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On Cofinal Extensions of Models of Fragments of Arithmetic

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Abstract We present a model-theoretic proof of Motohashi's preservation theorem for cofinal extensions, and examine various criteria for a model of a fragment of PA to have a proper elementary cofinal extension. Using these criteria we answer a question of Roman Kossak's, exhibiting for each $n \geq 0$ countable models M and N of $I\Sigma_n + \exp + \neg B\Sigma_{n+1}$ such that: (i) M has no proper elementary cofinal extensions and (ii) N does have proper elementary cofinal extensions.

Introduction Let $\mathcal{L}_A$ be the usual first-order language of arithmetic with nonlogical symbols $0, 1, +, \cdot, <$, and let PA^- be the $\mathcal{L}_A$-theory of the nonnegative parts of discretely ordered rings. The theories $I\Delta_0 + \exp$, $I\Sigma_n$, and $B\Sigma_n$ ($n \in \mathbb{N}$) are the usual fragments of Peano Arithmetic (PA). More specifically, $I\Sigma_n$ is axiomatized by PA^- together with the scheme of *Σ_n-induction*,

$$\forall \bar{a}(\theta(0,\bar{a}) \wedge \forall x(\theta(x,\bar{a}) \rightarrow \theta(x+1,\bar{a})) \rightarrow \forall x\theta(x,\bar{a})),$$

for all Σ_n formulas $\theta(x,\bar{a})$ (see Paris & Kirby [8]). The theory $I\Delta_0 + \exp$ is $I\Delta_0$ ($=I\Sigma_0$) together with a single axiom exp stating that the exponential function x^y is total (see Gaifman & Dimitracopoulos [2] for details in how this can be expressed in $\mathcal{L}_A$). The theory $B\Sigma_n$ is $I\Delta_0$ together with the scheme of *Σ_n-collection*,

$$\forall \bar{a}, t(\forall x < t \exists y\theta(x,y,\bar{a}) \rightarrow \exists z \forall x < t \exists y < z\theta(x,y,\bar{a})),$$

for all Σ_n formulas $\theta(x,y,\bar{a})$ (see [8]). With a certain convenient abuse of notation, we will write '$M \models I\Sigma_n + \neg B\Sigma_{n+1}$' to mean '$M \models I\Sigma_n$ and $M \not\models B\Sigma_{n+1}$', similarly for $B\Sigma_n + \neg I\Sigma_n$. Parsons [9] showed that $I\Sigma_{n+1} \vdash B\Sigma_{n+1} \vdash I\Sigma_n$ for all $n \geq 0$, and that models of $I\Sigma_n + \neg B\Sigma_{n+1}$ exist for all $n \geq 0$; Paris and Kirby [8] and (independently) Lessan [5] showed that models of $B\Sigma_n + \neg I\Sigma_n$ exist for all $n \geq 1$.

If M and N are models of PA^- and $M \subseteq N$ we say M is *cofinal in N*, $M \subseteq_{\mathrm{cf}} N$, iff $\forall a \in N \exists b \in M$ ($N \models b > a$); N is an *end-extension* of M, M is an initial

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segment of N, $M \subseteq_e N$, iff $\forall a \in N \forall b \in M$ ($N \models a < b \Rightarrow a \in M$). One of the most basic results on cofinal extensions is the theorem of Gaifman [1] which says that if M and N are models of PA and $M \subseteq_{cf} N$ then $M \prec N$. Versions of this result are also known for M and N satisfying certain fragments of PA (see Fact 3 below). However, despite the great amount of information Gaifman's theorem gives us, many interesting questions concerning cofinal extensions are still unsolved.

Roman Kossak has asked whether all countable models of $I\Sigma_n + \neg B\Sigma_{n+1}$ have proper elementary cofinal extensions, or indeed if any such models exist. (It is known that, for each $n \geq 1$, every countable model M of $B\Sigma_n + \exp + \neg I\Sigma_n$ has a proper elementary extension $K \succ_{cf} M$ (see Kaye [4]). In this paper, we answer Kossak's question by exhibiting a large class of models of $I\Sigma_n + \neg B\Sigma_{n+1}$ that do have proper elementary cofinal extensions, and also by providing for each $n \in \mathbb{N}$ an example of a model $M \models I\Sigma_n + \neg B\Sigma_{n+1}$ without such an extension. The problem of finding a nice alternative characterization of the countable models M of PA^- having a proper elementary cofinal extension is still unsolved. (The word 'nice' is important here: exact characterizations—albeit rather unwieldy ones that give little extra information—*can* be obtained using infinitary sentences.)

