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Almost Hugeness and a Related Notion

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Abstract We consider a natural weakening of hugeness. In contrast to the supercompact situation, this notion fits into a nice hierarchy with almost hugeness, hugeness, and n-hugeness.

Preliminaries We work in ZFC. Our set theoretic notation is standard. V denotes the universe of all sets. Greek letters α and β denote ordinals, while γ, δ, η, θ, κ, λ, and σ are reserved for cardinals. By the term "inner model", we shall always mean a transitive class which satisfies ZFC. If M is an inner model, and λ is an infinite cardinal, we say that M is closed under λ-sequences if and only if for any $x \subseteq M$, if $|x| \leq \lambda$ then $x \in M$. M is closed under $< \lambda$-sequences if and only if M is closed under γ-sequences for each $\gamma < \lambda$. V_α denotes the collection of all sets of rank less than α, and H_γ denotes the collection of all sets hereditarily of cardinal less than γ. $\lambda^{\underline{\kappa}}$ denotes $\sup_{\gamma<\kappa}(\lambda^\gamma)$.

We shall always use the term "inaccessible" to mean "strongly inaccessible". A cardinal λ is Mahlo if and only if the inaccessibles below λ form a stationary subset of λ.

For $\kappa \leq \lambda$, $P_\kappa(\lambda) = \{x \subseteq \lambda : |x| < \kappa\}$, and, for $\kappa < \lambda$, $P_{=\kappa}(\lambda) = \{x \subseteq \lambda : |x| = \kappa\}$. Then κ is λ-supercompact if and only if there is a normal, fine, κ-complete ultrafilter on $P_\kappa(\lambda)$, and κ is huge with target λ if and only if there is a normal, fine, κ-complete ultrafilter on $P_{=\kappa}(\lambda)$. In either case, we shall refer to such ultrafilters simply as normal ultrafilters. We say that κ is $< \lambda$-supercompact if and only if κ is γ-supercompact for unboundedly many $\gamma < \lambda$.

Supercompactness and hugeness can also be characterized by embedding properties. κ is λ-supercompact if and only if there is an elementary embedding $i: V \to M$, where M is an inner model closed under λ-sequences, κ is the critical point of i, and $i(\kappa) > \lambda$. κ is huge with target λ if and only if there exists an elementary embedding $i: V \to M$ as above, except that $i(\kappa) = \lambda$. For the basic facts

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and methods involving supercompact and huge cardinals, the reader is referred to Solovay, Reinhardt, and Kanamari [11].

If U is a normal ultrafilter on $P_\kappa(\lambda)$, M_u denotes the transitive collapse of the ultrapower $\Pi V/U$. We note that M_u is closed under λ-sequences. We let $\pi_u : \Pi V/U \to M_u$ denote the collapsing isomorphism, and $i_u : V \to M_u$ denote the canonical elementary embedding. Standard methods (see, e.g. Barbanel [1], p. 85) show that $2^{\lambda^\kappa} < i_u(\kappa) < (2^{\lambda^\kappa})^+$. Generally, we shall abuse notation slightly, and consider the domain of π_u to consist of functions from $P_\kappa(\lambda)$ into V instead of equivalence classes of such functions.

Suppose $\kappa \le \gamma < \delta$, U_γ is a normal ultrafilter on $P_\kappa(\gamma)$, and U_δ is a normal ultrafilter on $P_\kappa(\delta)$. We say that U_γ is a restriction of U_δ to γ, and write $U_\gamma = U_\delta \upharpoonright \gamma$ if and only if for every $A \subseteq P_\kappa(\gamma)$, $A \in U_\gamma$ if and only if $\{x \in P_\kappa(\delta) : x \cap \gamma \in A\} \in U_\delta$. For λ a limit cardinal, we say that $\langle U_\gamma : \kappa \le \gamma < \lambda \rangle$ is a coherent sequence of normal ultrafilters (CSNU) if and only if for each γ with $\kappa \le \gamma < \lambda$, U_γ is a normal ultrafilter on $P_\kappa(\gamma)$ and, whenever $\kappa \le \gamma < \delta < \lambda$, $U_\gamma = U_\delta \upharpoonright \gamma$. When dealing with such a coherent sequence, we shall simplify notation by writing M_γ, π_γ, and i_γ, instead of $M_{u\gamma}$, $\pi_{u\gamma}$ and $i_{u\gamma}$, respectively.

There are natural elementary embeddings between the inner models associated with a CSNU. Suppose $\langle U_\gamma : \kappa \le \gamma < \lambda \rangle$ is a CSNU, and $\kappa \le \gamma \le \delta < \lambda$. Define $l_{\gamma\delta} : \Pi V/U_\gamma \to \Pi V/U_\delta$ as follows:

> For $f : P_\kappa(\gamma) \to V$, $l_{\gamma\delta}(f) : P_\kappa(\delta) \to V$ is defined by $(l_{\gamma\delta}(f)(x) = f(x \cap \gamma)$, for each $x \in P_\kappa(\delta)$.

It is straightforward to check that $l_{\gamma\delta}$ is elementary. Then, we can define $k_{\gamma\delta} : M_\gamma \to M_\delta$ by $k_{\gamma\delta} = \pi_\delta \circ l_{\gamma\delta} \circ \pi_\gamma^{-1}$. It should be noted that we are again abusing notation slightly in considering the domain and range of $l_{\gamma\delta}$ to consist of certain functions, instead of equivalence classes of such functions. If the specific CSNU that we are using is not clear by context, we shall write $k^B_{\gamma\delta}$ to mean the $k_{\gamma\delta}$ defined as above, using the CSNU B.

