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Notre Dame Journal of Formal Logic
Volume 31, Number 4, Fall 1990

Some Syntactical Properties of Intermediate Predicate Logics

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Abstract In a previous paper the author introduced a syntactical property, which he calls the pseudo-relevance property, for the sake of studying a certain semantical aspect. An intermediate predicate logic **L** is said to have the *pseudo-relevance property* if for all formulas A and B which contain no predicate variable in common, either not-A or B is provable in **L** whenever A implies B is provable in **L**. The pseudo-relevance property can be regarded as a weak version of Craig's interpolation property. From the same point of view, one can see the similarity between Hallden-completeness and the disjunction property. We treat these syntactical properties and their weak versions, and study the relationships between them.

Introduction In [1], Komori proved that every intermediate *propositional* logic **L** has the property that for all formulas A and B which have no propositional variables in common, $A \supset B \in \mathbf{L}$ implies either $\neg A \in \mathbf{L}$ or $B \in \mathbf{L}$. The author [4] showed that the situation changes in intermediate predicate logics. In fact, there are uncountably many intermediate predicate logics without (the predicate version of) the above property. We call here this property the *pseudo-relevance property* (PRP). It is easily seen that PRP can be regarded as a weak version of Craig's interpolation property (see, e.g., Ono [3]). From the same point of view, one can see the similarity between Halldén-completeness and the disjunction property; namely, **L** is said to be H-complete if for all formulas A and B which have no predicate variables in common, $A \vee B \in \mathbf{L}$ implies either $A \in \mathbf{L}$ or $B \in \mathbf{L}$. Wroński [7] gave a necessary and sufficient condition for the H-completeness of an intermediate propositional logic by making use of alge-

*The author would like to express his thanks to Professor M. Takano for his comments on an earlier version of this paper. He also wishes to thank a referee for this journal, whose suggestions made this article more readable than earlier versions.

Received July 7, 1989; revised September 11, 1989

braic semantics. That is, an intermediate propositional logic is H-complete if and only if it is characterized by a *strongly compact* pseudo-Boolean algebra. We will treat here PRP, H-completeness, and their weak versions—we call them PRP* and H*-completeness—of intermediate predicate logics and study the relationships between them.

In Section 1, we will define PRP, H-completeness, and other properties, and prove some basic lemmas. Our aim can be proposed in a concrete fashion here; that is, to determine whether one property implies another or not (see Figure 1). We will prepare semantical tools also in this section. The readers will find that some of our techniques are quite similar to those used in connection with propositional logics. In Section 2, we will achieve our aim by making use of these tools.

In the following we assume the readers' familiarity with Ono [2].

1 Pseudo-relevance property and Halldén-completeness As usual, we fix a first-order language $\mathcal{L}$ which contains neither constants nor function symbols. Sometimes, we will identify $\mathcal{L}$ with the set of all predicate variables of $\mathcal{L}$.

Definition 1.1 Let $\mathbf{L}$ be an intermediate predicate logic (or more simply, a *logic*).

(i) $\mathbf{L}$ is said to have PRP (PRP*) if for all formulas A and B (in $\mathcal{L}$) which contain no predicate variables in common, $A \supset B \in \mathbf{L}$ implies either $\neg A \in \mathbf{L}$ or $B \in \mathbf{L}$ (either $\neg A \in \mathbf{L}$ or $\neg\neg B \in \mathbf{L}$, respectively).
(ii) $\mathbf{L}$ is said to be Halldén-complete (H*-complete) if for all formulas A and B (in $\mathcal{L}$) which contain no predicate variables in common, $A \vee B \in \mathbf{L}$ implies either $A \in \mathbf{L}$ or $B \in \mathbf{L}$ (either $\neg\neg A \in \mathbf{L}$ or $\neg\neg B \in \mathbf{L}$, respectively).

Komori [1] proved that every intermediate *propositional* logic has PRP. Wroński [7] proved that an intermediate *propositional* logic is H-complete if and only if it is characterized by a pseudo-Boolean algebra which has the second greatest element. Such a pseudo-Boolean algebra is said to be *strongly compact*. Thus, the relationship between these two properties in the propositional case has been completely clarified. In this paper, we will study the situation in the predicate case. Our next lemmas can be easily seen to be true.

Lemma 1.2 *Let* $\mathbf{L}$ *be a logic.*
(i) *If* $\mathbf{L}$ *has* PRP *then* $\mathbf{L}$ *has* PRP*.
(ii) *If* $\mathbf{L}$ *has* PRP* *then* $\mathbf{L}$ *is* H*-*complete.*
(iii) *If* $\mathbf{L}$ *is* H-*complete then* $\mathbf{L}$ *is* H*-*complete.*

Proof: It is well-known that $A \supset \neg\neg A$ is provable in the intuitionistic predicate logic $\mathbf{LJ}$. From this, (i) and (iii) follow. To show (ii), recall that $A \vee B \supset (\neg A \supset B)$ is provable in the intuitionistic logic. Suppose that $A \vee B \in \mathbf{L}$; then $\neg A \supset B \in \mathbf{L}$. Since $\mathbf{L}$ has PRP*, either $\neg\neg A \in \mathbf{L}$ or $\neg\neg B \in \mathbf{L}$. This is what was to be proven.