We shall also give a simple model-theoretic proof of the preservation theorem for cofinal extensions in Motohashi [6].

The original motivation for this work was in trying to develop machinery to solve certain questions left over from Kaye [4]. In particular, it was noted there that if $M \prec K \models I\Sigma_n$ is *not* cofinal then the unique initial segment I of K which is a cofinal extension of M satisfies $B\Sigma_{n+1}$, and so if $M \prec I$ then both M and K also satisfy $B\Sigma_{n+1}$. This suggests the following question:

Question Is there a countable model $M \models B\Sigma_{n+1}$ such that $M \not\models I\Sigma_{n+1}$ and whenever $K \succ M$ there is an intermediate model $I \subseteq K$ such that $M \prec_{cf} I \subseteq_e K$?

This question (and also several variations of it) is still open, but the construction of such M (if any exist) would seem to require detailed knowledge of formulas preserved in cofinal extensions and also of the properties of models with many elementary cofinal extensions.

The preservation theorem We now give our proof of Motohashi's preservation theorem for models of PA^-. (For convenience we shall state it for languages $\mathcal{L}$ extending $\mathcal{L}_A$ and models of PA^-, although a similar result would hold for any language containing $<$ over a base theory that implies $<$ is a linear order with no greatest element (see Motohashi [6]).

Definition Let $\mathcal{L} \supseteq \mathcal{L}_A$. The class $\exists_{cf}$ of $\mathcal{L}$-formulas is the least class containing all quantifier-free $\mathcal{L}$-formulas and satisfying:

i. if $\phi(\bar{x}), \psi(\bar{x}) \in \exists_{cf}$ then $\phi(\bar{x}) \vee \psi(\bar{x}), \phi(\bar{x}) \wedge \psi(\bar{x}) \in \exists_{cf}$
ii. if $\phi(\bar{x}, y) \in \exists_{cf}$ then $\exists y \phi(\bar{x}, y) \in \exists_{cf}$
iii. if $\phi(\bar{x}, y) \in \exists_{cf}$ then $Qy\phi(\bar{x}, y) \in \exists_{cf}$, where $Qy\phi(\bar{x}, y)$ is $\forall z \exists y(z < y \wedge \phi(\bar{x}, y))$ for some suitable variable z not occurring elsewhere.

Dually, $\forall_{cf}$ is the least class of $\mathcal{L}$-formulas containing all quantifier-free $\mathcal{L}$-formulas and closed under $\wedge$, $\vee$, $\forall$, and the quantifier Q^*, where $Q^*y\phi(\bar{x}, y)$ is

$\exists z \forall y(y > z \rightarrow \phi(\bar{x}, y))$ for some suitable new variable z. Notice that $\forall_{cf}$ formulas are equivalent to negations of $\exists_{cf}$ formulas, and vice versa.

It is easy to check that $\exists_{cf}$ formulas are preserved upwards in cofinal extensions and, dually, $\forall_{cf}$ formulas are preserved downwards. In fact we have:

Theorem 1 *Suppose T is an $\mathcal{L}'$-theory, where $\mathcal{L}'$ is a countable language, $\mathcal{L}' \supseteq \mathcal{L} \supseteq \mathcal{L}_A$, and $T \vdash \mathrm{PA}^-$, and suppose M is a suitably saturated countable $\mathcal{L}$-structure (see below for a definition of 'suitably saturated') such that, for all $\mathcal{L}$-sentences $\sigma \in \forall_{cf}$,*

$$T \vdash \sigma \Rightarrow M \vDash \sigma$$

then M has a cofinal extension $N \vDash T$.

Definition If $M \vDash \mathrm{PA}^-$ is an $\mathcal{L}$-structure, where $\mathcal{L} \supseteq \mathcal{L}_A$ is a recursive first-order language and Γ is a recursive class of $\mathcal{L}$-formulas, we say M is Γ-*tall* iff for any recursive sequence of formulas $(\phi_n(x, \bar{y}))_{n \in \mathbb{N}}$ from Γ and any $\bar{a} \in M$, if

$$M \vDash Q^* \phi_n(x, \bar{a})$$

and

$$M \vDash \forall x(\phi_{n+1}(x, \bar{a}) \rightarrow \phi_n(x, \bar{a}))$$

for all $n \in \mathbb{N}$ then $\{\phi_n(x, \bar{a}) \mid n \in \mathbb{N}\}$ is realized in M.

