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An Equivalent of the Axiom of Choice in Finite Models of the Powerset Axiom

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Abstract It is shown that in a finite model for the set-theoretical Powerset axiom every set s has a Choice set iff every set s has a Meet set $\cap s$. Moreover, the Choice set of s is unique and is equal to $\cap s$, where $\cap s$ is a singleton and $\cap s \in s$.

Let $(F, \in)$ be a finite model for the set-theoretical Powerset axiom, i.e., in $(F, \in)$ every set has a powerset.

For instance, let us consider the finite model $(M, \in)$ whose domain consists of the three sets a, b, c and where the $\in$-relation is defined by:

(1) $a = \{b\}, \qquad b = \{a\}, \qquad c = \{a, b, c\}.$

It can be readily verified that $(M, \in)$ is a model for the Powerset axiom. Indeed, we have:

(2) $\mathcal{P}(a) = b, \qquad \mathcal{P}(b) = a, \qquad \mathcal{P}(c) = c$

where $\mathcal{P}(x)$ stands for the Powerset of x, i.e., the set of all subsets (needless to say, which exist in $(M, \in)$) of x.

We verify (2), say, for c. From (1) it follows that each one of the three sets a, b, c is a subset of c. Moreover, since a, b, c are collected by c, it follows that c is the set of all the subsets of c in $(M, \in)$. Hence $\mathcal{P}(c) = c$ in $(M, \in)$.

In [1] it is shown that in a finite model for the Powerset axiom the set-theoretical Extensionality axiom also holds. Thus, the notions of "uniqueness" and "equality" used in the above, and the notations introduced in (1) and (2), are justified.

Also, in [1] it is shown that in a finite model $(F, \in)$ of the Powerset axiom, for every set x and y

(3) $x \subseteq y$ iff $\mathcal{P}(x) \subseteq \mathcal{P}(y)$, and
(4) Every set of $(F, \in)$ is a powerset of some set of $(F, \in)$ and thus there is no empty set in $(F, \in)$.

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Let us recall that a set is called *disjointed* iff no pairwise distinct elements of it have an element in common. Also, a set c is called a *Choice set* of a set s, iff *c has one and only one element in common with every element of s and every element of c is an element of some element of s.*

Let us consider the following two statements, of which the first is the usual Axiom of Choice ([2], p. 55).

(AC_1) Every disjointed set none of whose elements is the empty set has a Choice set.

(AC_2) Every set none of whose elements is the empty set has a Choice set.

Clearly, (AC_2) need not be valid in every model of ZF + AC_1, as shown below.

In a finite model for the Powerset axiom the situation is as follows. As shown in [1], in any finite model for the Powerset axiom, AC_1 is automatically valid; but AC_2 need not be valid. Indeed, the finite model $(M, \in)$ defined by (1) and (2) is a model for the Powerset axiom, nevertheless c has no Choice set in the model $(M, \in)$. This is because none of the sets $\{a\}, \{b\}, \{a,b,c\}$ can possibly be a Choice set of the set $c = \{a,b,c\} = \{\{a\},\{b\},\{a,b,c\}\}$. On the other hand, Theorem 2 below shows that in a finite model for the Powerset axiom if every set s has a Meet set $\cap s$ (i.e., *the set of all the common elements of the elements of s*) then AC_2 is valid in that model. Clearly, again this does not hold in every model of ZF + AC_1, even though in the latter every set has a Meet set (we take $\cap\varnothing = \varnothing$).

We observe that in a finite model for the Powerset axiom it is not necessarily the case that every set has a Meet set. For instance, c in the above model $(M, \in)$ has no Meet set.

Lemma 1 *Let $(F, \in)$ be a finite model for the Powerset axiom. If s in $(F, \in)$ has a Choice set c then $c \in s$. Moreover, c is a singleton and $c = \cap s$.*

Proof: As mentioned in (4), since $(F, \in)$ has no empty set and since every set in $(F, \in)$ is the powerset of some set, we let

(5) $s = \{s_1, \ldots, s_n\} = \mathcal{P}(s_1)$.

Now, let c be a Choice set of s. From (5) it follows that every element of s is a subset of s_1 and therefore, by the definition of a Choice set, $c \subseteq s_1$, which again by (5) implies that $c \in s$. Again, from (5) it follows that c cannot have more than one element, since $c \in s$ and c is a Choice set of s. Therefore, c is a singleton, since $(F, \in)$ has no empty set. But then obviously $c = \cap s$.

