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Notre Dame Journal of Formal Logic
Volume 32, Number 3, Summer 1991

Minimal Satisfaction Classes with an Application to Rigid Models of Peano Arithmetic

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Abstract For each regular κ, models of Peano Arithmetic are constructed which are rigid, recursively saturated, and κ-like. The construction relies on a theorem asserting that countable, recursively saturated models of PA have many minimal, inductive satisfaction classes.

After Kaufmann's rather classless model, i.e. an ω_1-like recursively saturated model of PA all of whose classes are definable (cf. Kaufmann [1]), other examples of ω_1-like recursively saturated models of PA, with properties different from those of countable recursively saturated models, are no longer surprising. However, it is still worthwhile to investigate questions about the existence of ω_1-like recursively saturated models with various second-order properties. One reason is that questions about ω_1-like models can usually be translated to questions about their countable elementary initial segments, and these questions often turn out to be interesting in their own right.

In this paper we construct an ω_1-like recursively saturated model of PA which is rigid (that is, it has no nontrivial automorphisms) and even has no nontrivial elementary embeddings into itself. A theorem asserting the existence of rigid ω_1-like recursively saturated models of PA was stated, without proof, in Kossak and Kotlarski [3] as a corollary of a result about automorphisms of countable recursively saturated models. That construction depended on the set-theoretic principle $\diamondsuit$. The construction presented here is based on a MacDowell–Specker type argument, using minimal inductive satisfaction classes, and needs no set-theoretic assumptions. We use it in Theorem 10 and Corollary 11 to construct rigid, κ-like recursively saturated models for all uncountable κ.

A satisfaction class S for a model M is minimal if (M, S) has no proper elementary substructures. We will prove a theorem showing the existence of many minimal inductive satisfaction classes for countable models of PA. A slightly weaker version of this result was stated first without proof in Kossak [2].

Received January 14, 1991; revised August 5, 1991

Let L be the language of PA. We denote by Q_n the closure of the set of all Σ_n formulas of L under negation, conjunction, and bounded quantification, and by Q_∞ the set of all formulas of L. If M is a model of PA and either $e = \infty$ or e is an element of M, then $Q_e(M)$ is the set of Q_e formulas in the sense of M (under a fixed arithmetization). A subset S of M is a Q_e-satisfaction class of M if S consists of (codes of) pairs of the form (φ, α), where φ is in $Q_e(M)$ and α is (a code of) a valuation for φ, and the usual conditions of Tarski for the definition of satisfaction hold.

Let L^* be an arbitrary finite language extending L. Then let PA* be the L^*-theory consisting of PA and all instances of the induction schema in the extended language. We say that S is an *inductive Q_e-satisfaction class*, or briefly a *Q_e-class*, for a model M, if S is a Q_e-satisfaction class for M and $(M, S) \models \mathrm{PA}^*$. Well-known simple facts about Q_e-classes are summarized in the next proposition.

Proposition 1 *Let M be a model of PA.*
(i) *For every standard n, there is a unique Q_n-class for M.*
(ii) *If M is countable, then M is recursively saturated iff for some nonstandard $e \in M$ there exists a Q_e-class for M.*
(iii) *If S and D are Q_e-classes for M and* $(M, S, D) \models \mathrm{PA}^*$, *then $S = D$. (In fact Π_1-induction is enough here.)*

We will say that a subset X of M is *minimal* if (M, X) has no proper elementary submodels. Subsets X, Y are *elementarily equivalent* if (M, X) and (M, Y) are elementarily equivalent. For $M \models \mathrm{PA}^*$ we let $\mathrm{Def}(M)$ be the set of all parametrically definable subsets of M. If M, N are models of PA, and M is an initial segment of N, then we say that the subset $X \subseteq M$ is *coded in N* if $X = Y \cap M$ for some $Y \in \mathrm{Def}(N)$. The standard system of a model M, denoted by $\mathrm{SSy}(M)$, is the set of subsets of ω which are coded in M. Recall that a Scott set χ is a set of subsets of ω which forms an ω-model of WKL_0. If $M \models \mathrm{PA}$, then $\mathrm{SSy}(M)$ is a Scott set.