Notice by the observation (known as "Craig's trick") that every r.e. set of formulas is equivalent (in the predicate calculus) to a recursive set of formulas, if M is Γ-recursively saturated (i.e., any finitely satisfiable recursive set of formulas $p(x, \bar{a}) \subseteq \Gamma$ with finitely many parameters $\bar{a} \in M$ is realized in M) and Γ is closed under $\wedge$, then M is Γ-tall. Thus, countable Γ-tall $\mathcal{L}$-structures $M \vDash S$ exist for any consistent theory S. In the theorem, 'suitably saturated' should be taken to mean '$\forall_{cf}$-tall' where, in the definition above, 'recursive' is replaced by 'recursive in oracles for suitable Gödel-numberings of $\mathcal{L}'$, Gödel-numberings of $\mathcal{L}$-formulas (as a subset of those for $\mathcal{L}'$), and an axiomatization of T'. Thus, for most applications where $\mathcal{L}$, $\mathcal{L}'$, and T are all recursive, we may take 'suitably saturated' to mean '$\forall_{cf}$-tall'.

Proof of Theorem 1: Let $\mathcal{L}(M)$, $\mathcal{L}'(M)$ denote the languages $\mathcal{L}$, $\mathcal{L}'$ respectively with constants added for each $a \in M$. We must find an $\mathcal{L}'(M)$-structure $N \vDash T$ satisfying $\phi(\bar{a})$ for all q.f. $\phi(\bar{a})$ in $\mathcal{L}(M)$ that is true in M, and omitting the type

$$p(x) = \{x > a \mid a \in M\}.$$

Thus, by the omitting types theorem, it is sufficient to find a complete consistent $\mathcal{L}'(M)$ theory, T^*, containing the above sentences, such that for all $\mathcal{L}'(M)$-formulas $\psi(x)$

$$T^* \vdash \exists x \psi(x) \Rightarrow T^* \vdash \exists x < a \psi(x)$$

for some $a \in M$. We build such a T^* as the union of finite extensions $T + \lambda(\bar{a})$ of T. We say $\lambda(\bar{a})$ is *extendible* iff $\lambda(\bar{a}) \in \mathcal{L}'(M)$ and

$$T + \lambda(\bar{a}) \vdash \sigma(\bar{a}) \Rightarrow M \vDash \sigma(\bar{a})$$

for all $\sigma(\bar{a}) \in \forall_{cf}$ in the language $\mathcal{L}(M)$. We shall prove that this notion of extendibility has the following two properties:

Property 1 *If* $\lambda(\bar{a})$ *is extendible and* $b \in M$ *then* $\lambda(\bar{a}) \wedge b = b$ *is extendible.*

Property 2 *If* $\lambda(\bar{a})$ *is extendible and* $\phi(x, \bar{y})$ *is any* $\mathcal{L}'$*-formula with the variables shown, then either* $\lambda(\bar{a}) \wedge \forall x \neg\phi(x, \bar{a})$ *is extendible or* $\lambda(\bar{a}) \wedge \exists x < b\phi(x, \bar{a})$ *is extendible for some* $b \in M$.

Proof of Property 1: If $b \notin \{\bar{a}\}$ and

$$T + \lambda(\bar{a}) \wedge b = b \vdash \sigma(\bar{a}, b) \in \forall_{\mathrm{cf}} \cap \mathcal{L}(M),$$

then

$$T + \lambda(\bar{a}) \vdash \forall y\sigma(\bar{a}, y) \in \forall_{\mathrm{cf}} \cap \mathcal{L}(M),$$

since $\forall_{\mathrm{cf}}$ is closed under $\forall$. Hence $M \vDash \forall y\sigma(\bar{a}, y)$, and so $M \vDash \sigma(\bar{a}, b)$.

Proof of Property 2: Suppose

$$T + \lambda(\bar{a}) + \forall x \neg\phi(x, \bar{a}) \vdash \sigma(\bar{a}) \in \forall_{\mathrm{cf}} \cap \mathcal{L}(M)$$

and, for all $b \in M \setminus \{\bar{a}\}$,

$$T + \lambda(\bar{a}) + \exists x < b\phi(x, \bar{a}) \vdash \tau_b(b, \bar{a}) \in \forall_{\mathrm{cf}} \cap \mathcal{L}(M),$$

but $M \nvDash \sigma(\bar{a})$ and $M \nvDash \tau_b(b, \bar{a})$ for all $b \in M \setminus \{\bar{a}\}$. We define a sequence of $\mathcal{L}$-formulas as follows:

$$\rho_0(x, \bar{y}) =_{\mathrm{def}} x = x$$

$$\rho_{i+1}(x, \bar{y}) =_{\mathrm{def}} \rho_i(x, \bar{y}) \wedge \theta(x, \bar{y}),$$

if i is the Gödel-number of a proof from the axioms of T of

$$\forall \bar{y}(\lambda(\bar{y}) \rightarrow \forall x(\exists z < x\phi(z, y) \rightarrow \theta(x, \bar{y}))$$

where $\theta(x, \bar{y}) \in \forall_{\mathrm{cf}} \cap \mathcal{L}$; $\rho_{i+1}(x, \bar{y}) =_{\mathrm{def}} \rho_i(x, \bar{y})$ otherwise.