We will need the following facts on these embeddings. All can be proved in a straightforward manner (see also Barbanel [2]).

Suppose $\kappa \le \gamma < \delta < \eta < \lambda$.

Fact 1 $k_{\gamma\eta} = k_{\delta\eta} \circ k_{\gamma\delta}$.
Fact 2 $i_\delta = k_{\gamma\delta} \circ i_\gamma$.
Fact 3 For $\alpha \le \gamma$, $k_{\gamma\delta}(\alpha) = \alpha$.

The following construction has been used in [2] and [11]. Assume that λ is a limit cardinal, and $\langle U_\alpha : \kappa \le \gamma < \lambda \rangle$ is a CSNU. Then, using Fact 1, it follows that $\langle (M_\gamma, k_{\gamma\delta}) : \kappa \le \gamma < \delta < \lambda \rangle$ is a directed system.

Assume, for the remainder of this section, that the direct limit of this system is well-founded. It is not hard to see that this follows if $\mathrm{cf}(\lambda) > \aleph_0$. However, as we shall see in the next section, we may still have a well-founded direct limit with $\mathrm{cf}(\lambda) = \aleph_0$.

Let M be the transitive collapse of this well-founded direct limit. For any γ with $\kappa \le \gamma < \lambda$, let $j_\gamma : M_\gamma \to M$ be the canonical elementary embedding. By Fact 3, for $\alpha \le \gamma$, $j_\gamma(\alpha) = \alpha$. Define $i : V \to M$ by $i = j_\gamma \circ i_\gamma$. It follows from Fact 2 that the definition of i is independent of the choice of γ.

For any $x \in M$, if $x \in \text{range}\, j_\gamma$, we define $x(\gamma) = j_\gamma^{-1}(x)$. Let $m(x)$ be the least γ such that $x(\gamma)$ is defined. Then, for any $x \in M$, $m(x)$ is always defined, and $x(\gamma)$ is defined if and only if $m(x) \le \gamma < \lambda$.

The proof of the following lemma is straightforward.

Closure Lemma *M is closed under $<\text{cf}(\lambda)$-sequences.*

κ is almost huge with target λ if and only if there is an elementary embedding $i : V \to M$, where M is an inner model closed under $<\lambda$-sequences, κ is the critical point of i, and $i(\kappa) = \lambda$. Almost hugeness is a natural weakening of hugeness. It was studied in [11] and has found important applications (see, for example, Section 17 of Kanamori and Madigor [8]).

Almost hugeness cannot (to our knowledge) be characterized by the existence of a single normal ultrafilter. However, we can characterize almost hugeness using CSNUs. First, we must introduce the following technical condition:

A CSNU has property EC if and only if whenever $\kappa \le \gamma < \lambda$ and $\gamma \le \sigma < i_\gamma(\kappa)$, there is a δ such that $\gamma \le \delta < \lambda$ and $k_{\gamma\delta}(\sigma) = \delta$.

The intuition here is that, with γ and σ as above, if we look at σ in M_γ and follow through its images in our directed system, these images are eventually constant (Fact 3 is used here). "EC" is meant to denote "eventually constant".

Lemma ([11]) *κ is almost huge with target λ if and only if λ is inacessible and there exists a* CSNU $\langle U_\gamma : \kappa \le \gamma < \lambda \rangle$ *satisfying* EC.

Much of this paper involves considering what happens if there exists a CSNU $\langle U_\gamma : \kappa \le \gamma < \lambda \rangle$ satisfying EC, but where λ is not assumed to be inaccessible. We note that a different technical condition associated with CSNUs and giving an equivalence with almost hugeness was used in Barbanel [4].

1 The main lemma In this section, we state and prove a lemma that will be used repeatedly in future sections. We adopt the notation developed for CSNUs and the associated directed systems of inner models and elementary embeddings.

Main Lemma *Let* $\langle U_\gamma : \kappa \le \gamma < \lambda \rangle$ *be a* CSNU *satisfying* EC. *In addition, assume that* $\text{cf}(\lambda) > \aleph_0$. *Then* $\{\eta < \lambda : \eta$ *is a limit cardinal, and the* CSNU $\langle U_\gamma : \kappa \le \gamma < \eta \rangle$ *satisfies* EC$\}$ *is a closed and unbounded subset of* λ.

We will need the fact that if there exists a CSNU $\langle U_\gamma : \kappa \le \gamma < \lambda \rangle$ satisfying EC, then λ is a strong limit cardinal. To see this, suppose that $\langle U_\gamma : \kappa \le \gamma < \lambda \rangle$ is a CSNU satisfying EC, and suppose, by way of contradiction, that for some γ with $\kappa \le \gamma < \lambda$, $2^\gamma \ge \lambda$. Then, since $2^\gamma < i_\gamma(\kappa)$ (as noted in the preliminaries), we have $\lambda < i_\gamma(\kappa)$. By EC, this implies that for some δ with $\gamma \le \delta < \lambda$, we have $k_{\gamma\delta}(\lambda) = \delta$. Thus, $k_{\gamma\delta}(\lambda) < \lambda$. This is a contradiction, since $k_{\gamma\delta}$ is an elementary embedding.

