Notre Dame Journal of Formal Logic
Volume 32, Number 1, Winter 1991

Relevance and Paraconsistency — A New Approach. Part III: Cut-Free Gentzen-Type Systems

ARNON AVRON*

Abstract The system RMI is a purely relevance logic based on the intuitive ideas of relevance domains and degrees of significance. In this paper, we show that unlike the systems of Anderson and Belnap, RMI has a corresponding cut-free, Gentzen-type version. This version manipulates *hypersequents* (i.e. finite sequences of ordinary sequents), and no translation of those hypersequents into the language of RMI is possible. This shows that RMI is multiple-conclusioned in nature and hints on possible applications of it to the study of parallelism.

1 Introduction and background The systems RMI and $\mathrm{RMI_{min}}$ are powerful, purely intentional relevance logics that were introduced in [4]. Semantically they correspond to the algebraic structures which have been developed in [3] following the intuitive ideas of relevance domains, relevance relations, and degrees of reality (or of significance). Our main goal in this paper is to show that, unlike the systems of Anderson and Belnap in [1], RMI and $\mathrm{RMI_{min}}$ have corresponding cut-free, Gentzen-type versions. The existence of such versions is significant from the proof-theoretical point of view and has obvious importance for the task of developing automated reasoning systems that will be sensitive to considerations of relevance and paraconsistency. Even more important, perhaps, is the fact that in the case of RMI, the corresponding Gentzen-type version manipulates *hypersequents* (i.e., finite sequences of ordinary sequents) rather than ordinary sequents. Unlike the case of RM (which we pursued in [5] using similar techniques), in the present case no translation of hypersequents into sentences of the language is possible. Together with the results of section E of [4]

*I would like to thank Prof. H. Gaifman and Dr. Y. Hirschfeld for their help and encouragement while working on this subject.

Received February 1986; revised April 1989

this shows that RMI is inherently multiple-conclusioned in nature. This fact might hint on possible applications (of the kind suggested in [7] for Linear Logic) of RMI to the study of parallelism in computer science (such applications of hypersequential systems in general is the topic of a forthcoming paper).

For the reader's convenience we review now the main notions and results from [4] and [2] that we shall need.

The system RMI

Primitive connectives: $\sim, \rightarrow, \wedge, \vee$.

Defined connectives:

$$A + B =_{Df} (\sim A \rightarrow B)$$

$$R^{+}(A,B) =_{Df} (A \rightarrow A) + (B \rightarrow B)$$

$$R^{\wedge}(A,B) =_{Df} (A \rightarrow A) \wedge (B \rightarrow B)$$

$$A \supset B =_{Df} B \vee (A \rightarrow B).$$

Axioms:

A1 $A \rightarrow (A \rightarrow A)$
A2 $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
A3 $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
A4 $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
A5 $\sim\sim A \rightarrow A$
A6 $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$
A7 $A \wedge B \rightarrow A$
A8 $A \wedge B \rightarrow B$
A9 $(A \rightarrow B) \wedge (A \rightarrow C) \rightarrow (A \rightarrow B \wedge C)$
A10 $A \rightarrow A \vee B$
A11 $B \rightarrow A \vee B$
A12 $(A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C)$.

Rules of inference:

RI $A, A \rightarrow B \vdash B$
RII $R^{+}(B,C), B, C \vdash B \wedge C$
RIII $R^{+}(B,C) \vdash A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$.

Some fundamental properties of RMI are summarized in the following propositions. The (easy) proofs can be found in [4].

Proposition 1
(1) *In the above formulation Rule II can be replaced by*:

(Re. Adj) $A \rightarrow B, A \rightarrow C \vdash A \rightarrow B \wedge C$.

(2) *Rule III can be replaced by the axiom*:

(RD) $R^{\wedge}(B,C) \supset [A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)]$.

(3) *As usual* $A \vee B$ *is equivalent to* $\sim(\sim A \wedge \sim B)$.
(4) *The classical deduction theorem holds in* RMI *for* $\supset$.

Important subsystems:

(1) The system $\mathrm{RMI}_{\min}$ is RMI without Rule III.

(2) The system $\mathrm{RMI}_{\overset{\sim}{\to}}$ is the system in the $\{\sim,\rightarrow\}$ language with Axioms 1–6 and Rule RI.

For the case of $\mathrm{RMI}_{\overset{\sim}{\to}}$ the following Gentzen-type system was already provided in [2]:

The system $GRMI_{\overset{\sim}{\to}}$

Axioms: $\overbrace{p,p,\ldots,p}^{m\text{ times}} \Rightarrow \overbrace{p,p,\ldots,p}^{n\text{ times}}$ (p atomic, $m > 0$, $n > 0$).

Rules of Inference: Exchange, contraction and the classical Gentzen's logical rules for negation ($\sim$) and implication ($\rightarrow$).

The most important results from [2] concerning $\mathrm{GRMI}_{\overset{\sim}{\to}}$ are reviewed in the next proposition. Their proofs in [2] are rather standard.

Proposition 2

(1) $\mathrm{GRMI}_{\overset{\sim}{\to}}$ *is closed under the following rules*:

(a) *Anticontraction*:

$$\frac{A,\Gamma \Rightarrow \Delta}{A,A,\Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta,A}{\Gamma \Rightarrow \Delta,A,A}$$

(b) *Weak expansion*:

$$\frac{\Gamma \Rightarrow \Delta}{A,\Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,A},$$

provided all atomic variables of A *occur in* $\Gamma \Rightarrow \Delta$.

(c) *Relevant mingle*:

$$\frac{\Gamma_1 \Rightarrow \Delta_1,A \qquad \Gamma_2 \Rightarrow \Delta_2,A}{\Gamma_1,\Gamma_2 \Rightarrow \Delta_1,\Delta_2,A} \quad \frac{A,\Gamma_1 \Rightarrow \Delta_1 \qquad A,\Gamma_2 \Rightarrow \Delta_2}{A,\Gamma_1,\Gamma_2 \Rightarrow \Delta_1,\Delta_2}$$

(d) *Relevant combining*:

$$\frac{\Gamma \Rightarrow \Delta \qquad \Gamma' \Rightarrow \Delta'}{\Gamma,\Gamma' \Rightarrow \Delta,\Delta'}$$

provided $\Gamma \Rightarrow \Delta$ *and* $\Gamma' \Rightarrow \Delta'$ *share a variable in common.*

(2) *The cut-elimination theorem holds for* $\mathrm{GRMI}_{\overset{\sim}{\to}}$

(3) $\mathrm{RMI}_{\overset{\sim}{\to}}$ *and* $\mathrm{GRMI}_{\overset{\sim}{\to}}$ *are equivalent.*

(4) *The interpolation theorem*: *if* $\vdash_{\mathrm{RMI}_{\overset{\sim}{\to}}} A \rightarrow B$ *then there is a sentence* C *having only variables common to* A *and* B *such that* $\vdash_{\mathrm{RMI}_{\overset{\sim}{\to}}} A \rightarrow C$ *and* $\vdash_{\mathrm{RMI}_{\overset{\sim}{\to}}} C \rightarrow B$.