It is clear that each ρ_i is $\forall_{\mathrm{cf}}$ and that the map $i \mapsto \ulcorner\rho_i(x, \bar{y})\urcorner$ is recursive in suitable oracles for T, $\mathcal{L}'$, and $\mathcal{L}$. Also

$$T + \lambda(\bar{a}) \vdash Q^*x(\rho_i(x, \bar{a}) \vee \sigma(\bar{a})) \text{ for each } i,$$

since

$$T + \lambda(\bar{a}) \vdash Q^*x(\forall w \neg\phi(w, \bar{a}) \vee \exists z < x\phi(z, \bar{a}));$$

hence

$$T + \lambda(\bar{a}) \vdash Q^*x(\sigma(\bar{a}) \vee \exists z < x\phi(z, \bar{a})).$$

Thus, as $\forall_{\mathrm{cf}}$ is closed under Q^* and $\vee$, and as $\lambda(\bar{a})$ is extendible,

$$M \vDash Q^*x(\rho_i(x, \bar{a}) \vee \sigma(\bar{a})) \text{ for each } i.$$

But $M \nvDash \sigma(\bar{a})$, hence $M \vDash Q^*x\rho_i(x, \bar{a})$ for each i. Therefore, by the appropriate notion of M being $\forall_{\mathrm{cf}}$-tall, there is some $b \in M$ such that $M \vDash \bigwedge_{i \in \mathbb{N}} \rho_i(b, \bar{a})$, and without loss of generality we may assume $b \notin \{\bar{a}\}$. But then $\tau_b(x, \bar{y})$ is a conjunct of some $\rho_i(x, \bar{y})$, hence $M \vDash \tau_b(b, \bar{a})$, a contradiction. Hence Property 2 holds.

Given Properties 1 and 2 we can construct our complete extension T^* as follows: enumerate all $\mathcal{L}'(M)$-formulas in one free-variable x as

$$\phi_0(x), \phi_1(x), \ldots, \phi_1(x), \ldots \quad (i \in \mathbb{N})$$

and define λ_0 to be $\exists y(y = y)$ (or any other trivially true $\mathcal{L}$-sentence). Then clearly λ_0 is extendible, by our hypothesis on T. For each $i \in \mathbb{N}$ define λ_{i+1} to be *either* $\lambda_i \wedge \forall x \neg\phi_i(x)$ *or* $\lambda_i \wedge \exists x < b\phi_i(x)$ for some $b \in M$, so that λ_{i+1} is extendible. (Properties 1 and 2 show that we can always find such a λ_{i+1}.) Then $T^* = T + \{\lambda_i \mid i \in \mathbb{N}\}$ is clearly complete, and does not disprove any $\mathcal{L}(M)$-sentence in the quantifier-free diagram of M (since $\forall_{\mathrm{cf}}$ contains all quantifier-free $\mathcal{L}$-formulas and by the definition of extendibility), and has the required form, so there is a model $N \models T^*$ omitting the type $p(x)$, by the omitting-types theorem.

Corollary 2 *Let $\mathcal{L} \supseteq \mathcal{L}_A$ be a complete first-order language, and T an $\mathcal{L}$-theory extending* PA^-. *Then a formula $\phi(\bar{x})$ of $\mathcal{L}$ is provably equivalent in T to a $\exists_{\mathrm{cf}}$ formula iff* for *all $M \subseteq_{\mathrm{cf}} N$ both satisfying T and all $\bar{a} \in M$, $M \models \phi(\bar{a}) \Rightarrow N \models \phi(\bar{a})$.*

Proof: One direction is trivial. For the other, if $\phi(\bar{x})$ is not equivalent in T to any $\exists_{\mathrm{cf}}$ formula then the following theory in the language $\mathcal{L} \cup \{\bar{c}\}$ is consistent:

$$T + \{\psi(\bar{c}) \in \forall_{\mathrm{cf}} \mid T + \neg\phi(\bar{c}) \vdash \psi(\bar{c})\} + \phi(\bar{c}).$$

Let $(M, \bar{c})$ be a countable, suitably saturated model of this theory. By the theorem $(M, \bar{c}) \models \phi(\bar{c})$ and M has a cofinal extension $N \models T + \neg\phi(\bar{c})$.