We say that an L^*-theory T *represents* a set $X \subseteq \omega$ if there is an L^*-formula $\varphi(x)$ such that for each $n \in \omega$,

$$n \in X \Leftrightarrow T \vdash \varphi(\underline{n}),$$

and

$$n \notin X \Leftrightarrow T \vdash \neg\varphi(\underline{n}).$$

$\mathrm{Rep}(T)$ is the set of sets represented by T. When needed, we will identify a theory T with the set of Gödel numbers of its sentences.

The following is essentially the basic result of Scott [6] on Scott sets.

Lemma 2 *Let $T_0 \supseteq \mathrm{PA}^*$ be an L^*-theory which represents itself, and let χ be a countable Scott set such that $T_0 \in \chi$. Then there is a complete, consistent L^*-theory $T \supseteq T_0$ such that $\chi = \mathrm{Rep}(T)$. Moreover, there are continuum many such theories.*

Our main result about minimal satisfaction classes is a direct application of Lemma 2.

Theorem 3 *If a nonstandard countable model $M \models \mathrm{PA}$ has a Q_e-class, where either $e = \infty$ or else $e \in M$ is nonstandard, then M has continuum many pairwise elementarily inequivalent minimal Q_e-classes.*

Proof: We consider only the case that $e \in M$, the case $e = \infty$ being very similar. Let $L^* = L \cup \{S, e\}$, and let $T_0 = \mathrm{Th}((M, e)) \cup \{$"$S$ is a Q_e-class"$\}$. Standard facts about recursive saturation imply that $T_0 \in \mathrm{Rep}(T_0) \cap \mathrm{SSy}(M)$. Applying Lemma 2, we get continuum many completions T of T_0 such that $\mathrm{Rep}(T) = \mathrm{SSy}(M)$. For each such T, let (M_T, e_T, S_T) be its minimal model. Then $\mathrm{SSy}(M_T) = \mathrm{SSy}(M)$ and M_T is recursively saturated since it has a Q_{e_T}-class, so there is an isomorphism $f: M_T \to M$ such that $f(e_T) = e$. Since e_T is definable in (M_T, S_T), it follows that S_T is a minimal Q_{e_T}-class, so that $f''S_T$ is a minimal Q_e-class for M. Clearly, different completions T yield elementarily inequivalent Q_e-classes S.

If S is a Q_e-class for M and $d < e$, then we denote by $S|d$ the restriction of S to Q_d formulas in the sense of M. Of course, for every $d < e$ the restriction $S|d$ is a Q_d-class for M. We denote by $\mathrm{Tr}_n(w, v)$ a Δ_{n+1} truth predicate for Q_n. The next lemma is due to Henryk Kotlarski.

Lemma 4 *If S is a Q_e-class for a model $M \models \mathrm{PA}$, and $d \in M$ is such that $d + \omega < e$, then the structure $(M, S|d)$ is recursively saturated (so that $\mathrm{Th}((M, S|d)) \in \mathrm{SSy}(M)$).*

Proof (sketch): By induction on φ we can show that for every $d < e$

$$(M, S) \models \forall \varphi \in Q_d \forall b (S(\varphi, b) \Leftrightarrow S(\mathrm{Tr}_d(w, v), (\varphi, b))).$$

Hence, by replacing every subformula of $S(\alpha, t)$ of ϕ by $Tr_d(\alpha, v)$, every formula ϕ of $L(S)$ is translated to a Q_{d+n} formula ϕ^* in the sense of M, for some $n < \omega$, such that for each $b \in M$,

$$(M, S_d) \models \phi(b) \quad \text{iff} \quad (M, S) \models S(\phi^*, b).$$

Then we can use the above equivalence and overspill in (M, S) to prove that $S|d$ is recursively saturated.

Notice that if S is a Q_e-class for M and $d + n = e$, where $n \in \omega$, then $S \in \mathrm{Def}(M, S|d)$. We remark that by use of Lemma 4, we can improve Theorem 3 as follows: If $M \models \mathrm{PA}$ is countable, S is a Q_e-class for M where either $e \in M$ or $e = \infty$, and $d + \omega < e$, then M has continuum many pairwise inequivalent Q_e-classes D such that $D|d = S|d$ and (M, D, d) is minimal.

The next lemma is a special case of Tarski's theorem on the undefinability of truth.

Lemma 5 *If X is a minimal subset of M and $(M, X) \models \mathrm{PA}^*$, then* $\mathrm{Th}((M, X)) \notin \mathrm{SSy}(M)$.