II Gentzen-type formulations for RMI_{min} We start with $\mathrm{RMI}_{\min}$. We present two alternative Gentzen-type calculi for it:

The system $GRMI^{(I)}_{min}$

Axioms: $A, A, \ldots, A \Rightarrow A, A, \ldots, A$ (A-any sentence).

Rules of Inference:
(1) All rules of $GRMI_{\rightleftarrows}$
(2) Cut
(3) $(\wedge \Rightarrow)$: $\dfrac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \quad \dfrac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$

(4) $(\Rightarrow \wedge)$: $\dfrac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$ (If $\Gamma \cup \Delta \neq \varnothing$).

Note: In contrast to $(+ \Rightarrow)$ or $(\rightarrow \Rightarrow)$, in $(\Rightarrow \wedge)$ the two premises should have the *same side formulas*, and in contrast to the corresponding rule of the sequential calculus for the system R without distribution, such side formulas *must exist* (i.e., from $\Rightarrow A$ and $\Rightarrow B$ we cannot infer $\Rightarrow A \wedge B$).

Lemma 1 $GRMI^{(I)}_{min}$ *is closed under the following rules:*
(a)

$$(\Rightarrow \vee)\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \quad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$(\vee \Rightarrow)\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \text{ (if } \Gamma \cup \Delta \neq \varnothing)$$

(b) Anticontraction (see Proposition 2 above).
(c) Relevant mingle (see Proposition 2 above).

Proposition 3 RMI_{min} *and* $GRMI^{(I)}_{min}$ *are equivalent.*

The proofs are left to the reader.

Note: Using full r.d.s.'s (see [3]) it can be easily shown that $GRMI^{(I)}_{min}$ is not closed under weak expansion and under relevant combining. For example, although $A \wedge B \Rightarrow A \wedge B$ and $A \wedge C \Rightarrow A \wedge C$ are provable, $A, A \wedge B \Rightarrow A \wedge B$ and $A \wedge B, A \wedge C \Rightarrow A \wedge B, A \wedge C$ are not.

The system $GRMI^{(I)}_{min}$ has a major drawback: Cut-elimination fails for it. For example: $A \wedge B \Rightarrow A, B$ is provable in it (since $\vdash \Rightarrow A \wedge B \rightarrow A$, $\vdash \Rightarrow A \wedge B \rightarrow B$ and since $\vdash_{GRMI_{\rightarrow}} C, C \rightarrow A, C \rightarrow B \Rightarrow A, B$), but it is easy to see that no cut-free proof of this sequent is possible. We introduce therefore another version, for which cut-elimination does obtain:

The system $GRMI_{min}$

Axioms: $p \Rightarrow p$ (p atomic).

Rules: As in $GRMI^{(I)}_{min}$; and in addition, also the two relevant mingle rules.

Lemma 2
(1) $\vdash_{GRMI_{min}} A \Rightarrow A$ *for any sentence A.*
(2) *Anticontraction is an admissible rule of* $GRMI_{min}$.

Proof: (1) By induction on the length of A. (2) By a mingle of $A,\Gamma \Rightarrow \Delta$ (or $\Gamma \Rightarrow \Delta, A$) with itself, followed by contractions.

Proposition 4 $\quad$ $\mathrm{GRMI_{min}}$ *and* $\mathrm{GRMI^{(I)}_{min}}$ *are equivalent.*

Proof: That $\mathrm{GRMI_{min}} \subseteq \mathrm{GRMI^{(I)}_{min}}$ follows from Lemma 1. For the converse we note that by Lemma 2 all axioms of $\mathrm{GRMI^{(I)}_{min}}$ are theorems of $\mathrm{GRMI_{min}}$. From this our proposition immediately follows.

Theorem 1 $\quad$ *The cut-elimination theorem is true for* $\mathrm{GRMI_{min}}$.

Proof: We start by showing that cut-elimination is equivalent in $\mathrm{GRMI_{min}}$ to mix-elimination. That cut-elimination entails mix-elimination is obvious, because of the rules of exchange and of contraction. For the converse the weakening rule is usually used, but in the present case it is not available. Instead we use the relevant mingle rule in the following way: Suppose $\vdash A,\Gamma_1 \Rightarrow \Delta_1$ and $\vdash \Gamma_2 \Rightarrow \Delta_2, A$. (Here '$\vdash$' means '$\underset{\mathrm{GRMI_{min}}}{\vdash}$'.) We repeatedly apply mingles of both sequents with $A \Rightarrow A$ (and contractions), until we derive $A,\overbrace{A,\ldots,A}^{n \text{ times}}, \Gamma_2 \Rightarrow \Delta_2, A$ and $A,\Gamma_1 \Rightarrow \Delta_1,\overbrace{A,\ldots,A}^{m \text{ times}}$, where m and n are the number of times A occurs in Δ_2 and Γ_1, respectively. A mix of both sequents, followed by exchanges, gives $\Gamma_1,\Gamma_2 \Rightarrow \Delta_1,\Delta_2$.

It remains, therefore, to prove that every mix is eliminable. Applying the usual method for this is not so simple, though, since the limitations on applications of the $(\Rightarrow \wedge)$ and mingle rules cause difficulties. For example: if $\Rightarrow\varphi$, $\varphi \Rightarrow A$ and $\varphi \Rightarrow B$ are all provable, we can apply $(\Rightarrow \wedge)$ first and then a cut to obtain $\Rightarrow A \wedge B$. We cannot reverse the order of these steps, however, since from $\Rightarrow A$ and $\Rightarrow B$ we cannot infer $\Rightarrow A \wedge B$. (The mingle rules cause similar problems.) The key for the solution of these problems is given in the following:

Definition $\quad$ By an *A-supermix* we mean the following rule of inference: From $\Gamma_1 \Rightarrow \Delta_1; \Gamma_1 \Rightarrow \Delta_2; \ldots; \Gamma_k \Rightarrow \Delta_k$, where $k > 0$ and $A \in \bigcap_{i=1}^{k} \Delta_i$, and from $\Gamma'_1 \Rightarrow \Delta'_1; \Gamma'_2 \Rightarrow \Delta'_2; \ldots; \Gamma'_n \Rightarrow \Delta'_n$, where $n > 0$ and $A \in \bigcap_{i=1}^{n} \Gamma'_i$, to infer: $\Gamma_1,\ldots,\Gamma_k, \Gamma_1'^{*},\ldots,\Gamma_n'^{*} \Rightarrow \Delta_1^{*},\Delta_2^{*},\ldots,\Delta_k^{*},\Delta'_1,\ldots,\Delta_{n'}$, where $\Gamma_i'^{*}(\Delta_i^{*})$ is $\Gamma'_i(\Delta_i)$ without some (perhaps all) of its A's.