Corollary *In a finite model for the Powerset axiom a set has at most one Choice set.*

Proof: The above Lemma implies that in a finite model for the Powerset axiom if a Choice set of s exists then it is uniquely determined by x as $\cap s$.

Next, we prove the following rather unexpected inverse of Lemma 1.

Lemma 2 *Let $(F, \in)$ be a finite model for the Powerset axiom. If s in $(F, \in)$ has a Meet set $\cap s$ then $\cap s \in s$. Moreover, $\cap s$ is a singleton and $\cap s$ is a Choice set of s.*

Proof: As in the proof of Lemma 1, let

(6) $s = \{s_1, \ldots, s_n\} = \mathcal{P}(s_1)$.

Now, let the Meet set $\cap s$ of s exist in $(F, \in)$. Clearly $\cap s \subseteq s_1$ so that by (6) we have $\cap s \in \mathcal{P}(s_1)$ and therefore $\cap s \in s$. From this and (6) it follows that

(7) $\cap s = s_i$, for some $s_i \in s$.

But then, just as in the case of s in (6), for s_i we have

(8) $s_i = \{t_1, \ldots, t_m\} = \mathcal{P}(t_1) \subseteq s_1$.

We prove that $\cap s$ is a singleton by showing that an arbitrary element t_j of s_i is equal to t_1. Indeed, let $t_j \in s_i$. But then, by (8), we have

(9) $t_j \subseteq t_1$.

Consequently, by (3), (9), and (8) we have $\mathcal{P}(t_j) \subseteq \mathcal{P}(t_1) \subseteq s_1$ which by (6) implies $\mathcal{P}(t_j) \in s$. Thus, $\cap s \subseteq \mathcal{P}(t_j)$ which, by (7) and (8), implies $\mathcal{P}(t_1) \subseteq \mathcal{P}(t_j)$. But then by (3) we have $t_1 \subseteq t_j$ which, in view of (9), implies $t_j = t_1$. Thus, $\cap s = \{t_1\}$, i.e., $\cap s$ is a singleton, and since $\cap s \in s$ we see that $\cap s$ is (by the above Corollary) the Choice set of s.

From Lemmas 1 and 2, we immediately derive:

Theorem 1 *In a finite model for the Powerset axiom, every set has a Choice set iff every set has a Meet set. Moreover, the Meet set $\cap s$ of a set s is such that $\cap s \in s$ and $\cap s$ is a singleton and is the unique Choice set of s.*

Let "The Meetset axiom" stand for the statement "every set has a Meet set". Then, in view of Theorem 1, we have:

Theorem 2 *In a finite model for the Powerset axiom it is the case that the Axiom of Choice* AC_2 *is valid iff the Meetset axiom is valid.*

From the above we see that in finite models for the Powerset axiom the Meetset axiom is equivalent to the stronger (than AC_1) version AC_2 of the Axiom of Choice.

Below we give two more results concerning finite models for the Powerset axiom.

Lemma 3 *Let $(F, \in)$ be a finite model for the Powerset axiom. Let s be a set of $(F, \in)$ with $n \geq 2$ elements. Then in $(F, \in)$ there exists a set with at most $n - 1 \geq 1$ elements.*

Proof: As in (5), let $s = \{s_1, \ldots, s_n\} = \mathcal{P}(s_1)$. Since s has ≥ 2 elements, let s_k be an element of s distinct from s_1. Thus, s_k is a proper subset of s_1 and by (3) we see that $\mathcal{P}(s_k)$ is a proper subset of $\mathcal{P}(s_1) = s$. Clearly, $\mathcal{P}(s_k)$ is an element of $(F, \in)$, and since $\mathcal{P}(s_k)$ is a proper subset of s we see that $\mathcal{P}(s_k)$ has at most $n - 1 \geq 1$ elements since there is no empty set in $(F, \in)$.

Based on Lemmas 2 and 3, we prove:

Theorem 3 *Let $(F, \in)$ be a finite model for the Powerset axiom. Then $(F, \in)$ always has a singleton. Moreover, every element of a singleton of $(F, \in)$ is itself a singleton.*

Proof: Let s be a set of $(F, \in)$ with $n \geq 2$ elements. By (4) there is no empty set in $(F, \in)$. Therefore, by applying Lemma 3 to s at most $n - 1$ times, it can be readily shown that $(F, \in)$ has a singleton.

Next, let $q = \{h\}$ be a singleton in $(F, \in)$. Obviously, $\bigcap q = h$ and therefore, by Lemma 2, we see that h is a singleton. Thus, every element of a singleton of $(F, \in)$ is itself a singleton, as desired.

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