We illustrate our situation in Figure 1.

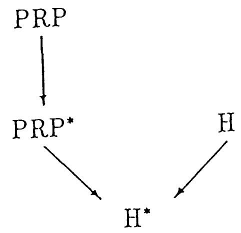

Figure 1.

Theorem 1.3 *Figure* 1 *describes completely the situation. That is, we cannot put any additional* $\rightarrow$*'s* (*i.e., arrows of implication*) *in Figure* 1.

The main aim of this paper is to prove Theorem 1.3. For each nonempty set D, we denote by $\mathcal{L}[D]$ the language obtained from $\mathcal{L}$ by adding the name $\bar{d}$ of each $d \in D$ as new constant. We sometimes identify $\mathcal{L}[D]$ with the set of all sentences of it. A pair $(\mathbf{P}, D)$ of a nondegenerate pseudo-Boolean algebra $\mathbf{P}$ and a nonempty set D is said to be an *algebraic frame*, if $\mathbf{P}$ is λ-complete, where λ is the cardinality of D. An *assignment* f of an algebraic frame $(\mathbf{P}, D)$ is a mapping from the set of all atomic sentences in $\mathcal{L}[D]$ to $\mathbf{P} = \langle P, \cap, \cup, \rightarrow, 0, 1\rangle$. We extend it to a mapping from $\mathcal{L}[D]$ to $\mathbf{P}$ as follows:

(A1) $f(A \wedge B) = f(A) \cap f(B)$
(A2) $f(A \vee B) = f(A) \cup f(B)$
(A3) $f(A \supset B) = f(A) \rightarrow f(B)$
(A4) $f(\neg A) = f(A) \rightarrow 0$
(A5) $f(\forall x A(x)) = \bigcap_{d \in D} f(A(\bar{d}))$
(A6) $f(\exists x A(x)) = \bigcup_{d \in D} f(A(\bar{d}))$.

A formula A (of $\mathcal{L}$) is said to be *valid* in an algebraic frame $(\mathbf{P}, D)$ if $f(\bar{A}) = 1$ for every assignment f of $(\mathbf{P}, D)$, where $\bar{A}$ is the universal closure of A. The set of all formulas (of $\mathcal{L}$) that are valid in an algebraic frame $(\mathbf{P}, D)$ is denoted by $L^+(\mathbf{P}, D)$. As is well-known, $L^+(\mathbf{P}, D)$ contains all formulas provable in the intuitionistic predicate logic **LJ**, and is closed under modus ponens, the rule of generalization and the rule of substitution. An algebraic frame $(\mathbf{P}, D)$ is said to be an ω^+*-algebraic frame* if D is an infinite set.

Now, we will introduce a key tool developed in [4]. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be mutually disjoint languages (i.e., they contain no predicate variables in common) such that $\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}$. Suppose we have assignments f_1 and f_2 of an algebraic frame $(\mathbf{P}, D)$. Define an assignment f of $(\mathbf{P}, D \times D)$ by

$$f(p(\overline{(d_1, e_1)}, \ldots, \overline{(d_n, e_n)})) = \begin{cases} f_1(p(\bar{d}_1, \ldots, \bar{d}_n)) & \text{if } p \in \mathcal{L}_1, \\ f_2(p(\bar{e}_1, \ldots, \bar{e}_n)) & \text{if } p \in \mathcal{L}_2, \end{cases}$$

for each n-ary predicate variable p and each $(d_i, e_i) \in D \times D$ $(i = 1, \ldots, n)$. Then, by induction, we have the following.

Lemma 1.4 *Let a formula* $A \equiv A(x_1, \ldots, x_m)$ *of* $\mathcal{L}$ *contain no free variables other than* $x_1, \ldots, x_m$. *For every* $(d_i, e_i) \in D \times D$ $(i = 1, \ldots, m)$,

(i) *if* A *is a formula of* $\mathcal{L}_1$, $f(A(\overline{(d_1,e_1)},\ldots,\overline{(d_m,e_m)})) = f_1(A(\overline{d_1},\ldots,\overline{d_m}))$,
(ii) *if* A *is a formula of* $\mathcal{L}_2$, $f(A(\overline{(d_1,e_1)},\ldots,\overline{(d_m,e_m)})) = f_2(A(\overline{e_1},\ldots,\overline{e_m}))$.

We say the above f is *made by splicing* f_1 and f_2 over $(\mathbf{P}, D \times D)$ with respect to $(\mathcal{L}_1, \mathcal{L}_2)$.

Lemma 1.5 *Let* $\mathbf{L}$ *be a logic in which* $\neg\neg\forall x(p(x) \vee \neg p(x))$ *is provable. Then* $\mathbf{L}$ *has* PRP*. *Moreover, if there exists a class* $\mathcal{C}$ *of* ω^+*-algebraic frames such that* $\mathbf{L} = \bigcap_{(\mathbf{P},D)\in\mathcal{C}} L^+(\mathbf{P},D)$, *then* $\mathbf{L}$ *has* PRP.

Proof: Our proof proceeds in a way that is similar to the proof given for the case of propositional logics in Komori [1]. First, note the following Fact 1.6 due to Umezawa [6].