Two other concepts that we need are defined as follows:

(1) The *complexity* of an A-supermix is the number of connectives occurring in A.
(2) The *rank* of an A-supermix is the sum of the *heights* of its premises, where the height of a sequent in a given proof is defined in such a way that the height of a conclusion of an inference is greater than the sum of the heights of its premises (the definitions exact details are not important).

We now prove by a double induction on the complexity and the rank of the supermix the following:

Main Lemma $\quad$ *If every premise of a given A-supermix has a mix-free proof, then its conclusion has a mix-free proof too.*

Case a: The last rule applied in the proof of one of the premises ($\Gamma_1' \Rightarrow \Delta_1'$, say) is neither a logical rule having A as the principal formula, nor an axiom $A \Rightarrow A$.

In a case like this we usually apply first the Induction Hypothesis (I.H.) separately to each of the premises of $\Gamma_1' \Rightarrow \Delta_1'$ containing A (together with the other premises of the supermix). In this way we get one or two supermixes, each of which has the same complexity as the given one, but a lower rank. We then apply the last inference in the proof of $\Gamma_1' \Rightarrow \Delta_1'$ to the conclusions of those supermixes. A problem arises, however, when this last step is no longer possible. This may happen in the following two cases, and so another procedure is needed for them:

Subcase a.1: $\Gamma_1' \Rightarrow \Delta_1'$ was derived by a mingle from $\Gamma_{1,1}' \Rightarrow \Delta_{1,1}'$ and $\Gamma_{1,2}' \Rightarrow \Delta_{1,2}'$, and A is the only sentence which makes this mingle possible.

In this case we apply the I.H. to the single A-supermix of $\Gamma_{1,1}' \Rightarrow \Delta_{1,1}'$, $\Gamma_{1,2}' \Rightarrow \Delta_{1,2}'$ and the other premises of the given supermix (this new supermix has the same complexity as the given one but a smaller rank, so the I.H. is indeed applicable). The conclusion of this supermix is identical to that of the given one.

Subcase a.2: $\Gamma_1' \Rightarrow \Delta_1'$ has the form $A,A,\dots,A \Rightarrow B \wedge C$ (and its premises are $A,A,\dots,A \Rightarrow B$ and $A,A,\dots,A \Rightarrow C$), while the other premises of the supermix are of the form $A,A,\dots,A \Rightarrow$ and $\Rightarrow A,A,\dots,A$.

In this case we first apply the I.H. to $A,A,\dots,A \Rightarrow B$ and to the other premises of the given supermix. We get $\Rightarrow B$. From this $\Rightarrow B,B$ follows by a relevant mingle. $\Rightarrow C,C$ can be obtained similarly from $A,A,\dots,A \Rightarrow C$. In addition, we obtain $\Rightarrow B,C$ by applying the I.H. to $A,A,\dots,A \Rightarrow B$ and $A,A,\dots,A \Rightarrow C$ simultaneously (together with the other premises of the given supermix). All three supermixes have a smaller rank than that of the given one. Now from $\Rightarrow B,B$ and $\Rightarrow B,C$ we derive $\Rightarrow B,B \wedge C$. $\Rightarrow C,B \wedge C$ is derived similarly. Applying ($\Rightarrow \wedge$) once more we get $\Rightarrow B \wedge C$, $B \wedge C$ and then $\Rightarrow B \wedge C$, as desired.

Case b: A is atomic and Case a does not obtain.

In this case all the premises of the given supermix are axioms of the form $A \Rightarrow A$, which therefore is also the supermix's conclusion.

Case c: A is not atomic and in all the proofs of the premises of the given supermix the last inference is logical with A as the principal formula.

There are many subcases to deal with in this case. As an illustration, we take the most difficult of them, namely: $A = B \wedge C$.

Subcase c.1: A occurs in the succedent of one of the two premises of $\Gamma_1 \Rightarrow \Delta_1$ (say). By applying then the I.H. separately to each of these premises (together with the other premises of the given supermix), we get $\Gamma \Rightarrow \Delta, B$ and $\Gamma \Rightarrow \Delta, C$ (where $\Gamma \Rightarrow \Delta$ is the conclusion of the given supermix). Suppose, for example, that $\Gamma_1' \Rightarrow \Delta_1'$ was inferred from $B,\Gamma_1'' \Rightarrow \Delta_1'$ (where $\Gamma_1' = B \wedge C,\Gamma_1''$). We may assume that $A \notin \Gamma_1''$ (otherwise we first apply a standard treatment of the kind given in [6]). It follows that a B-mix of $B,\Gamma_1'' \Rightarrow \Delta_1'$ and of $\Gamma \Rightarrow \Delta, B$ (followed by some exchanges and contractions) gives $\Gamma \Rightarrow \Delta$ (note that $\Gamma_1'' \subseteq \Gamma$, $\Delta_1' \subseteq \Delta$). Since this mix has a smaller complexity than the given one, we can eliminate it by the I.H.

Subcase c.2: All the $\Gamma_i \Rightarrow \Delta_i$ are of the form $\Gamma_i \Rightarrow \Delta_i^*, A$ (where $A \notin \Delta_i^*$) and were derived by $\Rightarrow \wedge$ from $\Gamma_i \Rightarrow \Delta_i^*, B$ and $\Gamma_i \Rightarrow \Delta_i^*, C$ (and so $\Gamma_i \cup \Delta_i^* \neq \emptyset$). $2k$ mingles will then give $\Gamma_1, \ldots, \Gamma_k \Rightarrow \Delta_1^*, \Delta_2^*, \ldots, \Delta_k^*, B$ and $\Gamma_1, \ldots, \Gamma_k \Rightarrow \Delta_1^*, \ldots, \Delta_k^*, C$ (where $\bigcup_{i=1}^k \Gamma_i \cup \bigcup_{i=1}^k \Delta_i^* \neq \emptyset$ and $A = B \wedge C \notin \bigcup_{i=1}^k \Delta_i^*$). Again we may assume that each $\Gamma_j' \Rightarrow \Delta_j' (j = 1, \ldots, n)$ was inferred by $\wedge \Rightarrow$ from either $B, \Gamma_j' \Rightarrow \Delta_j'$ or $C, \Gamma_j'^* \Rightarrow \Delta_j'$, where $A \notin \Gamma_j'^*$. We can apply, therefore, a B-supermix to all the sequents of the form $B, \Gamma_j'^* \Rightarrow \Delta_j'$ (if such exist) and to $\Gamma_1, \ldots, \Gamma_k \Rightarrow \Delta_1^* \ldots \Delta_k^*, B$. A similar treatment, using a C-supermix, can be given to the premises of the form $C, \Gamma_j'^* \Rightarrow \Delta_j'$. Finally, by applying mingle to the two sequents obtained (if really there are two), followed by some exchanges and contractions, we get $\Gamma \Rightarrow \Delta$ (the mingle is possible since $\bigcup_{i=1}^k \Gamma_i \cup \bigcup_{i=1}^k \Delta_i^* \neq \emptyset$, and its sentences are common to the two sequents involved).