It follows from this corollary and Gaifman's result that every formula is equivalent in PA to both a $\exists_{\mathrm{cf}}$ formula and a $\forall_{\mathrm{cf}}$ formula. (It would be interesting to know if there are any other consistent theories extending $I\Delta_0 + \exp$ with this property.) In fact we have

Fact 3 *If $M \subseteq_{\mathrm{cf}} N$ are both models of $B\Sigma_n$* + exp, *where $n \geq 1$, then $M \prec_{\Sigma_{n+1}} N$.*

Proof: $M \prec_{\Sigma_1} N$, by the proof of the MRDP theorem in $I\Delta_0 + \exp$ (see [2]). $M \prec_{\Sigma_{n+1}} N$ is proved by induction on n: if $N \models \exists x \forall y \psi(x, y, \bar{a})$ with $\psi \in \Sigma_{n-1}$ and $\bar{a} \in M$, then $N \models \exists x < b\, \forall y \psi(x, y, \bar{a})$ for some $b \in M$. Hence $N \models \forall z \exists x < b \forall y < z \psi(x, y, \bar{a})$. By $B\Sigma_{n-1}$ in N this is equivalent to a Π_n formula, so by $M \prec_{\Sigma_n} N$, $M \models \forall z \exists x < b \forall y < z \psi(x, y, \bar{a})$, hence by $B\Sigma_n$ in M, $M \models \exists x \forall y \psi(x, y, \bar{a})$.

Corollary 4 *Every Σ_{n+1} formula is equivalent in $B\Sigma_n$* + exp *to both a $\exists_{\mathrm{cf}}$ and a $\forall_{\mathrm{cf}}$ formula.*

It is a natural question to ask about the strength of the theories $I\forall_{\mathrm{cf}}$ and $I\exists_{\mathrm{cf}}$ (= PA^- + induction on $\forall_{\mathrm{cf}}$ or $\exists_{\mathrm{cf}}$ formulas, respectively.) This has a straightforward and slightly disappointing answer.

Theorem 5 $I\forall_{\mathrm{cf}} = I\exists_{\mathrm{cf}} = \mathrm{PA}$.

Proof: $I\forall_{\mathrm{cf}} = I\exists_{\mathrm{cf}}$, by an argument similar to that in Paris and Kirby [8] showing $I\Sigma_n = I\Pi_n$. From Kaye [3] we have $I\exists_{\mathrm{cf}} \vdash I\exists_1 \vdash I\Sigma_1$, hence $I\exists_{\mathrm{cf}} \vdash B\Sigma_1$. By Corollary 4, every Σ_2 formula is equivalent in $B\Sigma_1$ to a $\exists_{\mathrm{cf}}$ formula, so $I\exists_{\mathrm{cf}} \vdash I\Sigma_2 \vdash B\Sigma_2$, and so on.

By contrast, $I\forall_{cf}^{-} \neq$ PA, where $I\forall_{cf}^{-}$ is *parameter-free* $\forall_{cf}$-induction (since any cofinal substructure of a model of true arithmetic satisfies $I\forall_{cf}^{-}$). It is not clear how strong the theory of parameter-free $\exists_{cf}$-induction is, however.

Elementary cofinal extensions We now turn to *elementary* cofinal extensions and Roman Kossak's question.

Theorem 6 *Let $M \models$ PA$^-$ be countable and Σ_n-tall for all $n \in \mathbb{N}$. Then M has a proper cofinal elementary extension.*

Proof: Let $\mathcal{L}_A(M) = \mathcal{L}_A \cup M$ and let ∞ be a new constant symbol. We say that a sentence $\phi(\infty, \bar{a})$ of $\mathcal{L}(M) = \mathcal{L}_A(M) \cup \{\infty\}$ is *extendible* iff there are infinitely many b such that $M \models \phi(b, \bar{a})$. Clearly '$\phi(\infty, \bar{a})$ is extendible' implies '$\phi(\infty, \bar{a}) \wedge b = b$ is extendible' for each $b \in M$.

Claim *If $\phi(\infty, \bar{a})$ is extendible and*

$$M \models \forall v(\phi(v, \bar{a}) \to \exists \bar{w}\theta(\bar{w}, v, \bar{a}))$$

for some $\mathcal{L}_A(M)$-formula $\theta(\bar{w}, v, \bar{a})$ then there is $b \in M$ such that $\phi(\infty, \bar{a}) \wedge \exists \bar{w} < b\theta(\bar{w}, \infty, \bar{a})$ is extendible.