Next, we introduce some notation. Assume $\langle U_\gamma : \kappa \le \gamma < \lambda \rangle$ is a CSNU satisfying EC. Fix γ and σ with $\kappa \le \gamma < \lambda$ and $\gamma \le \sigma < i_\gamma(\kappa)$. Let $\text{ec}(\sigma, \gamma)$ be the cardinal satisfying the condition given by EC. That is, $\gamma \le \text{ec}(\sigma, \gamma) < \lambda$, and $k_{\gamma, \text{ec}(\sigma,\gamma)}(\sigma) = \text{ec}(\sigma, \gamma)$. It follows from Fact 3 that $\text{ec}(\sigma, \gamma)$ is well-defined and that if $\gamma \le \sigma_1 < \sigma_2 < i_\gamma(\kappa)$, then $\text{ec}(\sigma_1, \gamma) < \text{ec}(\sigma_2, \gamma)$.

Proof: Assume that $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ is a CSNU satisfying EC. Let $C = \{\eta < \lambda : \eta$ is a limit cardinal and the CSNU $\langle U_\gamma : \kappa \leq \gamma < \eta \rangle$ satisfies EC$\}$.

The proof that C is closed is straightforward. To show that C is unbounded, fix γ_0 with $\kappa \leq \gamma_0 < \lambda$. For all $n < \omega$, define γ_{n+1} as follows:

$$\gamma_{n+1} = \sup\{\text{ec}(\sigma, \gamma_n) : \gamma_n \leq \sigma < i_{\gamma_n}(\kappa)\}.$$

We must establish that this definition makes sense, by showing that if $\gamma_n < \lambda$, then $\gamma_{n+1} < \lambda$. As noted in the preliminaries, $i_{\gamma_n}(\kappa) \leq (2^{\gamma_n^{\kappa}})^+$. Hence, $|i_{\gamma_n}(\kappa)| < 2^{\gamma_n^{\kappa}} < \lambda$. Since λ is a strong limit cardinal, $2^{\gamma_n^{\kappa}} < \lambda$. Pick η with $2^{\gamma_n^{\kappa}} < \eta < \lambda$. $M_\eta \models i_\eta \kappa$ is inaccessible. Since M_η is closed under η-sequences, it follows that (in V), $\text{cf}(i_\eta(\kappa)) > \eta$. Let $B = \{k_{\gamma_n \eta}(\sigma) : \gamma_n \leq \sigma < i_{\gamma_n}(\kappa)\}$. Clearly, $B \subseteq i_\eta(\kappa)$, and $|B| = |i_\gamma(\kappa)| \leq 2^{\gamma_n^{\kappa}} < \eta < \text{cf}(i_\eta(\kappa))$. Hence, B is not unbounded in $i_\eta(\kappa)$. Pick θ such that $\theta > \sup(B)$, $\theta > \eta$, and $\theta < i_\eta(\kappa)$. Then clearly for each σ with $\gamma_n \leq \sigma < i_{\gamma_n}(\kappa)$, $\text{ec}(\sigma, \gamma_n) < \text{ec}(\theta, \eta)$. Hence, $\gamma_{n+1} < \text{ec}(\theta, \eta) < \lambda$.

Let $\gamma^* = \sup\{\gamma_n : n < \omega\}$. Since $\text{cf}(\lambda) > \aleph_0$, $\gamma^* < \lambda$. It follows easily that $\gamma^* \in C$. Since $\gamma^* \geq \gamma_0$, this establishes that C is unbounded.

As an immediate application of the main lemma, we will study the large cardinal strength of the targets of almost huge cardinals. Before doing so, we note the analogous, but easier, result for huge cardinals. If κ is huge with target λ, then κ is measurable but, if λ is the least target for κ, then λ is not 2^λ-supercompact. The proof is straightforward.

Let us now suppose that κ is almost huge with target λ, and that this is witnessed by $i : V \to M$. Since κ is inaccessible, $M \models \lambda$ is inaccessible. It is straightforward to check that the fact that M is closed under $< \lambda$-sequences is enough to guarantee that λ really is (in V) inaccessible. Similarly, λ is hyper-inaccessible, hyper-hyperinaccessible, etc. (see Drake [7] for definitions). However, λ is not, in general, Mahlo.

Theorem 1 *If κ is almost huge with target λ and λ is Mahlo, then $\{\eta < \lambda : \kappa$ is almost huge with target $\eta\}$ is a stationary subset of λ.*

Proof: Immediate from the main lemma.

Corollary *If κ is almost huge with λ its first target, then λ is not Mahlo.*

Proof: Immediate from the theorem.

2 A natural weakening of almost hugeness Although most large cardinal properties fit into a nice hierarchy, it is well-known that the relationship between hugeness and supercompactness is not so nice (for a discussion of this relationship, see [3]). Taking either the elementary embedding or the normal ultrafilter definitions, it is clear that supercompactness is the right generalization of measurability. The importance of the exact target of a huge cardinal makes hugeness a very different type of notion than that of supercompactness or measurability. In this section, we wish to consider a huge-type hierarchy that runs alongside, but is not precisely compatible with, the usual large cardinal hierarchy.