Corollaries
(1) $GRMI_{min}$ *has the subformula property.*
(2) RMI_{min} *is decidable.*
(3) *The interpolation theorem of Proposition* 2.4 *is true also for* RMI_{min}.
(4) RMI_{min} *has the variable-sharing property with respect to* $+$, $\rightarrow$, *and* $\wedge$.

Proof: The proofs of (1) and (2) are standard.

(3) By Maehara method (see [6]). We illustrate here the case of the mingle rule. Suppose that $C, \Gamma_1', \Gamma_2', \Gamma_1'', \Gamma_2'' \Rightarrow \Delta_1', \Delta_2', \Delta_1'', \Delta_2''$ is inferred from $C, \Gamma_1', \Gamma_1'' \Rightarrow \Delta_1', \Delta_1''$ and $C, \Gamma_2', \Gamma_2'' \Rightarrow \Delta_2', \Delta_2''$ by a mingle, and that $\Gamma_1'' \cup \Gamma_2'' \cup \Delta_1'' \cup \Delta_2'' \neq 0$. We show how to construct an interpolant for $C, \Gamma_1', \Gamma_2' \Rightarrow \Delta_1', \Delta_2'$ and $\Gamma_1'', \Gamma_2'' \Rightarrow \Delta_1'', \Delta_2''$. Without a loss in generality we may assume that $\Gamma_1'' \cup \Delta_1'' \neq \emptyset$. By the I.H. (applied to $C, \Gamma_2', \Gamma_2'' \Rightarrow \Delta_2', \Delta_2''$) there exists an interpolant A such that

(i) $\vdash_{GRMI_{min}} A, \Gamma_1'' \Rightarrow \Delta_1''$
(ii) $\vdash_{GRMI_{min}} C, \Gamma_1' \Rightarrow \Delta_1', A$.

Now, if $\Gamma_2'' \cup \Delta_2'' = \emptyset$, then $\vdash C, \Gamma_2' \Rightarrow \Delta_2'$, and a mingle of this sequent and of (ii) yields: $\vdash C, \Gamma_1', \Gamma_2' \Rightarrow \Delta_1', \Delta_2'$. Together with (i) this implies that A is an appropriate interpolant.

If, on the other hand, $\Gamma_2'' \cup \Delta_2'' \neq \emptyset$ then by I.H. there exists an interpolant B such that:

(iii) $\vdash B, \Gamma_2'' \Rightarrow \Delta_2''$ and
(iv) $\vdash C, \Gamma_2' \Rightarrow \Delta_2', B$.

A mingle of (ii) and of (iv), followed by $(\Rightarrow +)$ and an application of $(+ \Rightarrow)$ to (i) and (iii) together yield a demonstration that $A + B$ is an appropriate interpolant in this case.

(4) This follows immediately from (3) in the case of $\rightarrow$ and $+$. The case of $\wedge$ follows from that of $+$, since $\vdash_{RMI_{min}} A \wedge B \rightarrow A + B$ (see the note after Proposition 3).

III A hypersequential formation of RMI In this section we finally present a Gentzen-type calculus for RMI. Although $GRMI_{min}$ *is* its "hard core", this new calculus is much more complex than $GRMI_{min}$, since it deals with finite *sequences of sequents*. The new calculus does not correspond directly to RMI, but rather to the stronger multiple-conclusioned version which we have introduced in section E of [4].

Definition

(1) A *hypersequent* is a formal creature of the form: $\Gamma_1 \Rightarrow \Delta_1 \between \Gamma_2 \Rightarrow \Delta_2 \between \cdots \between \Gamma_n \Rightarrow \Delta_n$ $(n \geq 0)$, where Γ_i and Δ_i are finite sequences of formulas in RMI language.

(2) We call $\Gamma_i \Rightarrow \Delta_i (i = 1, \ldots, n)$ the *components* of the hypersequent $\Gamma_1 \Rightarrow \Delta_1 \between \Gamma_2 \Rightarrow \Delta_2 \between \ldots \between \Gamma_n \Rightarrow \Delta_n$.

Note: Our definition of a hypersequent allows a hypersequent with no components. We call this hypersequent the *empty hypersequent*. This should be distinguished from the empty *sequent*.

Notation: We use L, K, G as metavariables for hypersequents.

The system GRMI

Axioms: $p \Rightarrow p$ (p atomic).

Rules of inference:

(I) All the rules of $\mathrm{GRMI}_{\min}$ (including cut) but with "side" hypersequents allowed. For example, $(+ \Rightarrow)$ takes the following form:

$$\frac{G_1 \between A, \Gamma_1 \Rightarrow \Delta_1 \between K_1 \qquad G_2 \between B, \Gamma_2 \Rightarrow \Delta_2 \between K_2}{G_1 \between G_2 \between A + B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \between K_1 \between K_2}.$$

(II) External structural rules:

(II.1) $$\frac{G \between L}{G \between K \between L} \quad \text{(External weakening)}$$

(II.2) $$\frac{G \between \Gamma_1 \Rightarrow \Delta_1 \between K}{G \between \Gamma_2 \Rightarrow \Delta_2 \between \Gamma_1 \Rightarrow \Delta_1 \between K} \quad \text{(External exchange)}$$

(II.3) $$\frac{G \between \Gamma \Rightarrow \Delta \between \Gamma \Rightarrow \Delta \between K}{G \between \Gamma \Rightarrow \Delta \between K} \quad \text{(External contraction)}$$

(III) Splitting:

$$\frac{G \between \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 \between K}{G \between \Gamma_1 \Rightarrow \Delta_1 \between \Gamma_2 \Rightarrow \Delta_2 \between K}.$$

Definition We say that a sentence A is *provable in* GRMI ($\vdash_{\mathrm{GRMI}} A$) iff the sequent $\Rightarrow A$ is derivable in GRMI.

Theorem 2 $\vdash_{\mathrm{RMI}} \varphi$ *iff* $\vdash_{\mathrm{GRMI}} \varphi$.

