $M, \vec{u}, f$ be as above. For any $\langle A, k\rangle$ in the branch,*

if $k = 1$, *then* $\vec{u}, u_0 \models A$

if $k = 0$, *then* $\vec{u}, u_0 \not\models A$.

Proof: Induction on the logical complexity of A.

Corollary 10 *If a tree for $\langle \Sigma, \Omega\rangle$ contains an open branch, then $\Sigma \not\models \Omega$.*

Proof: $M, \vec{u}, f$ as defined above support Σ and reject Ω, according to Lemma 9.

Obviously, if Σ, Ω contain no index, the index reading f is superfluous. We can now establish our completeness theorem:

Theorem 11 (Completeness theorem for $S_{\mathbb{L}}(@)$) *Let Σ, Ω be sets of wffs of $\mathbb{L}(@)$. $\Sigma \models \Omega$ iff there are finite sets $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$.*

Proof: Let $\Sigma \models \Omega$. A tree for $\langle \Sigma, \Omega\rangle$ must close, according to Corollary 10. Then, by Lemma 8 there are finite sets $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ such that $\Gamma \vdash \Delta$.

We can prove a compactness theorem as well:

Theorem 12 (Compactness theorem) *Let Σ, Ω be two sets of wffs of $\mathbb{L}(@)$. $\Sigma \models \Omega$ iff there are finite sets $\Gamma \subseteq \Sigma$, $\Delta \subseteq \Omega$ such that $\Gamma \models \Delta$.*

Proof: Similar to that of Theorem 6.

Some possible modifications We could change the semantic clauses for @ in such a way that, for any M, $\vec{u}$, and v, either $\vec{u}, v \models @A$ or $\vec{u}, v \mathrel{=\!\!|} @A$. For this purpose, we should stipulate that

$$\vec{u}, u \mathrel{=\!\!|} @A \text{ iff } \vec{u}(\mathbb{a}), \mathbb{a} \not\models A \text{ for every } \mathbb{a} \in \mathbb{A}.$$

The system $S_{\mathbb{L}}(@)$ should be modified substituting the Rules 24′ and 25′ below for the original Rules 24 and 25:

24′ $\dfrac{\Gamma \vdash @A, \Delta}{\Gamma, \neg @A \vdash \Delta}$ **25′** $\dfrac{\Gamma, @A \vdash \Delta}{\Gamma \vdash \neg @A, \Delta}$

(with the same proviso).

The resulting system would be sound and complete, as can be checked introducing some obvious changes in the proofs.

A more interesting change would be to stipulate that $\mathbb{A}$ should contain a maximal element, that is, some $\mathbb{a}^*$ such that for every $\mathbb{a} \in \mathbb{A}$, $\mathbb{a} \sqsubseteq \mathbb{a}^*$. In this case, the index set E would be $E = \{n : n \text{ is a natural number}\} \cup \{\omega\}$. Index readings should satisfy the following conditions:

$$f(e) \in \mathbb{A}, \text{ for every } e \in E, \text{ and}$$

$$f(\omega) = \mathbb{a}^*.$$

The proviso added to Rules 22 and 25 should be:

$$e \neq \omega, \ e \text{ does not occur in } \Gamma, \Delta, A,$$

and 37′ and 38′ would substitute for the old Rules 37 and 38:

37′ $\dfrac{\Gamma, @_\omega A \vdash \Delta}{\Gamma, @_e A \vdash \Delta}$ **38′** $\dfrac{\Gamma \vdash @_e A, \Delta}{\Gamma \vdash @_\omega A, \Delta}$

with the same conditions on A.

Instructions 37 and 38 for the construction of the tree should be modified in order to ensure that whenever a node $\langle @_e A, 1\rangle$ is in a branch, so is $\langle @_\omega A, 1\rangle$, and whenever $\langle @_\omega A, 0\rangle$ is in a branch, so is $\langle @_e A, 0\rangle$ for each e occurring in it, for A as stipulated in 37 and 38. When constructing the model M we should define an element $\mathbb{a}^* \in \mathbb{A}$ corresponding to the index ω. $\mathbb{a}^*$ is the maximal element of $\mathbb{A}$: whenever $\langle @_e c = c, 1\rangle$ appears in the branch, so does $\langle @_\omega c = c, 1\rangle$, and so if $|c| \in D(\mathbb{a}_e)$ then $|c| \in D(\mathbb{a}^*)$, for any $\mathbb{a}_e \in \mathbb{A}$. Moreover, if $\langle @_e A, 1\rangle$ belongs to the branch (where A is $(\neg)Pc_1 \ldots c_n$), so does $\langle @_\omega A, 1\rangle$ and so $S(\mathbb{a}_e) \subseteq S(\mathbb{a}^*)$. Thus, $\mathbb{a}_e \sqsubseteq \mathbb{a}^*$, for every $\mathbb{a}_e \in \mathbb{A}$, and $\mathbb{a}^*$ is the required element.

This maximal element of $\mathbb{A}$ is not a world in the sense in which Barwise and Perry use this word because we have not required that for any A of the form $Pr_1 \ldots r_n$ either $A \in \mathbb{a}^*$ or $\neg A \in \mathbb{a}^*$. We could modify our semantics, rules, and proofs in order to meet such a requirement. However, this task is left to the reader.

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