We say that κ is $R(\lambda)$-huge ("R" for "range") if and only if there is an inner model M, and an elementary embedding $i : V \to M$ such that κ is the critical point of i, $i(\kappa) = \lambda$, and, given any $x \subseteq$ range i with $|x| < \lambda$, we have $x \in M$. κ is R-huge if and only if κ is $R(\lambda)$-huge for some λ. Note that if κ is almost huge with target λ, then κ is $R(\lambda)$-huge. We shall soon show that the converse is true if and only if λ is inaccessible.

We will soon establish that there are inner models associated with R-hugeness which possess nice closure properties in addition to those given in the definition, and in the next section, we shall study a natural hierarchy involving R-huge cardinals. $R(\lambda)$-hugeness can be characterized by the existence of a certain CSNU, as long as $\mathrm{cf}(\lambda) > \aleph_0$.

Theorem 2

(a) *If κ is $R(\lambda)$-huge, then there exists a* CSNU $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ *satisfying* EC.

(b) *If* $\mathrm{cf}(\lambda) > \aleph_0$ *and there exists a* CSNU $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ *satisfying* EC, *then κ is $R(\lambda)$-huge.*

Proof: The proof of Part a is essentially the same as the analogous proof for almost hugeness in [11].

For Part b, assume that $\mathrm{cf}(\lambda) > \aleph_0$, $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ is a CSNU satisfying EC, and $\langle (M_\gamma, k_{\gamma\delta}) : \kappa \leq \gamma < \lambda \rangle$ is the corresponding directed system of inner models and elementary embeddings. Let $i : V \to M$ be the inner model and elementary embedding obtained from this system. We note the fact that $\mathrm{cf}(\lambda) > \aleph_0$ is used here to establish that this system is well-founded and hence that M exists. We must show that $i : V \to M$ witnesses that κ is $R(\lambda)$-huge.

It follows easily that κ is the critical point of i. The proof that $i(\kappa) = \lambda$ is again as in the analogous proof for almost hugeness in [11]. Suppose then that $x \subseteq$ range i and $|x| < \lambda$. We must show that $x \in M$. Fix some γ with $\kappa \leq \gamma < \lambda$ and $\gamma > |x|$. Clearly, for each $a \in x$, $j_\gamma^{-1}(a)$ is defined, since $a \in$ range i. Let $y = \{j_\gamma^{-1}(a) : a \in x\}$. Then $y \subseteq M_\gamma$ and $|y| = |x| < \gamma$. Hence, since M_γ is closed under γ-sequences, we have $y \in M_\gamma$. Then, using Fact 3, it follows that $x = j_\gamma(y)$. Hence, $x \in M$.

Corollary *κ is almost huge with target λ if and only if κ is $R(\lambda)$-huge and λ is inaccessible.*

Proof: This is immediate from the theorem and the characterization of almost hugeness given by the lemma in the Preliminaries.

Corollary *If κ is $R(\lambda)$-huge, then λ is a strong limit cardinal.*

Proof: This is immediate from the theorem and the comments following the statement of the Main Lemma.

We do not know whether our assumption that $\mathrm{cf}(\lambda) > \aleph_0$ is necessary in Part b of Theorem 2. In general, we may ask the following:

Open question If $\mathrm{cf}(\lambda) = \aleph_0$ and $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ is a CSNU, must the associated direct limit be well-founded?

The following theorem answers this question for the special case where the CSNU comes from an elementary embedding witnessing that κ is $R(\lambda)$-huge.

Theorem 3 *Suppose $i : V \to M$ witnesses that κ is $R(\lambda)$-huge, and $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ is the* CSNU *induced by i. Then, the direct limit of the associated directed* *system of inner models and elementary embeddings is well-founded. Hence, the existence of such a* CSNU *witnesses the $R(\lambda)$-hugeness of κ.*

Proof: Let $i: V \to M$ and $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ be as in the statement of the theorem, and let $\langle (M_\gamma, k_{\gamma\delta}) : \kappa \leq \gamma < \delta < \lambda \rangle$ be the directed system corresponding to $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$.

For each γ with $\kappa \leq \gamma < \lambda$, define $h_\gamma : M_\gamma \to M$ by $h_\gamma(x) = (i(\pi_\gamma^{-1}(x)))(i[\gamma])$ for all $x \in M_\gamma$. It is straightforward to verify that each h_γ is an elementary embedding, and that for $\kappa \leq \gamma < \delta < \lambda$, $h_\delta \circ k_{\gamma\delta} = h_\gamma$. Hence the h_γ's yield an elementary embedding from the direct limit of the system $\langle (M_\gamma, k_{\gamma\delta}) : \kappa \leq \gamma < \delta < \lambda \rangle$ into the well-founded model M. It follows that the direct limit is well-founded.

To see that $\langle U_\alpha : \kappa \leq \gamma < \lambda \rangle$ witnesses the $R(\lambda)$-hugeness of κ, let N be the transitive collapse of the (well-founded) direct limit of $\langle (M_\gamma, k_{\gamma\delta}) : \kappa \leq \gamma < \delta < \lambda \rangle$ and let $j: V \to N$ be the canonical embedding. That $j: V \to N$ witnesses the $R(\lambda)$-hugeness of κ follows precisely as in the proof of Theorem 2b.