Proof: To show that $\vdash_{\mathrm{RMI}} \varphi$ implies $\vdash_{\mathrm{GRMI}} \varphi$ it suffices to derive all the axioms in GRMI, and to show that the set of sentences provable in GRMI is closed under RMI's rules. Now the axioms are easily derivable already in $\mathrm{GRMI}_{\min}$. Also provable there is the sequent $A, A \rightarrow B \Rightarrow B$. Hence it is obvious (using cuts) that the above set is closed under M.P. for $\rightarrow$.

Suppose now that $\vdash_{\mathrm{GRMI}} A$, $\vdash_{\mathrm{GRMI}} B$ and $\vdash_{\mathrm{GRMI}} R^+(A,B)$; we show $\vdash_{\mathrm{GRMI}}$

$A \wedge B$. Now $\vdash_{GRMI} R^+(A,B)$ means (using cuts) that $\vdash_{GRMI} A,B \Rightarrow A,B$, and $\vdash_{GRMI} A$, $\vdash_{GRMI} B$ means that $\vdash_{GRMI} \Rightarrow A$ and $\vdash_{GRMI} \Rightarrow B$. Using cuts and anticontraction we can derive from these three sequents $\vdash_{GRMI} \Rightarrow A,B$; $\vdash_{GRMI} \Rightarrow A,A$ and $\vdash_{GRMI} \Rightarrow B,B$. From this $\vdash_{GRMI} \Rightarrow A \wedge B, A \wedge B$ (and then $\vdash_{GRMI} A \wedge B$) easily follows.

For the third and last rule we must show that if $\vdash_{GRMI} R^+(B,C)$ (equivalently: if $\vdash_{GRMI} B,C \Rightarrow B,C$) then $\vdash_{GRMI} A \wedge (B \vee C) \rightarrow. (A \wedge B) \vee (A \wedge C)$. Denote the last sentence by D. The following is a proof in GRMI of $\Rightarrow D$. starting from $B,C \Rightarrow B,C$:

$$B,C \Rightarrow B,C$$

(Splitting)

$$B \Rightarrow C \veebar C \Rightarrow B \qquad B \Rightarrow B$$

$(\vee \Rightarrow)$

$$B \Rightarrow C \veebar B \vee C \Rightarrow B \qquad C \Rightarrow C$$

$(\vee \Rightarrow)$

$$B \vee C \Rightarrow C \veebar B \vee C \Rightarrow B \qquad\qquad A \Rightarrow A$$

$(\wedge \Rightarrow)$

$$A \wedge (B \vee C) \Rightarrow C \veebar A \wedge (B \vee C) \Rightarrow B \qquad A \wedge (B \vee C) \Rightarrow A$$

$(\Rightarrow \wedge)$

$$A \wedge (B \vee C) \Rightarrow A \wedge C \veebar A \wedge (B \vee C) \Rightarrow A \wedge B$$

$(\Rightarrow \vee)$

$$A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C) \veebar A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$$

$(\Rightarrow \rightarrow)$

$$\Rightarrow D \veebar \Rightarrow D$$

(External contraction)

$$\Rightarrow D.$$

For the converse, suppose $\vdash_{GRMI} \Rightarrow A$. We show that $\vdash_{RMI} A$. For this we associate with every hypersequent G, not containing the empty sequent $(\Rightarrow)$ as a component, an interpretation, φ_G, defined as follows: If $\Gamma \Rightarrow \Delta$ is $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ then $\varphi_{\Gamma \Rightarrow \Delta}$ is $\sim A_1 + \sim A_2 + \cdots + \sim A_m + B_1 + \cdots + B_n$. If $G = \Gamma_1 \Rightarrow \Delta_1 \veebar \cdots \veebar \Gamma_k \Rightarrow \Delta_k$ then $\varphi_G = \varphi_{\Gamma_1 \Rightarrow \Delta_1} \vee \cdots \vee \varphi_{\Gamma_n \Rightarrow \Delta_n}$. (Note that φ_G is a sentence in the language of RMI.) We now show by induction that if $\vdash_{GRMI} G$ (G as above) then $\vdash_{RMI} \varphi_G$. From this our claim immediately follows. Now, for the rules of group (I) we just need to use the fact that if $A_1, \ldots, A_n \vdash B$ then $C \vee A_1, C \vee A_2, \ldots, C \vee A_n \vdash_{RMI} C \vee B$ (see the proof in [4] of the completeness theorem for RMI). In the case of the cut rule the assumption that G does not contain the empty sequent is essential for this. The rules of group II are also easy to deal with. Finally, for splitting we use the above fact and the fact that $\vdash_{RMI} (A + B) \rightarrow A \vee B$.

Note: It may happen that $\vdash_{\mathrm{RMI}} \varphi_G$ but $\nvdash_{\mathrm{GRMI}} G$. For example: $\vdash_{\mathrm{RMI}} (p \supset q) \vee (q \supset p)$ but $\nvdash_{\mathrm{GRMI}} \Rightarrow p \supset q \vee\!\!\!\vee \Rightarrow q \supset p$.

Theorem 3 *GRMI admits cut-elimination.*

Proof: The proof is based, essentially, on that for $\mathrm{GRMI}_{\min}$, but the additional rules (especially splitting and external contraction) cause new serious complications. In order to overcome these new difficulties, we introduce a new concept: Let D be a proof of $G \vee\!\!\!\vee \Gamma \Rightarrow \Delta \vee\!\!\!\vee K$, and let $\Gamma^0 \subseteq \Gamma, \Delta^0 \subseteq \Delta$. Then $D^{G/\Gamma\Rightarrow\Delta/K}_{\Gamma^0\Rightarrow\Delta^0}$, the *history* of $\Gamma^0 \Rightarrow \Delta^0$ *in* D, is defined as follows:

Case a: $G \vee\!\!\!\vee \Gamma \Rightarrow \Delta \vee\!\!\!\vee K$ is an axiom, or it results from $G \vee\!\!\!\vee K$ by an external weakening. Then $D^{G/\Gamma\Rightarrow\Delta/K}_{\Gamma^0\Rightarrow\Delta^0}$ is $\Gamma^0 \Rightarrow \Delta^0$.

Case b: The indicated $\Gamma \Rightarrow \Delta$ is not involved in the last inference of D, or this inference is a splitting or an external exchange, or the sentences of Γ^0 and Δ^0 are not involved in this inference, or this inference is an internal exchange of a sentence in Γ^0 (or Δ^0) with a sentence not in Γ^0 (or Δ^0). Then $D^{G/\Gamma\Rightarrow\Delta/K}_{\Gamma^0\Rightarrow\Delta^0}$ is the same as the history of $\Gamma^0 \Rightarrow \Delta^0$ in the subproof of D of that premise of the last inference of D, from which the indicated $\Gamma \Rightarrow \Delta$ comes.