Proof of Claim: Let $\psi_n(x, \bar{a})$ be the formula

$$\exists v_1, v_2, \ldots, v_n\left(\bigwedge_{i \neq j} v_i \neq v_j \wedge \bigwedge_{i} \phi(v_i, \bar{a}) \wedge \exists \bar{w} < x\theta(\bar{w}, v_i, \bar{a})\right).$$

Then, since there are infinitely many b such that $M \models \phi(v, \bar{a})$ and $M \models \forall v(\phi(v, \bar{a}) \to \exists \bar{w}\theta(\bar{w}, v, \bar{a}))$, $M \models Q^*x\psi_n(x, a)$ for each $n \in \mathbb{N}$. Thus (by tallness) there is some b such that $M \models \bigwedge_{n \in \mathbb{N}} \psi_n(b, \bar{a})$, i.e., $\phi(\infty, a) \wedge \exists \bar{w} < b\theta(\bar{w}, \infty, \bar{a})$ is extendible.

Now enumerate all $\mathcal{L}_A(M)$-formulas in the two free-variables x, y by

$$\phi_0(x, y), \phi_1(x, y), \ldots, \phi_i(x, y), \ldots \quad (i \in \mathbb{N}).$$

Let $\lambda_0(\infty)$ be '$\infty = \infty$'. (This is trivially extendible.) Assume $\lambda_i(\infty)$ has been defined, and is extendible; let $\lambda_{i+1}(\infty)$ be $\lambda_i(\infty) \wedge \forall x\, \neg\phi_i(x, \infty)$, unless this is not extendible, in which case there are only finitely many $v \in M$ such that

$$M \models \lambda_i(v) \wedge \forall x \neg\phi_i(x, v).$$

Let this finite set of v's be $A = \{a_1, \ldots, a_k\}$. Then clearly $\lambda_i(\infty) \wedge \infty \neq a_1 \wedge \cdots \wedge \infty \neq a_k$ is extendible, since $\lambda_i(\infty)$ is, and

$$M \models \forall v(\lambda_i(v) \wedge v \neq a_1 \wedge \cdots \wedge v \neq a_k \to \exists x\phi_i(x, v)).$$

Thus by the claim there is $b \in M$ such that

$$\lambda_i(\infty) \wedge \infty \neq a_1 \wedge \cdots \wedge \infty \neq a_k \wedge \exists x < b\phi_i(x, \infty)$$

is extendible, and we let $\lambda_{i+1}(\infty)$ be this sentence.

When all the λ_is have been constructed, $T = \text{Th}(M, a)_{a\in M} + \{\lambda_i(\infty) \mid i \in \mathbb{N}\}$ is a complete theory, $T \vdash \infty \neq a$ for all $a \in M$, and by the omitting types theorem T has a model $K > M$ omitting

$$p(x) = \{x > a \mid a \in M\}$$

i.e., $K >_{\text{cf}} M$, as required.

It follows from this that any consistent extension $T \supseteq \text{PA}^-$ has a model with a proper cofinal elementary extension, in particular there are models M and N of $I\Sigma_n + \neg B\Sigma_{n+1}$ such that $M \subseteq_{\text{cf}} N$ and $M \neq N$, giving part of the answer to Roman Kossak's question. The condition that M is Σ_n-tall for all $n \in \mathbb{N}$ is not *necessary* for the existence of proper elementary cofinal extensions, however, since if $M \models \text{PA} + \neg\text{Con(PA)}$, $n \geq 1$, and $I = I^n(M) = \{x \in M \mid \exists y \in M \models x < y \text{ and } y \text{ is } \Sigma_n\text{-definable in } M\}$ then $I \models \Pi_{n+1} - Th(M)$ and $I \models B\Sigma_n + \exp + \neg I\Sigma_n$ (see [5]) so I has a proper elementary cofinal extension (see [4]), but I is not Σ_{n+1}-tall since

$$p(x) = \{\exists y < x(\forall z \neg\theta(z) \vee \theta(y)) \mid \theta \in \Sigma_n\}$$

is not realized in I. Reviewing the proof of Theorem 6, however, we see that it actually shows that if M is Σ_n-tall for all $n \in \mathbb{N}$ and $\phi(\infty, \bar{a})$ is *any* extendible sentence of $\mathcal{L}(M)$ then M has an extension $K >_{\text{cf}} M$ with $\infty \in K\backslash M$ and $K \models \phi(\infty, \bar{a})$; this statement *does* have a converse, at least for theories extending $I\Delta_0 + \exp$.