Fact 1.6 (Umezawa [6]) *The following three conditions on a logic* $\mathbf{L}$ *are equivalent*:
(i) *The sentence* $\neg\neg\forall x(p(x) \vee \neg p(x))$ *is provable in* $\mathbf{L}$.
(ii) *For every formula* A, $\neg A$ *is provable in the classical predicate logic* $\mathbf{LK}$ *if and only if* $\neg A$ *is provable in* $\mathbf{L}$.
(iii) *For every formula* A, A *is provable in* $\mathbf{LK}$ *if and only if* $\neg\neg A$ *is provable in* $\mathbf{L}$.

Suppose formulas A and B contain no predicate variable in common, and $A \supset B$ is provable in $\mathbf{L}$. Of course, $A \supset B$ is provable also in $\mathbf{LK}$. By Craig's interpolation theorem for $\mathbf{LK}$, either $\neg A$ or B is provable in $\mathbf{LK}$. By Fact 1.6, either $\neg A$ or $\neg\neg B$ is provable in $\mathbf{L}$. Hence, $\mathbf{L}$ has PRP*.

Next, assume in addition that there exists a class $\mathcal{C}$ of ω^+-algebraic frames such that $\mathbf{L} = \bigcap_{(\mathbf{P},D)\in\mathcal{C}} L^+(\mathbf{P},D)$. Suppose that $\neg A$ and B contain no predicate variable in common, and none of them are provable in $\mathbf{L}$. Without loss of generality, we may assume that A and B contain no free variables other than $x_1,\ldots,x_n$. We write $A(x_1,\ldots,x_n)$ and $B(x_1,\ldots,x_n)$ for A and for B respectively. Then, there exist an ω^+-algebraic frame $(\mathbf{P},D) \in \mathcal{C}$, an assignment f_1 of $(\mathbf{P},D)$, and elements $d_1,\ldots,d_n \in D$ such that

(1) $\mathbf{L} \subset L^+(\mathbf{P},D)$, and
(2) $f_1(B(\overline{d_1},\ldots,\overline{d_n})) \neq 1_{\mathbf{P}}$,

where $1_{\mathbf{P}}$ is the greatest element of $\mathbf{P}$. By Fact 1.6, $\neg A(x_1,\ldots,x_n)$ is not provable in $\mathbf{LK}$. By Gödel's completeness theorem, $\mathbf{LK}$ is characterized by the algebraic frame $(\{0,1\},D)$, since D is infinite. Hence, there exists an assignment g of $(\{0,1\},D)$ and elements $e_1,\ldots,e_n \in D$ such that

(3) $g(A(\overline{e_1},\ldots,\overline{e_n})) = 1$.

Next, we define a valuation f_2 of $(\mathbf{P},D)$ by

$$f_2(p(\overline{c_1},\ldots,\overline{c_m})) = \begin{cases} 1_{\mathbf{P}} & \text{if } g(p(\overline{c_1},\ldots,\overline{c_m})) = 1, \\ 0_{\mathbf{P}} & \text{otherwise,} \end{cases}$$

for every m-ary predicate variable p and every $c_i \in D$ $(i = 1,\ldots,m)$, where $0_{\mathbf{P}}$ is the least element of $\mathbf{P}$. By induction, we have

$$f_2(C) = \begin{cases} 1_{\mathbf{P}} & \text{if } g(C) = 1, \\ 0_{\mathbf{P}} & \text{otherwise,} \end{cases}$$

for every sentence C of $\mathcal{L}[D]$. Hence,

(4) $f_2(A(\overline{e_1}, \ldots, \overline{e_n})) = 1_{\mathbf{P}}$.

Let $\mathcal{L}_1$ be the set of predicate variables appearing in B, and suppose that $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Clearly, every predicate variable in A belongs to $\mathcal{L}_2$. Now, make the assignment f by splicing f_1 and f_2 over $(\mathbf{P}, D \times D)$ with respect to $(\mathcal{L}_1, \mathcal{L}_2)$. By Lemma 1.4 and (2),

$$f(B(\overline{(d_1, e_1)}, \ldots, \overline{(d_n, e_n)})) \neq 1_{\mathbf{P}}.$$

By (4),

$$f(A(\overline{(d_1, e_1)}, \ldots, \overline{(d_n, e_n)})) = 1_{\mathbf{P}}.$$

Thus, we have

$$f((A \supset B)(\overline{(d_1, e_1)}, \ldots, \overline{(d_n, e_n)})) \neq 1_{\mathbf{P}}.$$

Therefore, $A \supset B$ is not valid in $(\mathbf{P}, D \times D)$. Since D is infinite, $D \times D$ can be identified with D as an individual domain. It follows from this that $A \supset B$ is not valid in $(\mathbf{P}, D)$. Thus, by (1), $A \supset B$ is not provable in $\mathbf{L}$.