If X is a minimal subset of M and $(M, X) \models \mathrm{PA}^*$, then every element of M is definable in (M, X); hence, (M, X) is rigid and, moreover, there is no nontrivial elementary embedding of (M, X) into itself.

Recall the theorem of Kotlarski [5] and of Schmerl [7] which asserts that if N is a cofinal extension of a model M and $(M, X) \models \mathrm{PA}^*$, then there is a unique

$\bar{X} \subseteq N$ such that $(M, X) \prec (N, \bar{X})$. In particular, if S is a Q_e-class for M, then $\bar{S}$ is a Q_e-class for N.

Lemma 6 *Suppose $N \prec M$ is a cofinal substructure, either $e \in N$ or $e = \infty$, and $S \subseteq N$ is a minimal Q_e-class for N. If $f: N \to M$ is a cofinal embedding such that either $f(e) \in N$ or $e = \infty$ and such that $\overline{f''S} \in \mathrm{Def}((M, \bar{S}))$, then f is the identity function.*

Proof: First consider the case that $e \in N$, and let $d = f(e)$. We will show that $d = e$. If not, then without loss of generality we can assume $d < e$ (for if $d > e$ then $f''S$ is a minimal Q_d-class for $f''N$, and $\overline{f^{-1}(f''S)} = \bar{S} = \overline{f''S} \mid e \in \mathrm{Def}((M, \overline{f''S}))$, so just consider $f^{-1}: f''N \to M$ instead). Consequently, we even get that $d + \omega < e$. By Lemma 5, $\mathrm{Th}((N, S)) \notin \mathrm{SSy}(N)$. On the other hand, by Proposition 1(iii), $\overline{f''S} = \bar{S} \mid d$, and then $\mathrm{Th}((N, S)) = \mathrm{Th}((f''N, f''S)) = \mathrm{Th}((M, \overline{f''S})) = \mathrm{Th}((M, \bar{S} \mid d)) = \mathrm{Th}((N, S \mid d)) \in \mathrm{SSy}(N)$, by Lemma 4. Thus we get $d = e$.

In either case ($d = e$ or $e = \infty$) we get that $\overline{f''S} = \bar{S}$, so that $(N, S) \prec (M, \bar{S})$ and $(f''N, f''S) \prec (M, \bar{S})$. Both of these substructures are minimal; consequently $(N, S) = (f''N, f''S)$, and f being an automorphism of (N, S) must be the identity.

We will need Lemma 6 to get a κ-like, recursively saturated model of PA having no nontrivial embeddings into itself for arbitrary uncountable regular κ. However, for $\kappa = \omega_1$, we can get by with just the following immediate corollary of Lemma 6.

Corollary 7 *If S is a minimal Q_e-class for M, where either $e \in M$ or $e = \infty$, then each cofinal embedding $f: M \to M$ for which $\overline{f''S} \in \mathrm{Def}((M, S))$ is the identity function.*

Corollary 8 *Let M be a countable, recursively saturated model of* PA. *Then there is a countable, elementary end extension N such that if N' is any elementary end extension of N and $f: N' \to N'$ is an elementary embedding such that $f''M$ is cofinal in M, then $f \mid M$ is the identity function.*

Proof: Let S be a minimal inductive satisfaction class for M. Let (N, D) be a conservative, countable, elementary end extension of (M, S). (Conservativeness means that every subset of M which is coded in N is definable in (M, S). The existence of such extensions follows from a suitable version of the MacDowell-Specker Theorem.) Thus, $\mathrm{Def}((M, S))$ is the family of subsets of M which are coded in N, and $\mathrm{Def}((M, S))$ is also the family of subsets of M which are coded in an elementary end extension N' of N. If $f: N' \to N'$ is an elementary embedding and $f''M$ is cofinal in M, then $\overline{f''S} \in \mathrm{Def}((M, S))$, so by Corollary 7, $f \mid M$ must be the identity.

Theorem 9 *Every countable, recursively saturated model of* PA *has an ω_1-like, recursively saturated, elementary end extension M which has no nontrivial elementary embeddings into itself.*

Proof: Let M_0 be a countable, recursively saturated model of PA. Using Corollary 8, obtain a continuous chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable, recursively saturated elementary end extensions, and let M be the union of this chain. Clearly

M is an ω_1-like, recursively saturated elementary end extension of M_0. Now suppose that $f: M \to M$ is an elementary embedding. There are arbitrarily large $\alpha < \omega_1$ such that $f''M_\alpha$ is a cofinal subset of M_α, so by Corollary 8, $f \mid M_\alpha$ is the identity function. Therefore f is the identity function.