Case c: D has the form:

$$\frac{D'\left\{G \vee\!\!\!\vee \Gamma^{0\prime};\Gamma_1 \overset{\vdots}{\Rightarrow} \Delta^{0\prime};\Delta_1 \vee\!\!\!\vee K\right.}{G \vee\!\!\!\vee \Gamma^0;\Gamma_1 \Rightarrow \Delta^0;\Delta_1 \vee\!\!\!\vee K}$$

(where $\Gamma^0;\Gamma^1$ is some merge of Γ^0 and Γ^1). Then $D^{G/\Gamma\Rightarrow\Delta/K}_{\Gamma^0\Rightarrow\Delta^0}$ is:

$$\begin{array}{c} D'^{\,G/\Gamma^{0\prime};\Gamma_1\Rightarrow\Delta^{0\prime};\Delta_1/K}_{\Gamma^{0\prime}\Rightarrow\Delta^{0\prime}} \\ | \\ \Gamma^0 \Rightarrow \Delta^0. \end{array}$$

Case d: D has the form:

$$\frac{D_1\left\{G_1 \vee\!\!\!\vee \Gamma^0_1;\Gamma'_1 \overset{\vdots}{\Rightarrow} \Delta^0_1;\Delta'_1 \vee\!\!\!\vee K_1\right. \qquad D_2\left\{G_2 \vee\!\!\!\vee \Gamma^0_2;\Gamma'_2 \overset{\vdots}{\Rightarrow} \Delta^0_2;\Delta'_2 \vee\!\!\!\vee K_2\right.}{G \vee\!\!\!\vee \Gamma \Rightarrow D \vee\!\!\!\vee K}$$

(where $G = G_1 \vee\!\!\!\vee G_2$, $K = K_1 \vee\!\!\!\vee K_2$, $(\Gamma \Rightarrow \Delta) = (\Gamma^0;\Gamma'_1;\Gamma'_2 \Rightarrow \Delta^0;\Delta'_1;\Delta'_2))$. Then $D^{G/\Gamma\Rightarrow\Delta/K}_{\Gamma^0\Rightarrow\Delta^0}$ is:

$$\begin{array}{ccc} D^{\Gamma_1/\Gamma^0_1;\Gamma'_1\Rightarrow\Delta^0_1;\Delta'_1/K_1}_{1_{r^0_1\Rightarrow\Delta^0_1}} & & D^{G_2/\Gamma^0_2;\Gamma'_2\Rightarrow\Delta^0_2;\Delta'_2/K_2}_{2_{\Gamma^0_2\Rightarrow\Delta^0_2}} \\ \diagdown & & \diagup \\ & \Gamma^0 \Rightarrow \Delta^0. & \end{array}$$

Case e: D has the form:

$$\frac{D'\left\{G \vee\!\!\!\vee \Gamma \Rightarrow \Delta \overset{\vdots}{\vee\!\!\!\vee} \Gamma \Rightarrow \Delta \vee\!\!\!\vee K\right.}{G \vee\!\!\!\vee \Gamma \Rightarrow \Delta \vee\!\!\!\vee K}.$$

We have here two subcases to consider:

Subcase e.1:

$$D'^{\,G\vee\!\!\!\vee\Gamma\Rightarrow\Delta/\Gamma\Rightarrow\Delta/K}_{\Gamma^0\Rightarrow\Delta^0} = D'^{\,G/\Gamma\Rightarrow\Delta/\Gamma\Rightarrow\Delta\vee\!\!\!\vee K}_{\Gamma^0\Rightarrow\Delta^0}.$$

Then $D^{G/\Gamma \Rightarrow \Delta/K}_{\Gamma^0 \Rightarrow \Delta^0}$ is identical to these two histories.

Subcase e.2: Not (e.1). Then $D^{G/\Gamma \Rightarrow \Delta/K}_{\Gamma^0 \Rightarrow \Delta^0}$ is:

$$\begin{array}{ccc} D'^{G \curlyvee \Gamma \Rightarrow \Delta/\Gamma \Rightarrow \Delta/K}_{\Gamma^0 \Rightarrow \Delta^0} & & D'^{G/\Gamma \Rightarrow \Delta/\Gamma \Rightarrow \Delta \curlyvee K}_{\Gamma^0 \Rightarrow \Delta^0} \\ & \diagdown \quad \diagup & \\ & \Gamma^0 \Rightarrow \Delta^0. & \end{array}$$

By this we have finished defining $D^{G/\Gamma \Rightarrow \Delta/K}_{\Gamma^0 \Rightarrow \Delta^0}$. It is easy to see that it has the form of an ordinary $\mathrm{GRMI}^-_{\min}$ proof of $\Gamma^0 \Rightarrow \Delta^0$, except that it may include repetitions of the form:

$$\begin{array}{ccc} \Gamma \Rightarrow \Delta & & \Gamma \Rightarrow \Delta \\ & \diagdown \quad \diagup & \\ & \Gamma \Rightarrow \Delta, & \end{array}$$

it need not start from axioms, and instead of mingles it may include combinings of the form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}.$$

We next define the concept of *a generalized A-supermix* as follows: Suppose $D_i (i = 1, \ldots, m)$ is a proof of:

$$G_i^1 \curlyvee \Gamma_i; \Gamma_i^1 \Rightarrow \Delta_i; \Delta_i^1 \curlyvee G_i^2 \curlyvee \Gamma_i; \Gamma_i^2 \Rightarrow \Delta_i; \Delta_i^2 \curlyvee \cdots \curlyvee G_i^{k_i} \curlyvee \Gamma_i; \Gamma_i^{k_i} \Rightarrow \Delta_i; \Delta_i^{k_i} \curlyvee G_i^{k_i+1}$$

in which all the indicated $\Gamma_i \Rightarrow \Delta_i$ have the same history in $D_i (i = 1, \ldots, m)$ and $A \in \bigcap_{i=1}^m \Delta_i$. Suppose also that $D'_j (j = 1, \ldots, n)$ is a proof of:

$$G_j'^1 \curlyvee \Gamma'_j; \Gamma_j'^1 \Rightarrow \Delta'_j; \Delta_j'^1 \curlyvee \cdots \curlyvee G_j'^{m_j} \curlyvee \Gamma'_j; \Gamma_j'^{m_j} \Rightarrow \Delta'_j; \Delta_j'^{m_j} \curlyvee G_j'^{m_j+1}$$

in which all the indicated $\Gamma'_j \Rightarrow \Delta'_j (j = 1, \ldots, n)$ have the same history in D'_j and $A \in \bigcap_{j=1}^n \Gamma'_j$. Then the generalized A-supermix of these hypersequents is:

$$\begin{aligned} &G_1^1 \curlyvee G_1^2 \curlyvee \cdots \curlyvee G_1^{k_1} \curlyvee G_2^1 \curlyvee \cdots \curlyvee G_2^{k_2} \curlyvee \cdots \curlyvee G_m^1 \curlyvee \cdots \curlyvee G_m^{k_m+1} \curlyvee G_1'^1 \curlyvee \cdots \curlyvee G_n'^{m_n+1} \\ &\curlyvee \Gamma_1^1 \Rightarrow \Delta_1^1 \curlyvee \Gamma_1^2 \Rightarrow \Delta_1^2 \curlyvee \cdots \curlyvee \Gamma_1^{k_1} \Rightarrow \Delta_1^{k_1} \curlyvee \Gamma_2^1 \Rightarrow \Delta_2^1 \curlyvee \cdots \curlyvee \Gamma_2^{k_2} \Rightarrow \Delta_2^{k_2} \curlyvee \cdots \curlyvee \Gamma_m^{k_m} \\ &\quad \Rightarrow \Delta_m^{k_m} \curlyvee \Gamma_1'^1 \Rightarrow \Delta_1'^1 \curlyvee \cdots \curlyvee \Gamma_n'^{m_n} \Rightarrow \Delta_n'^{m_n} \curlyvee \Gamma_1, \Gamma_2, \ldots, \Gamma_m, \Gamma_1'^*, \Gamma_2'^* \cdots \Gamma_n'^* \\ &\quad \Rightarrow \Delta_1^*, \Delta_2^* \cdots \Delta_m^*, \Delta'_1, \ldots, \Delta'_n. \end{aligned}$$

Our main lemma is now: Every generalized A-supermix can be eliminated in such a way, so that if $\Gamma_0 \Rightarrow \Delta_0$ is a subsequent of one of the components of the result of this supermix, then it has in the new proof the same history it has in the old one.

The proof of the main lemma is by a threefold induction on: (1) The complexity of the supermix formula A, (2) The sum of the lengths of the common history of the $\Gamma_i \Rightarrow \Delta_i$ $(i = 1, \ldots, m)$ and that of the $\Gamma'_j \Rightarrow \Delta'_j (j = 1, \ldots, n)$, and (3) the sum of the lengths of the proofs of the premises of the given generalized supermix. Now, the definition of a generalized supermix makes the case of splitting trivial in this induction. Most of the other cases are not difficult either, but a full presentation of them all is long and tedious. As an illustration we shall do the case of an external contraction (which was found to be the most problematic one). To simplify notation, we let $m = 1$, $n = 1$, $k_2 = 2$, and $m_1 = 1$ (in the definition of a generalized supermix). We have accordingly, two proofs D_1 and D'_1 with the following form:

$$D_1 \left\{ \frac{D_1^* \left\{ G_1 \Vee \Gamma_1^1, \Gamma_1 \Rightarrow \Delta_1^1, \overset{\vdots}{\Delta_1} \Vee \Gamma_1^2, \Gamma_1 \Rightarrow \Delta_1^2, \Delta_1 \Vee \Gamma_1^2, \Gamma_1 \Rightarrow \Delta_1^2, \Delta_1 \Vee G_2 \right.}{G_1 \Vee \Gamma_1^1, \Gamma_1 \Rightarrow \Delta_1^1, \Delta_1 \Vee \Gamma_1^2, \Gamma_1 \Rightarrow \Delta_1^2, \Delta_1 \Vee G_2} \right.$$

$$D'_1 \left\{ \begin{array}{c} \vdots \\ K_1 \Vee \Gamma_2^1, \Gamma_2 \Rightarrow \Delta_2^1, \Delta_2 \Vee K_2 \end{array} \right.$$

(where the two $\Gamma_1 \Rightarrow \Delta_1$'s which are indicated in the conclusion of D_1 have the same histories in D_1—see the definition of a generalized supermix!). We now show how to construct from D_1 and D'_1 a supermix-free proof D as requested. For this we have two subcases to consider:

(i) In D_1^* all the three indicated $\Gamma_1 \Rightarrow \Delta_1$'s have the same history (which is identical to the one they have in D_1).

In this case we apply the I.H. to D'_1 and D_1^* and get a proof with the required properties of

$$G_1 \Vee G_2 \Vee K_1 \Vee K_2 \Vee \Gamma_1^1 \Rightarrow \Delta_1^1 \Vee \Gamma_1^2 \Rightarrow \Delta_1^2 \Vee \Gamma_1^2 \Rightarrow \Delta_1^2 \Vee \Gamma_1^2 \Rightarrow \Delta_1^2 \Vee \Gamma_2^1 \Rightarrow \Delta_2^1 \Vee \Gamma_1, \Gamma_2^* \Rightarrow \Delta_1^*, \Delta_2.$$

An external contraction of the $\Gamma_1^2 \Rightarrow \Delta_1^2$'s (which, by I.H., have the same history in the new proof) yields D.

(ii) Not case (i). This means that the second and the third $\Gamma_1 \Rightarrow \Delta_1$'s (in the final conclusion of D_1^*) have there different histories, which are both shorter than that of the first $\Gamma_1 \Rightarrow \Delta_1$ (which has in D_1^* the same history which the two $\Gamma_1 \Rightarrow \Delta_1$'s indicated in the conclusion of D_1 have there).

In this case we apply first the I.H. to D_1^* and D'_1, in order to eliminate the first $\Gamma_1 \Rightarrow \Delta_1$. We then get an appropriate proof, D_3, of:

$$G_1 \Vee \Gamma_1^2, \Gamma_1 \Rightarrow \Delta_1^2, \Delta_1 \Vee \Gamma_1^2, \Gamma_1 \Rightarrow \Delta_1^2, \Delta_1 \Vee G_2 \Vee K_1 \Vee K_2 \Vee \Gamma_1^1 \Rightarrow \Delta_1^1 \Vee \Gamma_2^1 \Rightarrow \Delta_2^1 \Vee \Gamma_1, \Gamma_2^* \Rightarrow \Delta_1^*, \Delta_2.$$

(In this step we use the fact that D_1^* is shorter than D_1.) We then apply the I.H. to D_3 and D'_1, and get a proof D_2 of:

$$G_1 \Vee \Gamma_1^2, \Gamma_1 \Rightarrow \Delta_1^2, \Delta_1 \Vee G_2 \Vee K_1 \Vee K_2 \Vee \Gamma_1^1 \Rightarrow \Delta_1^1 \Vee \Gamma_2^1 \Rightarrow \Delta_2^1 \Vee \Gamma_1, \Gamma_2^* \Rightarrow \Delta_1^*, \Delta_2 \Vee K_1 \Vee K_2 \Vee \Gamma_1^2 \Rightarrow \Delta_1^2 \Vee \Gamma_2^1 \Rightarrow \Delta_2^1 \Vee \Gamma_1, \Gamma_2^* \Rightarrow \Delta_1^*, \Delta_2.$$

In this step we use the fact that the common history in D_3 of the $\Gamma_1 \Rightarrow \Delta_1$'s is shorter than that which the $\Gamma_1 \Rightarrow \Delta_1$'s have in D_1.) Now in D_2 all the indicated identical components of the final conclusion have the same histories. We can apply therefore the I.H. once more, this time to D_2 and D'_1. This yields (with the help of some external exchanges and contractions which are history-preserving) a proof D as desired.