We make a remark here that the intuitionistic predicate logic **LJ** has PRP but the sentence $\neg\neg\forall x(p(x) \vee \neg p(x))$ is not provable in **LJ**. A logic $\mathbf{L}$ is said to be *(algebraically)* ω^+*-complete* if there exists a class $\mathcal{C}$ of ω^+-algebraic frames such that $\mathbf{L} = \bigcap_{(\mathbf{P}, D) \in \mathcal{C}} L^+(\mathbf{P}, D)$. Some relations between PRP and ω^+-completeness can be found in [4]. For example, the converse of Lemma 1.5 holds for algebraically complete logics. That is, every algebraically complete logic with PRP is ω^+-complete.

We can prove similar results concerning Kripke frames. We denote by $\langle \mathbf{M}, D \rangle$ the Kripke frame $(\mathbf{M}, U)$ with constant domain D, i.e., $U(a) = D$ for every $a \in \mathbf{M}$. A Kripke frame $(\mathbf{M}, U)$ is said to be an ω^+*-Kripke frame* if $U(a)$ is infinite for every $a \in \mathbf{M}$. Note the following Fact 1.7 due to Ono [2].

Fact 1.7 (Ono [2]) *For every ω^+-Kripke frame $\langle \mathbf{M}, D \rangle$ with constant domain, there exists a class $\mathcal{C}$ of ω^+-algebraic frames such that $L\langle \mathbf{M}, D \rangle = \bigcap_{(\mathbf{P}, V) \in \mathcal{C}} L^+(\mathbf{P}, V)$.*

By using this fact, we can obtain the following, which is a Kripke frame version of Lemma 1.5.

Lemma 1.5′ *Let $\mathbf{L}$ be a logic in which $\neg\neg\forall x(p(x) \vee \neg p(x))$ is provable. Then $\mathbf{L}$ has* PRP*. *Moreover, if there exists a class $\mathcal{C}$ of ω^+-Kripke frames with constant domain such that $\mathbf{L} = \bigcap_{\langle \mathbf{M}, D \rangle \in \mathcal{C}} L\langle \mathbf{M}, D \rangle$, then $\mathbf{L}$ has* PRP.

For H-completeness, we have the following Lemma 1.8. Observe that the strong-compactness of pseudo-Boolean algebra plays an important role also in the case of predicate logic (cf. [7]).

Lemma 1.8 *Let $\mathbf{L}$ be a logic. If there exists an ω^+-algebraic frame $(\mathbf{P}, D)$ such that $\mathbf{P}$ is strongly compact and $\mathbf{L} = L^+(\mathbf{P}, D)$, then $\mathbf{L}$ is* H*-complete.*

Proof: We can prove this lemma in a way that is quite similar to that used in connection with Lemma 1.5. Suppose that A and B contain no predicate variables in common, and none of them are provable in $\mathbf{L}$. Then there exist assignments f_1 and f_2 of $(\mathbf{P},D)$ such that

$$f_1(A') \leq a_0 \quad \text{and} \quad f_2(B') \leq a_0,$$

where a_0 is the second greatest element of $\mathbf{P}$ and A' (and B') is the sentence obtained from A (and B, respectively) by replacing every free occurrence of a variable by some constant. Take the assignment f of $(\mathbf{P},D \times D)$ made by splicing f_1 and f_2 as in Lemma 1.5. It is easy to see that f sends (an instance of) $A \vee B$ to an element $\leq a_0$. Hence, $A \vee B$ is not valid in $(\mathbf{P},D \times D)$. It follows from this that $A \vee B$ is not valid in $(\mathbf{P},D)$. Therefore, $A \vee B \notin \mathbf{L}$.

The strong-compactness of pseudo-Boolean algebra corresponds to the existence of the least element of a Kripke frame. Thus, we have

Lemma 1.8′ *Let* $\mathbf{L}$ *be a logic. If there exists an* ω^+*-frame* $\langle \mathbf{M},D\rangle$ *with constant domain and having the least element such that* $\mathbf{L} = L\langle \mathbf{M},D\rangle$, *then* $\mathbf{L}$ *is* H-*complete.*

Professor M. Takano informed the author of the existence of a counterexample for the converse of Lemma 1.8′ (in a personal communication). Here we will give his proof. Let ω be the first infinite ordinal which is identified with the set $\{i : i < \omega\}$ and $\leq^*$ be the dual of the canonical order $\leq$ on ω. The set ω together with $\leq^*$ forms an ordered set without the least element. We denote by ω^* this ordered set. Then Takano proved

Theorem 1.9
(i) *The logic* $L\langle \omega^*, \omega\rangle$ *is* H-*complete.*
(ii) *No Kripke frame having the least element characterizes* $L\langle \omega^*, \omega\rangle$.