Notice that by using elementarily inequivalent satisfaction classes in the construction in the proof of Corollary 8, we can obtain $2^{\aleph_1}$ models M satisfying the conditions of Theorem 9 no one of which is embeddable in another one. To construct these models, first let $\{X_\nu : \nu < 2^{\aleph_1}\}$ be a set of stationary subsets of ω_1 which are distinct modulo the filter of closed, unbounded subsets of ω_1. That is, whenever $C \subseteq \omega_1$ is closed and unbounded, and $\mu < \nu < 2^{\aleph_1}$, then $C \cap X_\mu \neq C \cap X_\nu$. Let M be a countable, recursively saturated model of PA and let nonstandard $e \in M$ be such that M has a Q_e-class. Let S_0, S_1 be elementarily inequivalent minimal Q_e-classes. Now obtain M^ν as the union of the chain $\langle M^\nu_\alpha : \alpha < \omega_1 \rangle$, as done in the proof of Theorem 9 where, at stage α, we use a minimal Q_e-class S of M^ν_α such that $(M^\nu_\alpha, S) \cong (M, S_0)$ if $\nu \in X_\alpha$ and $(M^\nu_\alpha, S) \cong (M, S_1)$ if $\nu \notin X_\alpha$.

Looking at the proofs of Theorem 9 and Corollary 8, we see that the model M of Theorem 9 was obtained as the union of a continuous chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable models, where for each $\alpha < \omega_1$ there is nonstandard $e_\alpha \in M_\alpha$, a minimal Q_{e_α}-class S_α for M_α and a Q_{e_α}-class S'_α for $M_{\alpha+1}$ such that $(M_{\alpha+1}, S'_\alpha)$ is a conservative extension of (M_α, S_α). By exercising some care, we can arrange for M to have some additional properties. We consider two examples.

We can obtain an ω_1-like recursively saturated M which has no nontrivial elementary embeddings into itself and which has no inductive satisfaction classes. To do this, we require that for each nonstandard $e \in M_0$, $\{\alpha < \omega_1 : e_\alpha < e\}$ be stationary. (By Theorem 3 this is possible.) Of course, Kaufmann's model is also an example of an ω_1-like, recursively saturated model without an inductive satisfaction class. Another construction, using different properties of satisfaction classes, will appear in [4].

To obtain an ω_1-like, recursively saturated M which has no nontrivial embeddings into itself but which does have an inductive satisfaction class, proceed as follows. Let S_0 be a minimal Q_{2e}-class for M_0, where $e \in M_0$ is nonstandard. Now just arrange that each S_α is a minimal Q_{2e}-class for $\beta < \alpha$, $(M, (S_\beta \mid e)) \prec (M, (S_\alpha \mid e))$. (By the remark following Lemma 4 this is possible.) Then M will have a Q_e-class which is $\bigcup \{S_\alpha \mid e : \alpha < \omega_1\}$.

We next extend Theorem 9 to all uncountable regular cardinals.

Theorem 10 *Let $M \models$ PA be recursively saturated and have countable cofinality, and suppose $\kappa > |M|$ is regular. Then M has a κ-like, recursively saturated, elementary end extension which has no nontrivial embeddings into itself.*

Proof: We will obtain an elementary chain $\langle M_\nu : \nu < \kappa \rangle$ of models with each M_ν having a universe which is an ordinal in κ. Without loss of generality we can assume that $M \in \kappa$. Let $M_0 = M$, and let $N_0 \prec M_0$ be countable, recursively saturated, and cofinal in M. Let $D_0 \subseteq N_0$ be a minimal Q_{2e}-class for N_0, where $e \in N_0$ is nonstandard. Let $S_0 \subseteq M_0$ be the unique class such that $(N_0, D_0) \prec (M_0, S_0)$. Let $\{X_\alpha : \alpha < \kappa\}$ be a partition of the stationary set $\{\nu < \kappa : \mathrm{cf}(\nu) = \omega\}$ into stationary sets, where $X_\alpha \neq X_\beta$ whenever $\alpha < \beta < \kappa$. Now obtain the chain

$\langle M_\nu : \nu < \kappa \rangle$ of models, with each M_ν having an inductive satisfaction class S_ν, as follows:

(1) If ν is a successor ordinal, or if ν is a limit ordinal and $\mathrm{cf}(\nu) > \omega$, then let $(M_{\nu+1}, S_{\nu+1})$ be any conservative extension of (M_ν, S_ν) such that $M_{\nu+1} \in \kappa$.
(2) If ν is a limit ordinal, then let $M_\nu = \bigcup_{\gamma<\nu} M_\gamma$ and $S_\nu = \bigcup_{\gamma<\nu} (S_\gamma | e)$.
(3) If ν is a limit ordinal and $\mathrm{cf}(\nu) = \omega$, then let α be such that $\nu \in X_\alpha$. Let $N_\nu < M_\nu$ be countable, recursively saturated and cofinal in M_ν such that $e \in N_\nu$ and $\alpha \in N_\nu$ provided $\alpha \in M_\nu$. Let D_ν be a minimal Q_{2e}-class for N_ν such that $D_\nu | e = S_\nu \cap N_\nu$, and then let $S = \overline{D_\nu}$. Finally, let $(M_{\nu+1}, S_{\nu+1})$ be a conservative extension of (M_ν, S) such that $M_{\nu+1} \in \kappa$.

The model $N = \bigcup_{\nu<\kappa} M_\nu$ will be the desired model. Clearly N is κ-like and it is an elementary end extension of M. Also, N is recursively saturated as $\bigcup_{\nu<\kappa} S_\nu | e$ is a Q_e-class for N. We need to show that N has no nontrivial elementary embeddings into itself. Let $f : N \to N$ be an embedding and consider some $\beta \in N$. Let $\alpha = \langle \beta, f(e) \rangle$, and let $\nu \in X_\alpha$ be such that $\alpha \in M_\nu$ and $f''M_\nu$ is a cofinal subset of M_ν. Then $\beta, f(e) \in N_\nu$ since $\alpha \in N_\nu$. Clearly, $f | N_\nu : N_\nu \to M_\nu$ is a cofinal embedding such that $f''D_\nu \in \mathrm{Def}((M, \overline{D_\nu}))$, so by Lemma 6, $f(\beta) = \beta$. Thus $f : N \to N$ is the identity.

Corollary 11 *Let $M \models \mathrm{PA}$ be recursively saturated and suppose $\kappa > |\mathrm{SSy}(M)|$. Then there is a rigid, κ-like, recursively saturated $N \equiv M$ such that* $\mathrm{SSy}(N) = \mathrm{SSy}(M)$.

Proof: Theorem 10 handles the case that κ is regular, so assume that κ is singular. Let $\lambda = \mathrm{cf}(\kappa)$, and let $\langle \kappa_\nu : \nu < \lambda \rangle$ be a continuous, increasing sequence of cardinals whose supremum is κ such that $\kappa_0 = |\mathrm{SSy}(M)|$ and $\kappa_{\nu+1}$ is regular for each $\nu < \lambda$. Let $M_0 < M$ be recursively saturated and have countable cofinality such that $|M_0| = \kappa_0$ and $\mathrm{SSy}(M_0) = \mathrm{SSy}(M)$. Let S be a Q_{2e}-class for M_0 for some nonstandard e, and let $S_0 = S | e$. By Lemma 4, (M_0, S_0) is recursively saturated. We will obtain a continuous, elementary chain $\langle M_\nu, S_\nu \rangle$ of recursively saturated structures, where M_ν is κ_ν-like whenever $0 < \nu < \lambda$. Suppose that $\nu < \lambda$, and that we already have (M_ν, S_ν). By (the proof of) Theorem 10, there is an elementary end-extension $(M_{\nu+1}, S_{\nu+1})$ which is rigid, recursively saturated, and $\kappa_{\nu+1}$-like. Let $N = \bigcup_{\nu<\lambda} M_\nu$. Clearly, N is also rigid and recursively saturated, and $N \equiv M$, N is κ-like and $\mathrm{SSy}(N) = \mathrm{SSy}(M)$.

Corollary 12 *Suppose $M \models$ PA and $\kappa > 2^{\aleph_0}$. Then there is rigid, κ-like, $\aleph_0$-saturated $N \equiv M$. Furthermore, if κ is regular, then N has no nontrivial embeddings into itself.*

There is a question left unanswered by the results here. If κ is singular, does there exist a κ-like, recursively saturated $N \models \mathrm{PA}$ which has no nontrivial elementary embeddings into itself?

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