An example Let RD^+ be the following version of RD:

$$\underline{RD^+}: R^+(B,C) \supset [A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)].$$

We show now how the cut-free mechanism of GRMI works by presenting a cut-free proof of RD^+.

We start with the following theorem of $\text{GRMI}_{\rightleftarrows}$:

$$R^+(B,C), R^+(B,C), B, C \Rightarrow B, C.$$

From this we derive, using splittings and contractions:

(1) $R^+(B,C),\ B \Rightarrow C \,\mathbin{\mid}\, C \Rightarrow B$
(2) $R^+(B,C),\ C \Rightarrow B \,\mathbin{\mid}\, B \Rightarrow C$
(3) $R^+(B,C), C \Rightarrow C \,\mathbin{\mid}\, R^+(B,C),\ B \Rightarrow B.$

Two applications of $\Rightarrow \vee$, first to (1) and (3) and then to the resulting hypersequent and to (2) yield (after some exchanges):

(4) $R^+(B,C),\ B \vee C \Rightarrow C \mathbin{\mid} R^+(B,C),\ B \vee C \Rightarrow B \mathbin{\mid} B \Rightarrow C \mathbin{\mid} C \Rightarrow B.$

We now show that RD^+ is derivable from each of the components of (4). It follows that if we start from (4) we can prove $\Rightarrow RD^+ \mathbin{\mid} \Rightarrow RD^+ \mathbin{\mid} \Rightarrow RD^+ \mathbin{\mid} \Rightarrow RD^+$, and from this $\Rightarrow RD^+$ follows, using external contractions.

Now, the proof of Theorem 2 includes a cut-free derivation of $\Rightarrow A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$ from $B \Rightarrow C \mathbin{\mid} C \Rightarrow B$. From this $\Rightarrow RD^+$ follows immediately, using $\Rightarrow \vee$. This finishes the case of the third and fourth components of (4).

Next, we start from $R^+(B,C),\ B \vee C \Rightarrow C$ (the first component) and infer from it (by $\wedge \Rightarrow$):

(5) $R^+(B,C),\ A \wedge (B \vee C) \Rightarrow C.$

A relevant mingle of (5) and of the GRMI_{min}-theorem $A \wedge (B \vee C),\ A \wedge (B \vee C) \Rightarrow A$, followed by a splitting, yields:

(6) $R^+(B,C),\ A \wedge (B \vee C) \Rightarrow A \mathbin{\mid} A \wedge (B \vee C) \Rightarrow C.$

Two applications of $\Rightarrow \wedge$, first to (5) and (6), then to the result and to $A \wedge (B \vee C) \Rightarrow A$, yield: $R^+(B,C),\ A \wedge (B \vee C) \Rightarrow A \wedge C \mathbin{\mid} A \wedge (B \vee C) \Rightarrow A \wedge C$. From this $\Rightarrow RD^+ \mathbin{\mid} \Rightarrow RD^+$ easily follows, using applications of $\Rightarrow \vee$ and $\Rightarrow \rightarrow$.

In a similar way, we can derive $\Rightarrow RD^+$ from the second component of (4).

Note: The idea of hypersequents, as well as the definition of the "history of $\Gamma^0 \Rightarrow \Delta^0$ in D", come from [5]. That paper includes a cut-free hypersequential formulation, GRM, of RM. GRM is quite similar to GRMI in its formulation. Yet the two systems radically differ with respect to the relations which exist in them between their hypersequents and their sentences: The hypersequents of GRMI cannot be faithfully translated into sentences of the language of RMI. This is an easy consequence of theorem E.4 of [4] and the obvious correspondence between GRMI and the multiple-conclusioned, Hilbert-type system which is presented in section E of [4] (details can be easily supplied by any reader of the two papers).

We end this paper with a corollary of the cut elimination theorem. It was proved already in [4] using semantical methods. The present proof, in contrast, is purely syntactic and also constructive.

Proposition 5 *RMI is a conservative extension of* $\mathrm{RMI}_{\rightrightarrows}$.

Proof: Using induction on cut-free proofs, we can easily show that if $\Gamma_i, \Delta_i (i = 1, \dots, n)$ consist solely of sentences in the language of $\mathrm{RMI}_{\rightrightarrows}$, then $\Gamma_1 \Rightarrow \Delta_1 \veebar \Gamma_2 \Rightarrow \Delta_2 \veebar \cdots \veebar \Gamma_n \Rightarrow \Delta_n$ is provable in GRMI iff for some indexes $i_1, \dots, i_k (1 \leq i_1 < \cdots < i_k \leq n)$ $\Gamma_{i_1}, \Gamma_{i_2}, \dots, \Gamma_{i_k} \Rightarrow \Delta_{i_1}, \Delta_{i_2}, \dots, \Delta_{i_k}$ is a theorem of $\mathrm{GRMI}_{\rightrightarrows}$. From this our proposition immediately follows.

REFERENCES

[1] Anderson, A. R. and N. P. Belnap, *Entailment—The Logic of Relevance and Necessity*, Vol. 1, Princeton University Press, Princeton, New Jersey, 1975.

[2] Avron, A., "Relevant entailment—Semantic and formal systems," *The Journal of Symbolic Logic*, vol. 9 (1984), pp. 334–341.

[3] Avron, A., "Relevance and paraconsistency—A new approach," *The Journal of Symbolic Logic*, vol. 55 (1990), pp. 707–731.

[4] Avron, A., "Relevance and paraconsistency—A new approach. Part II: The formal systems," *Notre Dame Journal of Formal Logic*, vol. 31 (1990), pp. 169–202.

[5] Avron, A., "A constructive analysis of RM," *The Journal of Symbolic Logic*, vol. 52 (1987), pp. 939–951.

[6] Takeuti, G., *Proof Theory*, North Holland, Amsterdam, 1975.

[7] Girard, J. Y., "Linear logic," *Theoretical Computer Science*, vol. 50 (1987), pp. 1–102.

School of Mathematical Sciences
Raymond and Beverly Sacker Faculty of Exact Sciences
Tel Aviv University
Ramat-Aviv, 69978, Israel