Proof: We can prove (i) in a way that is quite similar to that used to prove Lemma 1.8′. To show (ii), suppose, on the contrary to the conclusion, that there exists a Kripke frame $(\mathbf{M}, U)$ having the least element a_0 such that $L\langle \omega^*, \omega\rangle = L(\mathbf{M}, U)$. Then it is routine to check that $\mathbf{M}$ is linearly ordered and U is a constant mapping. Hence $(\mathbf{M}, U) = \langle \mathbf{M}, U(a_0)\rangle$. Since $L\langle \mathbf{M}, U(a_0)\rangle = L\langle \omega^*, \omega\rangle \subset \mathbf{LK}$, $U(a_0)$ must be infinite. It can be easily verified that the sentences $\exists x(\exists y p(y) \supset p(x))$ and $\exists x(p(x) \supset \forall y p(y))$ are valid in $\langle \omega^*, \omega\rangle$. Hence $\mathbf{M}$ has neither an infinite ascending chain, nor an infinite descending chain. (See, e.g., Ono [3].) Therefore $\mathbf{M}$ is a finite chain. Let n be the cardinality of $\mathbf{M}$. Then we have

$$p_0 \vee (p_0 \supset p_1) \vee \cdots \vee (p_{n-1} \supset p_n) \in L\langle \mathbf{M}, U(a_0)\rangle.$$

This contradicts the fact that

$$p_0 \vee (p_0 \supset p_1) \vee \cdots \vee (p_{n-1} \supset p_n) \notin L\langle \omega^*, \omega\rangle.$$

It is easy to see that the above logic $L\langle \omega^*, \omega\rangle$ is a counterexample also to the converse of Lemma 1.8. Since Kripke frames are easy to handle, we will use them in the next section. In the rest of this section, we will give a purely technical tool which will be needed in the next section.

Definition 1.10 (See Ono [3]) Let $\mathbf{M} = \langle M, \leq_{\mathbf{M}} \rangle$ and $\mathbf{N} = \langle N, \leq_{\mathbf{N}} \rangle$ be partially ordered sets where M and N are underlying sets of $\mathbf{M}$ and $\mathbf{N}$ respectively. We denote by $\mathbf{M} \bigtriangledown \mathbf{N}$ the ordered set $\langle \mathrm{M} \bigtriangledown \mathrm{N}, \leq_{\mathbf{M}\bigtriangledown\mathbf{N}} \rangle$ defined as

$$M \bigtriangledown N = M + N + \{0\},$$
$a \leq_{\mathbf{M}\bigtriangledown\mathbf{N}} b$ if and only if
(1) $a \in M$, $b \in M$ and $a \leq_{\mathbf{M}} b$, or
(2) $a \in N$, $b \in N$ and $a \leq_{\mathbf{N}} b$, or
(3) $a = 0$,

where $+$ is the disjoint union of sets. Then $\mathbf{M} \bigtriangledown \mathbf{N}$ is a partially ordered set with the least element 0.

Lemma 1.11 *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be partially ordered sets, and let* D *be a nonempty set. Then it is the case that* $L\langle \mathbf{M} \bigtriangledown \mathbf{N}, D\rangle \subset L\langle \mathbf{M}, D\rangle \cap L\langle \mathbf{N}, D\rangle$.

Proof: Suppose that a sentence A of $\mathcal{L}[D]$ does not belong to $L\langle \mathbf{M}, D\rangle$. Then there exists an element $a \in \mathbf{M}$ and a valuation V of $\langle \mathbf{M}, D\rangle$ such that $V(A,a) = f$. Define valuation W of $\langle \mathbf{M} \bigtriangledown \mathbf{N}, D\rangle$ by

$$W(B,b) = \begin{cases} V(B,b) & \text{if } b \in \mathbf{M}. \\ f & \textit{otherwise}, \end{cases}$$

for every atomic sentence B of $\mathcal{L}[D]$. We can check that W is well-defined. By induction, we can show that $W(C,b) = V(C,b)$ for every sentence $C \in \mathcal{L}[D]$ and every $b \in \mathbf{M}$. Hence, $W(A,a) = V(A,a) = f$. *Thus*, $A \notin L\langle \mathbf{M} \bigtriangledown \mathbf{N}, D\rangle$. It follows from this that $L\langle \mathbf{M} \bigtriangledown \mathbf{N}, D\rangle \subset L\langle \mathbf{M}, D\rangle$. Similarly, $L\langle \mathbf{M} \bigtriangledown \mathbf{N}, D\rangle \subset L\langle \mathbf{N}, D\rangle$. Therefore, $L\langle \mathbf{M} \bigtriangledown \mathbf{N}, D\rangle \subset L\langle \mathbf{M}, D\rangle \cap L\langle \mathbf{N}, D\rangle$.

2 Separation of properties In this section, we will complete the proof of Theorem 1.3. We will do this by proving Lemmas 2.1, 2.2 and 2.3. At the end of this paper, we will prove that even H*-completeness is not "common" to all intermediate predicate logics.

Lemma 2.1 *There exists a logic with* PRP* *but without* PRP. *Thus*, PRP* *does not imply* PRP.

Proof: Let $\mathbf{K}$ be an intermediate propositional logic other than the classical one. The maximum predicate extension $\mathbf{K}^*$ of $\mathbf{K}$ is axiomatized as

$$\mathbf{K}^* = \mathbf{LJ} + \mathbf{K} + (\exists x p(x) \supset \forall x p(x)) \vee (q \vee \neg q),$$

where p is a unary predicate variable and q is a propositional variable (see Ono [2]). It is easily verified that the sentence $\neg\neg\forall x(p(x) \vee \neg p(x))$ is provable in $\mathbf{K}^*$. Hence, $\mathbf{K}^*$ has PRP* by Lemma 1.5′. Since the formula

$$\{(\exists x p(x) \supset \forall x p(x)) \vee (q \vee \neg q)\} \supset \{(\exists x p(x) \wedge \exists x \neg p(x)) \supset (q \vee \neg q)\}$$

is provable in $\mathbf{LJ}$, $(\exists x p(x) \wedge \exists x \neg p(x)) \supset (q \vee \neg q)$ is provable in $\mathbf{K}^*$. On the other hand, neither $\neg(\exists x p(x) \wedge \exists x \neg p(x))$ nor $q \vee \neg q$ is provable in $\mathbf{K}^*$, since $\mathbf{K}^* \subsetneq \mathbf{LK}$. Thus, $\mathbf{K}^*$ does not have PRP.