The characterization of R-hugeness given by Theorem 2 tells us that if κ is $R(\lambda)$-huge, then there are large cardinals unbounded below λ. For example, we have the following result:

Theorem 4 *Suppose κ is $R(\lambda)$-huge. Then, $\{\gamma < \lambda : \gamma$ is measurable$\}$ is an unbounded subset of λ.*

Proof: Assume κ is $R(\lambda)$-huge and $i: V \to M$ is an elementary embedding and inner model which witnesses that κ is $R(\lambda)$-huge, and which is obtained from a CSNU $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ and corresponding directed system of inner models and elementary embeddings $\langle (M_\gamma, k_{\gamma\delta}) : \kappa \leq \gamma < \delta < \lambda \rangle$, as in the proof of Theorem 2 (if $\mathrm{cf}(\lambda) > \aleph_0$) or Theorem 3 (regardless of the value of $\mathrm{cf}(\lambda)$).

Fix any cardinal $\gamma < \lambda$. We claim that γ is measurable if and only if $M \models \gamma$ is measurable. By the second corollary of Theorem 2, λ is a strong limit cardinal. Hence, $2^\gamma < \lambda$. Fix any cardinal δ with $\max\{2^\gamma, \kappa\} \leq \delta < \lambda$. Then, by closure considerations, γ is measurable if and only if $M_\delta \models \gamma$ is measurable. By the elementarity of $j_\delta : M_\delta \to M$, and the fact that $j_\delta(\gamma) = \gamma$ (by Fact 3), we have that $M_\delta \models \gamma$ is measurable if and only if $M \models \gamma$ is measurable. Hence, we have shown that γ is measurable if and only if $M \models \gamma$ is measurable.

Clearly, κ is 2^κ-supercompact. Therefore, $\{\gamma < \kappa : \gamma$ is measurable$\}$ is an unbounded subset of κ. By the elementarity of $i: V \to M$, $M \models \{\gamma < \lambda : \gamma$ is measurable$\}$ is an unbounded subset of λ. Then, by the argument in the preceding paragraph, we conclude that $\{\gamma < \lambda : \gamma$ is measurable$\}$ is an unbounded subset of λ.

We note that Theorem 4 is very much in the spirit of hugeness, and very much not in the spirit of supercompactness.

Next we study some additional closure properties associated with R-hugeness.

Theorem 5 *Suppose κ is $R(\lambda)$-huge. Then, there exists $i: V \to M$ witnessing that κ is $R(\lambda)$-huge and satisfying the following:*

(a) *M is closed under $<$* cf(λ)*-sequences.*
(b) *$V_\lambda \subseteq M$ and $V_\gamma \in M$.*
(c) *For any $z \subseteq M$, if* cf$(\lambda) <$ cf$|z|$ *and* $|z| < \lambda$, *then there exists $y \subseteq z$ such that $|y| = |z|$ and $y \in M$.*

Proof: Assume that κ is $R(\lambda)$-huge. As in the proof of Theorem 4, let $i: V \to M$ be an elementary embedding and inner model witnessing that κ is $R(\lambda)$-huge, which is obtained from a CSNU $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ and corresponding directed system of inner models and elementary embeddings $\langle (M_\gamma, k_{\gamma\delta}) : \kappa \leq \gamma < \delta < \lambda \rangle$.

Part a is our Closure Lemma from the preliminaries.

For Part b, we first note that by Theorem 4, there are measurable, and hence inaccessible, cardinals unbounded below λ. Fix any inaccessible γ with $\kappa \leq \gamma < \lambda$. By closure considerations, and a straightforward induction, we see that $H_\gamma \subseteq M_\gamma$. By Fact 3, for $\alpha \leq \gamma$, $j_\gamma(\alpha) = \alpha$. Then, another straightforward induction tells us that for any $x \in H_\gamma$, $j_\gamma(x) = x$. Hence $H_\gamma \subseteq M$. Since γ is inaccessible, $V_\gamma = H_\gamma$, and we have $V_\gamma \subseteq M$. Then, using the fact that γ was an arbitrary inaccessible with $\kappa \leq \gamma < \lambda$, and the fact that the inaccessibles are unbounded below λ, we have $V_\lambda = \bigcup\{V_\gamma : \kappa \leq \gamma < \lambda \text{ and } \gamma \text{ is inaccessible}\} \subseteq M$.

Clearly $(V_\lambda)_M \subseteq V_\lambda$. On the other hand, if $x \in V_\lambda$, then, since $V_\lambda \subseteq M$, $x \in M$. Then certainly $M \models x \in V_\lambda$, and we have shown that $V_\lambda \subseteq (V_\lambda)_M$. This establishes that $V_\lambda = (V_\lambda)_M$, and that therefore $V_\lambda \in M$.

For Part c, fix any $z \subseteq M$ with $\text{cf}(\lambda) < \text{cf}|z|$ and $|z| < \lambda$. Let $f: \text{cf}(\lambda) \to \lambda$ be a cofinal mapping. Then, $z = \bigcup_{\beta < \text{cf}(\lambda)} \{x \in z : m(x) \leq f(\beta)\}$. Since $\text{cf}(\lambda) < \text{cf}|z|$, it follows that for some fixed $\beta < \text{cf}(\lambda)$, $|\{x \in z : m(x) \leq f(\beta)\}| = |z|$. Fix any γ with $\gamma \geq \max\{f(\beta), |z|\}$. Then $|\{x \in z : m(x) \leq \gamma\}| = |z| \leq \gamma$. Let $w = \{x(\gamma) : x \in z \text{ and } m(x) \leq \gamma\}$. Then $w \subseteq M_\gamma$ and $|w| = |z| \leq \gamma$. Hence $w \in M_\gamma$. Let $y = j_\gamma(w)$. It follows, using Fact 3, that $y \subseteq z$ and $|y| = |z|$.