Theorem 7 *Let $M \models I\Delta_0 + \exp$ be countable. Then M is Σ_n-tall for all $n \in \mathbb{N}$ iff for all $\mathcal{L}_A$-formulas $\phi(x, \bar{y})$ and all $\bar{a} \in M$*

$$\{v \in M \mid M \models \phi(v, \bar{a})\} \textit{ is infinite}$$
$$\Rightarrow \exists K >_{\text{cf}} M \textit{ with } b \in K\backslash M \textit{ such that } K \models \phi(b, \bar{a}).$$

Proof: We have already proved one direction. For the converse, recall from Paris and Dimitracopoulos [7] that for some carefully chosen Gödel-numbering there are Σ_n-complete formulas $\text{Sat}_{\Sigma_n}(\ulcorner\theta\urcorner, x)$ for each $n \in \mathbb{N}$, such that $I\Delta_0 + \exp$ proves $\forall\bar{x}(\theta(\bar{x}) \leftrightarrow \text{Sat}_{\Sigma_n}(\ulcorner\theta\urcorner, \langle\bar{x}\rangle))$ for each $\theta \in \Sigma_n$, where $\langle\bar{x}\rangle$ denotes some suitable function coding tuples as single numbers. Now if M is nonstandard but not Σ_n-tall, suppose

$$\{\psi_k(x, \bar{a}) \mid k \in \mathbb{N}\} \subseteq \Sigma_n$$

witnesses this failure of Σ_n-tallness.

By replacing n with $n + 2$ and each $\psi_k(y, \bar{a})$ with $\forall x > y\ \psi_k(x, \bar{a})$ if necessary we may assume that

(1) $M \models \exists y \psi_k(y, \bar{a})$,
(2) $M \models \forall x, y(\psi_k(y, \bar{a}) \wedge x > y \rightarrow \psi_k(x, \bar{a}))$, and
(3) $M \models \forall y(\psi_{k+1}(y, \bar{a}) \rightarrow \psi_k(y, \bar{a}))$

for all $k \in \mathbb{N}$, and that no $b \in M$ satisfies $M \models \bigwedge\!\!\!\bigwedge_{k\in\mathbb{N}} \psi_k(b, \bar{a})$. Using the fact that $(\psi_k(x, \bar{a}))_{k\in\mathbb{N}}$ is a recursive sequence, by an easy overspill argument there is $c \in M$ such that

$$M \models (c)_0 = \ulcorner\psi_0(x, \bar{y})\urcorner$$

and

$$M \models (c)_k = \ulcorner (c)_0 \wedge (c)_1 \wedge \cdots \wedge (c)_{k-1} \wedge \psi_k(x, \bar{y}) \urcorner$$

for each $k \geq 1$, where $(c)_k$ is the kth element of the sequence coded by c. By overspill again there is a nonstandard $\nu \in M$ such that

$$M \models \forall r, s < \nu (r < s \rightarrow (c)_r \text{ is a conjunct of } (c)_s);$$

hence

(4) $M \models \forall x, \bar{y} \forall r, s < \nu (r < s \wedge \mathrm{Sat}_{\Sigma_n}((c)_s, \langle x, \bar{y} \rangle) \rightarrow \mathrm{Sat}_{\Sigma_n}((c)_r, \langle x, \bar{y} \rangle))$.

By (1) and (3), the formula

$$\phi(v, a, c, \nu) =_{\mathrm{Def}} v < \nu \wedge \exists x\, \mathrm{Sat}_{\Sigma_n}((c)_v, \langle x, \bar{a} \rangle)$$

is satisfied by infinitely many $b \in M$ (namely exactly those standard $b \in M$). But if $K \models \phi(b, a, c, \nu)$ with $\mathbb{N} < b < \nu$ and $K >_{\mathrm{cf}} M$ then

$$K \models \mathrm{Sat}_{\Sigma_n}((c)_b, \langle d, \bar{a} \rangle)$$

for some $d \in K$. Since $K > M$, from (4) we have

(5) $K \models \psi_k(d, \bar{a})$

for all $k \in \mathbb{N}$. Since $K \supseteq_{\mathrm{cf}} M$ there is $e \in M$ with $e > d$. But then by (2) and (5)

$$K \models \psi_k(e, \bar{a})$$

for all $k \in \mathbb{N}$; hence $M \models \bigwedge\!\!\!\bigwedge_{k \in \mathbb{N}} \psi_k(e, \bar{a})$, a contradiction.

Our last result completes the answer to Roman Kossak's question.