Lemma 2.2 *There exists a logic with* PRP *which is not* H-*complete. Thus,* PRP *does not imply* H-*completeness.*

Proof: Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the Kripke bases defined in Figure 2.

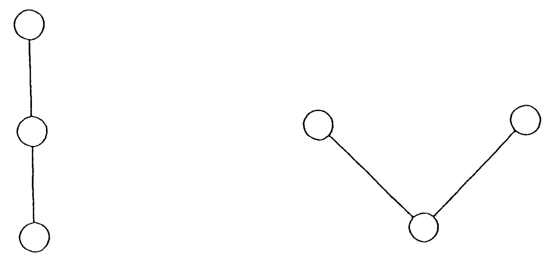

Figure 2

Let $\mathbf{L}_1 = L\langle \mathbf{M}_1, \omega\rangle \cap L\langle \mathbf{M}_2, \omega\rangle$, where ω is the first infinite ordinal which is identified with the set $\{i : i < \omega\}$. By making use of Lemma 1.5′, we can easily see that $\mathbf{L}_1$ has PRP. We denote by *Lin* and P_2 the sentences $(p \supset q) \vee (q \supset p)$ and $r \vee (r \supset (s \vee \neg s))$, respectively, where p, q, r and s are propositional variables. Observe that *Lin* and P_2 contain no predicate variables in common. It is obvious that $Lin \in L\langle \mathbf{M}_1, \omega\rangle$ and $P_2 \in L\langle \mathbf{M}_2, \omega\rangle$. Hence, $Lin \vee P_2$ is provable in $\mathbf{L}_1$. It can be easily verified that $Lin \notin L\langle \mathbf{M}_2, \omega\rangle$ and $P_2 \notin L\langle \mathbf{M}_1, \omega\rangle$. Hence, neither *Lin* nor P_2 is provable in $\mathbf{L}_1$. Thus, $\mathbf{L}_1$ is not H-complete.

From Lemma 2.2, it immediately follows that neither PRP* nor H*-completeness implies H-completeness. It remains for us to show that H-completeness implies neither PRP nor PRP*. We have only to prove

Lemma 2.3 *There exists an* H-*complete logic without* PRP*. *Thus,* H-*completeness does not imply* PRP*.

Proof: Define an infinite partially ordered set $\mathbf{M}$ as indicated in Figure 3. Let $\mathbf{Q}^* = [0,\infty) \cap \mathbf{Q}$, where $\mathbf{Q}$ is the set of rational numbers. Clearly, $\mathbf{Q}^*$ is a partially ordered set with the least element. By Lemma 1.8′, $\mathbf{L}_2 = L\langle \mathbf{M} \nabla \mathbf{Q}^*, \omega\rangle$ is H-complete. Define formulas Lin^*, T, $T'(w)$ and T^* by

$$Lin^* \equiv \forall x \forall y((p(x) \supset p(y)) \vee (p(y) \supset p(x))),$$

$$T \equiv (\forall x p(x) \supset \exists x q(x)) \supset \{\exists y(p(x) \supset \exists z q(z)) \vee \exists y(\exists z q(z) \supset q(y))\}$$

(Takano's axiom [5]),

$$T'(w) \equiv (\forall x r(x,w) \supset \exists x s(x,w)) \supset \{\exists y(r(x,w) \supset \exists z s(z,w)) \vee \exists y(\exists z s(z,w) \supset s(y,w))\},$$

$$T^* \equiv \forall w T'(w),$$

where p and q are unary predicate variables and r and s are binary predicate variables. Clearly, Lin^* and T^* contain no predicate variables in common. We

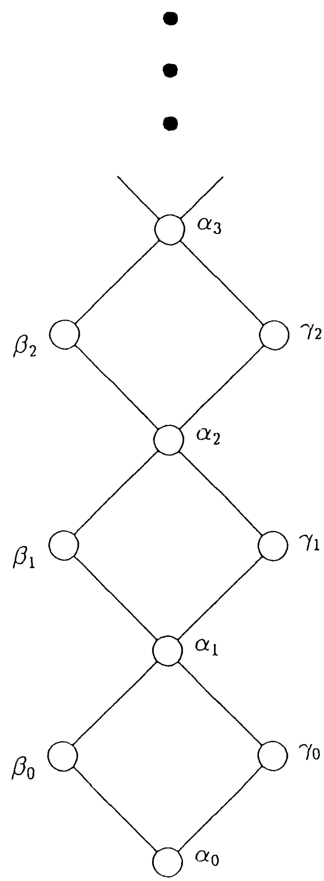

Figure 3

claim that $\neg Lin^* \supset T^*$ is provable in $\mathbf{L}_2$, but neither $\neg\neg Lin^*$ nor $\neg\neg T^*$ is provable in $\mathbf{L}_2$. First, we prove some Sublemmas.