We close this section by pointing out why two "$<$'s" in the theorem cannot be changed to "$\leq$'s". In Part a, "$<$ cf(λ)-sequences" cannot be strengthened to cf(λ)-sequences". Suppose, for example, that κ is the first R-huge cardinal, and λ is the first R-huge target for κ. (It is not hard to show that the first R-huge cardinal in fact has precisely one target.) By Theorem 3, let $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$ be a CSNU witnessing that κ is $R(\lambda)$-huge, and let $i: V \to M$ be the corresponding elementary embedding and inner model. Fix $x \subseteq \lambda$ such that x is unbounded in λ, and order-type $(x) = \text{cf}(\lambda)$. Let $z = \{U_\gamma : \gamma \in x\}$. Then $z \subseteq M$ and $|z| = |x| = \text{cf}(\lambda)$. Assume by way of contradiction that $z \in M$. Then, by taking all restrictions of elements of z, it follows that $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle \in M$. Hence $M \models \kappa$ is $R(\lambda)$-huge. But, by elementarity, $M \models i(\kappa)$ is the least R-huge cardinal. This is a contradiction, since $i(\kappa) > \kappa$.

A similar argument shows that in Part c, "cf$(\lambda) <$ cf$|z|$" cannot be strengthened to "cf$(\lambda) \leq$ cf$|z|$". Let κ, λ, $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$, i, M, x and z be as above. Since $|x| = \text{cf}(\lambda)$, $|x|$ is regular. Hence, we have $\text{cf}|z| = \text{cf}|x| = |x| = \text{cf}(\lambda)$. Suppose by way of contradiction that there exists $y \subseteq z$ with $|y| = |z|$ and $y \in M$. The fact that $|y| = |z|$, together with the fact that order-type $(x) = \text{cf}(\lambda)$, tells us that the elements of y are unbounded in the sequence $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$. Hence, since $y \in M$, it follows, by taking all restrictions of elements of y, that $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle \in M$. We then proceed exactly as above to obtain a contradiction.

3 The hierarchy The n-huge cardinals result from natural strengthenings of hugeness. They fit into a nice hierarchy in the following two ways, where we assume $m < n < \omega$, and $i: V \to M$ witnesses that κ is n-huge (see [11] for relevant definitions and techniques):

a. $\{\gamma < \kappa : \gamma \text{ is } m\text{-huge}\} \in U$, where U is the normal ultrafilter on κ induced by i.
b. Call $\lambda = i(\kappa)$ the first target for κ. Then, $\{\eta < \lambda : \eta$ is a first target for κ by some elementary embedding witnessing that κ is m-huge$\} \in W$, where W is a normal ultrafilter on λ which is induced by i in a natural way. Similar results hold for the second target, third target, etc.

In this section, we shall show that R-hugeness yields an analogous hierarchy below hugeness. We first note that standard methods show that hugeness and almost hugeness are related in the above manner. That is, if κ is huge with target λ, then almost every $\gamma < \kappa$ is almost huge, and for almost every $\eta < \lambda$, κ is almost huge with target η.

Theorem 6 *Suppose κ is $R(\lambda)$-huge, where* $\mathrm{cf}(\lambda) > \aleph_0$. *Let $i: V \to M$ be the elementary embedding and inner model obtained from some CSNU $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$, and let U be the associated normal ultrafilter on κ. Then,*
(a) i. $\{\gamma < \kappa : \{\eta < \kappa : \gamma$ *is* $R(\eta)$*-huge*$\}$ *contains a closed and unbounded subset of* $\kappa\} \in U$.
ii. $\{\gamma < \kappa : \{\eta < \lambda : \gamma$ *is* $R(\eta)$*-huge*$\}$ *contains a closed and unbounded subset of* $\lambda\} \in U$.
(b) $\{\eta < \lambda : \kappa$ *is* $R(\eta)$*-huge*$\}$ *contains a closed and unbounded subset of* λ.

Proof: Let $i: V \to M$, $\langle U_\gamma : \kappa \leq \gamma < \lambda \rangle$, and U be as in the statement of the theorem.

We prove Part b first. Let $C = \{\eta < \lambda : \eta$ is a limit cardinal and $\langle U_\gamma : \kappa \leq \gamma < \eta \rangle$ satisfies EC$\}$. By the Main Lemma, C is closed and unbounded.

It suffices to show that if $\eta \in C$, then κ is $R(\eta)$-huge. Fix some $\eta \in C$. Then $\langle U_\gamma : \kappa \leq \gamma < \eta \rangle$ is a CSNU satisfying EC. If $\mathrm{cf}(\eta) > \aleph_0$, then, by Theorem 2, κ is $R(\eta)$-huge. If $\mathrm{cf}(\eta) = \aleph_0$, then we must show that the direct limit of the system of inner models and elementary embeddings associated with $\langle U_\gamma : \kappa \leq \gamma < \eta \rangle$ is well-founded. This is straightforward, since this direct limit embeds in a natural way into the well-founded model, M_η. Hence, the canonical elementary embedding from V into the transitive collapse of this direct limit witnesses that κ is $R(\eta)$-huge. This established b.