Theorem 8[1] *For each $n \in \mathbb{N}$ there is a countable model $K \models I\Sigma_n + \exp + \neg B\Sigma_{n+1}$ with no elementary cofinal extensions.*

Proof: We first recall the sequence of functions f_n from [4]:

$$f_0(x) \text{ is } x^x, \text{ and, for } n \geq 1,$$

$$f_n(x) = (\mu z)\{\forall u, v < x \forall y (\mathrm{Sat}_{\Sigma_n}(u, \langle v, y \rangle) \rightarrow \exists y' \leq z\, \mathrm{Sat}_{\Sigma_n}(u, \langle v, y' \rangle))\}.$$

It was shown in [4] that, for each $n \in \mathbb{N}$, there are Σ_{n+1} formulas $f_n(x) = y$ and $f_n^{(z)}(x) = y$ such that $I\Sigma_n + \exp$ proves

$$\forall x \exists ! y f_n(x) = y$$

$$\forall x, x', y, y' (x \leq x' \wedge f_n(x) = y \wedge f_n(x') = y' \rightarrow y \leq y')$$

$$\forall x, y, z, z' (f_n^{(y)}(x) = z \wedge f_n^{(y)}(x) = z' \rightarrow z = z')$$

$$\forall x f_n^{(0)}(x) = x$$

$$\forall x, y, z, w (f_n^{(y)}(x) = z \wedge f_n(z) = w \rightarrow f_n^{(y+1)}(x) = w)$$

and, whenever $I \subseteq_e M \models I\Sigma_n$ satisfies $a \in I \Rightarrow \exists b \in I$ such that $M \models f_n(a) = b$, then $I \prec_{\Sigma_n} M$ and $I \models I\Sigma_n + \exp$.

Now let $M \models \mathrm{PA} + \neg\mathrm{Con(PA)}$, let $a \in M$ be the least proof of $0 = 1$ from PA (so a is nonstandard and Σ_1-definable in M) and let

$$I = \{x \in M \mid \exists y \in M \models x \leq y \wedge y = f_n^{(k)}(a) \text{ for some } k \in \mathbb{N}\},$$

so $I <_{\Sigma_n} M$ and $I \models I\Sigma_n + \exp$. Now let $K = K^{n+1}(I) =$ the set of all Σ_{n+1}-definable elements of I. It is straightforward to check (using $I \models I\Sigma_n + \exp$) that $K <_{\Sigma_{n+1}} I$, $K \models I\Sigma_n + \exp$, and that for all $k \in \mathbb{N}$

$$K \models b = f_n^{(k)}(a) \Leftrightarrow I \models b = f_n^{(k)}(a) \Leftrightarrow M \models b = f_n^{(k)}(a),$$

so $\{f_n^{(k)}(a) \mid k \in \mathbb{N}\} \subseteq_{\mathrm{cf}} K$. We claim K has no elementary cofinal extensions.

Suppose $K <_{\mathrm{cf}} J$ with $w \in J \backslash K$. Consider the formula $\theta(u, v) =_{\mathrm{def}}$

$$\exists k, b[b = f_n^{(k)}(a) \wedge b < v \wedge \exists j < k(\mathrm{Sat}_{\Sigma_{n+1}}(j, u) \wedge \forall x(\mathrm{Sat}_{\Sigma_{n+1}}(j, x) \rightarrow x = u))].$$

Thus,

1. $K \models \forall u \exists v \theta(u, v)$,
2. $\{u \in K \mid K \models \theta(u, v)\}$ is finite for all $v \in K$, and
3. $K \models \forall v, v', u\ (v \leq v' \wedge \theta(u, v) \rightarrow \theta(u, v'))$.

Now let $c \in J \models \theta(w, c)$ (c exists by (1) and $J > K$). By (3) and $J \supseteq_{\mathrm{cf}} K$ we may assume without loss of generality that $c \in K$. But $w \notin K$ so

$$\mathrm{card}\{x \in J \mid J \models \theta(x, c)\} > \mathrm{card}\{x \in K \mid K \models \theta(x, c)\} \in \mathbb{N}$$

which contradicts $J > K$.

Problem The above example fails to satisfy the following necessary condition for M to have a proper elementary cofinal extension:

> If $\theta(x, y)$ is an $\mathcal{L}_A$-formula, possibly containing parameters from M, such that $M \models \forall x Q^* y \theta(x, y)$, then for some $c \in M$ $\{x \in M \mid M \models \theta(x, c)\}$ is infinite.

Is this condition also sufficient?

NOTE

1. Since writing the first draft of this paper, C. Dimitracopoulos has informed me that Theorem 8 already appears in H. Lessan's thesis (Lessan [5]), and so should be attributed accordingly.

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