Sublemma 2.3.1
(i) $\neg\neg Lin^*$ *is not valid in* $\langle \mathbf{M}, \omega \rangle$.
(ii) $\neg\neg T^*$ *is not valid in* $\langle \mathbf{Q}^*, \omega \rangle$.

Proof: (i): Define a valuation V of $\langle \mathbf{M}, \omega \rangle$ by

$$V(p(\bar{i}), a) = \begin{cases} t & \text{if } a = \alpha_n \text{ and } i \leq 2n, \\ t & \text{if } a = \beta_n \text{ and } i \leq 2n+1, \\ t & \text{If } a = \gamma_n,\ i \leq 2n+2, \text{ and } i \neq 2n+1, \\ f & \text{otherwise,} \end{cases}$$

for each $i \in \omega$ and each $a \in \mathbf{M}$. Then, for every $n \in \omega$, $V(p(\overline{2n+1}) \supset p(\overline{2n+2}), \beta_n) = f$ and $V(p(\overline{2n+2}) \supset p(\overline{2n+1}), \gamma_n) = f$. Hence, for every

$n \in \omega$, $V(Lin^*, \alpha_n) = f$. It follows from this that for every $a \in \mathbf{M}$, $V(Lin^*, a) = f$. Thus, $V(\neg Lin^*, \alpha_0) = t$. Therefore, $V(\neg\neg Lin^*, \alpha_0) = f$.

(ii): Take two sequences $\{x_i\}_{i\in\omega}$ and $\{y_i\}_{i\in\omega}$ of rational numbers such that

(1) $0 < x_0 < x_1 < x_2 < \ldots < \sqrt{2}/2$, $\lim_{i\to\infty} x_i = \sqrt{2}/2$,

(2) $1 > y_0 > y_1 > y_2 > \ldots > \sqrt{2}/2$, $\lim_{i\to\infty} y_i = \sqrt{2}/2$.

Define a valuation W of $\langle \mathbf{Q}^*, \omega \rangle$ by

$$W(r(\bar{i},\bar{j}),a) = \begin{cases} t & \text{if } j + x_i \leq a, \\ f & \text{otherwise,} \end{cases}$$

$$W(s(\bar{i},\bar{j}),a) = \begin{cases} t & \text{if } j + y_i < a, \\ f & \text{otherwise,} \end{cases}$$

for each $i \in \omega$, each $j \in \omega$ and each $a \in \mathbf{Q}^*$. Take an arbitrary $j \in \omega$. We will show that for every $j \in \omega$,

(3) $W(\forall x r(x,\bar{j}) \supset \exists x s(x,\bar{j}), j) = t$.

Take any $a \geq j$ such that $W(\forall x r(x,\bar{j}), a) = t$. Then $j + x_i \leq a$ for every $i \in \omega$. By (1), $j + \sqrt{2}/2 \leq a$. Since $a \in \mathbf{Q}^*$, $j + \sqrt{2}/2 < a$. By (2), there exists an $i_0 \in \omega$ such that $j + y_{i_0} < a$. Hence, $W(s(\bar{i}_0,\bar{j}),a) = t$. It follows that $W(\exists x s(x,\bar{j}),a) = t$. Thus we have that for every $j \in \omega$, (3) holds. Next, it is easily seen that

(4) for every $i \in \omega$, $W(r(\bar{i},\bar{j}), j + x_i) = t$.

Note that each x_i is a lower bound of $\{y_i : i \in \omega\}$. Hence, we have

(5) for every $i \in \omega$, $W(\exists z s(z,\bar{j}), j + x_i) = f$.

By (4) and (5),

(6) $W(\forall y(r(y,\bar{j}) \supset \exists z s(z,\bar{j})), j) = f$.

By the definition of W, it can be easily checked that $W(\exists z s(z,\bar{j}), j + y_i) = t$ and $W(s(\bar{i},\bar{j}), j + y_i) = f$. Hence,

(7) $W(\exists y(\exists z s(z,\bar{j}) \supset s(y,\bar{j})), j) = f$.

By (3), (6), and (7), we have $W(T'(\bar{j}), j) = f$. Thus, for every $a \in \mathbf{Q}^*$, $W(T^*, a) = f$. It follows that $W(\neg T^*, 0) = t$, where 0 is the least element of $\mathbf{Q}^*$. Therefore, $W(\neg\neg T^*, 0) = f$. That is, $\neg\neg T^*$ is not valid in $\langle \mathbf{Q}^*, \omega \rangle$.

Sublemma 2.3.2
(i) *Lin^* is valid in $\langle \mathbf{Q}^*, \omega \rangle$.*
(ii) *T^* is valid in $\langle \mathbf{M}, \omega \rangle$.*

Proof: (i): It suffices to show that *Lin* is valid in $\langle \mathbf{Q}^*, \omega \rangle$. It is well-known that *Lin* is valid in every Kripke frame whose base is linear (see, e.g., Ono [3]). Hence, *Lin* is valid in $\langle \mathbf{Q}^*, \omega \rangle$.