For Part a, we first note that if $\eta < \lambda$, then it follows immediately from Theorem 2 and Theorem 5b that κ is $R(\eta)$-huge if and only if $M \models \kappa$ is $R(\eta)$-huge. Thus, Part b implies that $M \models \{\eta < \lambda : \kappa$ is $R(\eta)$-huge$\}$ contains a closed and unbounded subset of λ. Hence, $\{\gamma < \kappa : \{\eta \leq \kappa : \gamma$ is $R(\eta)$-huge$\}$ contains a closed and unbounded subset of $\kappa\} \in U$. This establishes a.i.

Let $A \in U$ be the set given by a.i. Fix for some $\gamma \in A$. Then, $\{\eta < \kappa : \gamma$ is $R(\eta)$-huge$\}$ contains a closed and unbounded subset of κ. It follows that $M \models \{\eta < \lambda : \gamma$ is $R(\eta)$-huge$\}$ contains a closed and unbounded subset of λ. Again using Theorem 2 and Theorem 5b, this implies that (in V), $\{\eta < \lambda : \gamma$ is

$R(\eta)$-huge} contains a closed and unbounded subset of λ. Since this is true for every $\gamma \in A$, and since $A \in U$, a.ii follows.

We close this section by pointing out that the relationship between R-hugeness and supercompactness is very similar to the relationship between hugeness and supercompactness. If κ is $R(\lambda)$-huge, then it follows immediately that κ is $<\lambda$-supercompact. On the other hand, no amount of supercompactness implies any R-hugeness. To see this, suppose κ is the first R-huge cardinal, λ is the first R-huge target for κ, and $i: V \to M$ witnesses that κ is $R(\lambda)$-huge. Since $M \models \lambda$ is inaccessible, $(V_\lambda)_M \models$ ZFC + κ is supercompact + there are no R-huge cardinals. This tells us that full supercompactness does not imply any R-hugeness.

Again as in the case of hugeness, if an R-huge cardinal exists, then there must be an R-huge cardinal below any supercompact cardinal. In fact, if there is a cardinal which is R-huge and is above some supercompact cardinal, then unboundedly many cardinals below this supercompact cardinal are R-huge. The techniques here are the same as those involving hugeness and supercompactness (see [3]).

4 < R-hugeness Let us say that κ is $< R(\lambda)$-huge if and only if $\{\eta < \lambda : \kappa$ is $R(\eta)$-huge$\}$ is an unbounded subset of λ, and that κ is $< R$-huge if and only if κ is $< R(\lambda)$-huge for some λ. Theorem 6a.ii implies that if κ is $R(\lambda)$-huge and $\mathrm{cf}(\lambda) > \aleph_0$, then almost every $\gamma < \kappa$ is $< R(\lambda)$-huge. Hence, $< R$-hugeness fits into our hierarchy in a nice way.

The main result of this section is that the $< R(\lambda)$-hugeness of κ, plus an appropriate large cardinal assumption on λ, implies that κ is $R(\lambda)$-huge. This is analogous to the following result on supercompactness (see, for example, DiPrisco [6]): If κ is $< \lambda$-supercompact, and λ is measurable, then κ is λ-supercompact.

Before stating our theorem, we must first discuss the notion of weak ineffability. Given any cardinal λ, we define the weakly ineffable ideal I on λ, by specifying I^+, the collection of sets of positive measure (that is, those sets not in I) as follows: For $A \subseteq \lambda$, $A \in I^+$ if and only if, given any sequence $\langle A_\alpha : \alpha \in A\rangle$ such that $A_\alpha \subseteq \alpha$ for each $\alpha \in A$, there exists $B \subseteq A$ with B an unbounded subset of λ, satisfying that if $\alpha, \beta \in B$ with $\alpha < \beta$, then $A_\alpha = A_\beta \cap \alpha$. λ is said to be weakly ineffable if and only if the weakly ineffable ideal on λ is nontrivial (that is, if and only if $\lambda \in I^+$).

Weakly ineffable cardinals and their associated ideals were studied by Baumgartner in [5], where these cardinals are the 1-almost ineffable cardinals. We shall need the following result, which is easily proved using the methods of [5]: If λ is weakly ineffable, then λ is inaccessible and, if I is the weakly ineffable ideal on λ, then $\{\eta < \lambda : \eta$ is inaccessible$\} \in I^*$, where I^* is the collection of all sets of measure one (that is, all sets whose complements are in I).

Theorem 7 *Suppose κ is $< R(\lambda)$-huge, λ is weakly ineffable, and I is the weakly ineffable ideal on λ. If $\{\eta < \lambda : \kappa$ is $R(\eta)$-huge$\} \in I^+$, then κ is $R(\lambda)$-huge.*

Before beginning the proof, we state and prove a lemma involving trees. Recall that a λ-tree is a tree of height λ, with each level having cardinality less than

λ. We differ slightly from standard usage by considering a branch to be any (not necessarily maximal) linearly ordered subset of the tree which is closed downward (we picture trees as growing upward). For $\gamma \leq \lambda$, a γ-branch of a λ-tree is a branch of length γ in the tree (which may be an initial segment of a longer branch).

Lemma *Suppose λ is weakly ineffable, I is the weakly ineffable ideal on λ, and T is a λ-tree. In addition, suppose that for some $A \in I^+$, we are given an η-branch B_η for each $\eta \in A$. Then, there is a λ-branch B in T such that $\{\eta < \lambda : B_\eta$ is an initial segment of $B\}$ is an unbounded subset of λ.*

Proof: Let λ, I, T, A and the B_η's be as in the statement of the lemma. Since λ is weakly ineffable, λ is inaccessible. Hence, T has λ many nodes and we may assume that the nodes are named by elements of λ. In particular, we may assume that the tree ordering is a subordering of the usual ordering of the ordinals. Thus, if $\alpha, \beta < \lambda$ and α is below β in the tree ordering, then $\alpha < \beta$.

Let $D = \{\eta < \lambda : \eta$ is inaccessible$\}$. Then $D \in I^*$. Hence $A \cap D \in I^+$. For $\eta \in D$, the first η many levels of T consist precisely of the elements of η. Hence, for each $\eta \in A \cap D$, $B_\eta \subseteq \eta$. It follows from the definition of I^+ that for some $E \subseteq A \cap D$, with E unbounded in λ, we have that $\gamma, \eta \in E$ with $\gamma < \eta$ implies that $B_\gamma = B_\eta \cap \gamma$. In terms of T, this says that B_γ is an initial segment of B_η. Let $B = \bigcup_{\eta \in E} B_\eta$. Then B is a branch through T. Since E is unbounded in λ, B is a λ-branch. For each $\eta \in E$, B_η is an initial segment of B, and we have established the lemma.

Proof (of Theorem 7): Assume that κ is $< R(\lambda)$-huge, λ is weakly ineffable, and I is the weakly ineffable ideal on λ. Let $A = \{\eta < \lambda : \kappa$ is $R(\eta)$-huge), and assume that $A \in I^+$. We must show that κ is $R(\lambda)$-huge.

Let T be the tree consisting of all normal ultrafilters on $P_\kappa(\eta)$ for every η such that $\kappa \leq \eta < \lambda$, with the restriction ordering. In other words, if $\kappa \leq \gamma < \eta < \lambda$, and U_γ and U_η are normal ultrafilters on $P_\kappa(\gamma)$ and $P_\kappa(\eta)$ respectively, then U_γ is below U_η in T if and only if $U_\gamma = U_\eta \upharpoonright \gamma$. Then, a branch in T is precisely the same as a CSNU.

Since λ is inaccessible, there are λ many cardinals η with $\kappa \leq \eta < \lambda$. Hence, T has height λ. Also, since for any such η, the number of normal ultrafilters on $P_\kappa(\eta)$ is $2^{2^{\eta^{\underline{\kappa}}}}$ (see Corollary 3.9 of [11]), it follows that every level of T has less than λ many nodes and hence T is a λ-tree.

We note that a level of T consists of all normal ultrafilters on $P_\kappa(\eta)$ for some fixed η. This does not necessarily correspond with the ηth level of T. For example, the third level of T consists of all normal ultrafilters on $P_\kappa(\kappa^{++})$. However, if $\kappa < \eta < \lambda$ and η is inaccessible, then it follows that the ηth level of T consists of all normal ultrafilters on $P_\kappa(\eta)$. Let $D = \{\eta : \kappa \leq \eta < \lambda$ and η is inaccessible$\}$. We shall work with $A \cap D$ instead of A. Since $D \in I^*$, $A \cap D \in I^+$.

For each $\eta \in A$, κ is $R(\eta)$-huge. Then, by Theorem 2a, we may fix a CSNU $B_\eta = \langle U^\eta_\gamma : \kappa \leq \gamma < \eta \rangle$ satisfying EC. Each such B_η is a branch in T. If $\eta \in A \cap D$ then B_η is an η-branch in T. Since $A \cap D \in I^+$, the lemma tells us that there exists a λ-branch B in T such that if $E = \{\eta < \lambda : B_\eta$ is an initial segment of $B\}$, then E is an unbounded subset of λ.

For $\kappa \leq \gamma < \lambda$, let $B(\gamma)$ be the node of B which is a normal ultrafilter on $P_\kappa(\gamma)$. Note that $B(\gamma)$ need not be the node of B on level γ.

B is certainly a CSNU. To establish that κ is $R(\lambda)$-huge, it suffices to show that B satisfies EC. Fix γ and σ with $\kappa \leq \gamma < \lambda$ and $\gamma \leq \sigma < i_{B(\gamma)}(\kappa)$. We must find a δ such that $\gamma \leq \delta < \lambda$ and $k^B_{\gamma\delta}(\sigma) = \delta$.

Since E is an unbounded subset of λ, we can pick some $\eta \in E$ with $\eta > \gamma$. Then, since B_η is a CSNU satisfying EC, there is a δ such that $\gamma \leq \delta < \eta$ and $k^{B_\eta}_{\gamma\delta}(\sigma) = \delta$. But, since B_η is an initial segment of B, $k^{B_\eta}_{\gamma\delta} = k_{\gamma\delta}$. Hence, we have found a δ with $\gamma \leq \delta < \lambda$ and $k^B_{\gamma\delta}(\sigma) = \delta$. This establishes that B satisfies EC, and hence that κ is $R(\lambda)$-huge.

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