(ii): We have only to show that Takano's axiom T is valid in $\langle \mathbf{M}, \omega \rangle$. Suppose otherwise. Then there exists a valuation V of $\langle \mathbf{M}, \omega \rangle$ and an element $a \in \mathbf{M}$ such that

(1) $V(\forall xp(x) \supset \exists xq(x), a) = t$,
(2) for every $i \in \omega$, there exists $b_i \geq a$ such that $V(p(\bar{i}), b_i) = t$ and $V(\exists zq(z), b_i) = f$,
(3) for every $i \in \omega$, there exists $c_i \geq a$ such that $V(\exists zq(z), c_i) = t$ and $V(q(\bar{i}), c_i) = f$.

By (2), $V(\exists zq(z), a) = f$. Take a c_i in (3). From the fact that $V(\exists zq(z), c_i) = t$, it follows that $a < c_i$. Since there are only finitely many elements between a and c_i, one of the following cases holds. (See Figure 3.)

Case I: There exists the least element d_0 of $\{d \geq a : V(\exists zq(z), d) = t\}$.

Case II: There exists an $n \in \omega$ such that $a \leq \alpha_n$, $V(\exists zq(z), \alpha_n) = f$ and $V(\exists zq(z), \beta_n) = V(\exists zq(z), \gamma_n) = t$.

Suppose that Case I holds. Then there exists a $j \in \omega$ such that $V(q(\bar{j}), d_0) = t$. Hence, $V(\exists zq(z) \supset q(\bar{j}), a) = t$. This contradicts (3). Next, suppose that Case II holds. Observe that $b_i \leq \alpha_n$ for every $i \in \omega$. Hence, for every $i \in \omega$, $V(p(\bar{i}), \alpha_n) = t$. It follows that $V(\forall xp(x), \alpha_n) = t$. By (1), $V(\exists zq(z), \alpha_n) = t$. This is a contradiction.

Now, we shall return to the proof of Lemma 2.3. By Sublemma 2.3.1 and Lemma 1.11, neither $\neg\neg Lin^*$ nor $\neg\neg T^*$ is provable in $\mathbf{L}_2$. Suppose $\neg Lin^* \supset T^*$ is not provable in $\mathbf{L}_2$. Then there exists a valuation V of $\langle \mathbf{M} \triangledown \mathbf{Q}^*, \omega \rangle$ and an element $a \in \mathbf{M} \triangledown \mathbf{Q}^*$ such that $V(\neg Lin^*, a) = t$ and $V(T^*, a) = f$. If there exists a $b \in \mathbf{Q}^*$ such that $a \leq_{\mathbf{M} \triangledown \mathbf{Q}^*} b$, then $V(Lin^*, b) = f$. If V' is the restriction of V to $\langle \mathbf{Q}^*, \omega \rangle$, then $V'(Lin^*, b) = f$. It follows that Lin^* is not valid in $\langle \mathbf{Q}^*, \omega \rangle$. This contradicts Sublemma 2.3.2(i). Therefore, $a \in \mathbf{M}$. From the fact that $V(T^*, a) = f$, it follows that T^* is not valid in $\langle \mathbf{M}, \omega \rangle$. This contradicts Sublemma 2.3.2(ii). Therefore, $\neg Lin^* \supset T^*$ is valid in $\langle \mathbf{M} \triangledown \mathbf{Q}^*, \omega \rangle$, and is provable in $\mathbf{L}_2$. This completes the proof of Lemma 2.3.

Thus we have finished our proof of Theorem 1.3.

By Theorem 1.3, H*-completeness is the most "common" property among those in Figure 1. However, even H*-completeness is not possessed by all intermediate predicate logics.

Theorem 2.4 *There exists a logic which is not* H*-*complete.*

Proof: Let $\mathbf{L}_3$ be $L\langle \mathbf{M}, \omega \rangle \cap L\langle \mathbf{Q}^*, \omega \rangle$, where $\mathbf{M}$, $\mathbf{Q}^*$ and ω are the same in the proof of Lemma 2.3. By Sublemma 2.3.2, $T^* \vee Lin^*$ is valid both in $\langle \mathbf{M}, \omega \rangle$ and in $\langle \mathbf{Q}^*, \omega \rangle$. By Sublemma 2.3.1, neither $\neg\neg T^*$ nor $\neg\neg Lin^*$ is provable in $L\langle \mathbf{M}, \omega \rangle \cap L\langle \mathbf{Q}^*, \omega \rangle$. Hence, $\mathbf{L}_3$ is not H*-complete.

In Lemma 1.5′, we proved that if a logic $\mathbf{L}$ is characterized by a class of ω^+-Kripke frames with constant domain and moreover if $\neg\neg\forall x(p(x) \vee \neg p(x))$ is provable in $\mathbf{L}$ then $\mathbf{L}$ has PRP. On the other hand, the proof of Theorem 2.4 shows that there exists a non-H*-complete logic which is characterized by a class of ω^+-Kripke frames with constant domain. That is, without the second assumption that $\neg\neg\forall x(p(x) \vee \neg p(x)) \in \mathbf{L}$, we cannot derive even H*-completeness.

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