Notre Dame Journal of Formal Logic
Volume 31, Number 2, Spring 1990

Natural Deduction in Normal Modal Logic

JOHN HAWTHORN

Abstract A natural deduction system for a wide range of normal modal logics is presented, which is based on Segerberg's idea that classical validity should be preserved "in any modal context". The resulting system has greater flexibility than the common Fitch-style systems.

In the introductory sections of Bull and Segerberg [2], Segerberg surveys the deductive methods available in modal logic, and finds them wanting (pp. 25–30). Hilbert systems are too clumsy,[1] Hintikka/Kripke tableaux methods become too complicated, and natural deduction methods, of either the Fitch or the Gentzen styles, are too restricted, being unable to handle the full range of normal modal logics. In response to this problem, he proposes a compromise solution; we should use a natural deduction formulation of **K**, the least normal modal logic, and then treat other systems as theories in **K**, formed by adding appropriate axioms to the natural deduction system. He goes on to propose a system which he claims is a version of **K**.

The purpose of this paper is to explore and extend Segerberg's system. I present a Fitch-based version of it, and show that it is indeed equivalent to **K**, and then compare it with Fitch's own modal systems. I extend the theory in two ways, first by liberalizing the rules, and then by using the liberalized version to give formulations of a wide range of modal logics, including, but not restricted to, the "standard" ones **T**, **D**, **B**, **S4**, and **S5**.

Segerberg's starting point is the following observation:

> The crux of the matter seems to be that any classically valid argument should remain valid *in any modal context*; the difficulty is to explicate the italicized phrase. The solution seems to be to require that whenever Γ tautologically implies A, then also $\Box^n\Gamma \vdash \Box^n A$. (p. 28)

Here $\Box^n\Gamma = \{\Box^n B : B \in \Gamma\}$, where $\Box^n$ abbreviates an n-long string of $\Box$'s. Segerberg then gives a set of inference rules following the Gentzen/Prawitz format, but does not give the necessary set of deduction rules.[2] Rather than follow

Received August 13, 1987; revised September 19, 1988

Segerberg slavishly, I shall give a Fitch-based version, in order to make comparison with the actual Fitch systems more direct.

The system to be described here presupposes a standard Fitch-style formulation of classical logic, **FCL**, with introduction and elimination rules for the connectives, $\sim$, $\vee$, &, and $\supset$.[3] In Prawitz's terminology ([8], p. 23), these rules are either *proper*, that is, rely only on sentences occurring earlier in the proof, or they are *improper*, that is, they rely on prior subproofs. For ease of exposition, I shall assume that the only improper rules in the system are $\supset$**I** and $\sim$**I**.[4] I shall also assume that such notions as (an occurrence of) a sentence or a subproof lying *immediately inside* a subproof are standardly defined, that the inference rules require that the items involved in an application of a rule all lie immediately inside the same subproof, and that Reiteration can be used only from one subproof to another immediately inside it. Finally, I assume the standard formation rules for a sentence in a language containing $\sim$, $\vee$, &, $\supset$, $\Box$, and $\Diamond$, and the standard definition of $\Diamond$ in terms of $\Box$ and $\sim$.

The System NDK

Proper Rules For each of the proper rules of **FCL**, schematically of the form A_1, (A_2), $(A_3) \vdash A$, the rule $\Box^n A_1$, $(\Box^n A_2)$, $(\Box^n A_3) \vdash \Box^n A$ is a rule of **NDK**.

Improper Rules Every subproof is flagged with $\Box^n$, for some $n \geq 0$. If a subproof headed by A and immediately containing B (or both B and $\sim B$) is flagged by $\Box^n$, then $\Box^n(A \supset B)$ (or $\Box^n \sim A$) is derivable in the (sub)proof immediately containing this subproof.

Reiteration If $\Box^n A$ lies immediately outside a subproof flagged by $\Box^n$, then A can be written immediately inside the flagged subproof.

Clearly, any proof in **NDK** in which $n = 0$ throughout is just a standard Fitch-style proof, so **NDK** contains all of classical logic, assuming that the standard Fitch-style systems do. If Th(**NDK**) is defined as the set of theses of **NDK**, i.e., those sentences which can be proved categorically rather than hypothetically, then we can give a suitable sense in which **NDK** is equivalent to **K**: Th(**NDK**) = **K**.

Theorem 1 Th(**NDK**) *is a normal modal logic.*

Proof: Every normal modal logic (NML) satisfies the following two conditions (in addition to having $\Diamond$ defined in terms of $\Box$ and $\sim$):[5]

(i) NML contains all tautologies and is closed under all classically valid inference forms.
(ii) NML is closed under the rule RK:

$$\text{RK} \qquad \frac{(A_1 \,\&\, A_2 \,\&\ldots\&\, A_n) \supset B}{(\Box A_1 \,\&\, \Box A_2 \,\&\ldots\&\, \Box A_n) \supset \Box B} \qquad n \geq 0.$$

Evidently, Th(**NDK**) satisfies condition (i), since **NDK** contains the classical Fitch system as a fragment. To see that it also satisfies condition (ii), consider the following schematic proof:

$$
\begin{array}{llr}
\vert\ \Box A_1 \,\&\, \Box A_2 \,\&\ldots\&\, \Box A_n & & \text{Ass.} \\
\vert\ \Box A_1 & & \&\text{E} \\
\vert\ \ \vdots & & \\
\vert\ \Box A_n & & \&\text{E} \\
\vert\ \Box\ \vert & \sim A & \text{Ass.} \\
\vert\ \quad\ \vert & A_1 & \text{Reit} \\
\vert\ \quad\ \vert & \ \vdots & \\
\vert\ \quad\ \vert & A_n & \text{Reit} \\
\vert\ \quad\ \vert & A_1 \,\&\, A_2 \,\&\ldots\&\, A_n & n\text{-fold }\&\text{I} \\
\vert\ \quad\ \vert & (A_1 \,\&\, A_2 \,\&\ldots\&\, A_n) \supset A & \text{Ex Hyp.} \\
\vert\ \quad\ \vert & A & \supset\text{E} \\
\vert\ \Box\sim\sim A & & \sim\text{I} \\
\vert\ \Box A & & \sim\text{E} \\
(\Box A_1 \,\&\, \Box A_2 \,\&\ldots\&\, \Box A_n) \supset \Box A & & \supset\text{I}
\end{array}
$$

In what follows, 'RK' will be used to refer either to the rule just given, or to the equivalent rule RK^n:

$$
\mathrm{RK}^n \qquad \frac{(A_1 \,\&\, A_2 \,\&\ldots\&\, A_m) \supset B}{(\Box^n A_1 \,\&\, \Box^n A_2 \,\&\ldots\&\, \Box^n A_m) \supset \Box^n B}.
$$

Theorem 2 *Every normal modal logic contains* Th(**NDK**).

Proof: We show that every deduction of a member A of Th(**NDK**) can be transformed into a sequence of sentences Σ with the following two properties: every sentence in Σ belongs to any normal modal logic (NML), and the last member of Σ is identical to the sentence A. Consequently, Th(**NDK**) is included in NML. The proof goes by induction on the length of the sequence. The sequence is formed in the following way. Let the sentences in the deduction be the sequence $\langle A_1, A_2, \ldots, A_n\rangle$; for any sentence in this sequence, A_i, let Ass_i be the assumption at the head of the subproof immediately inside of which A_i lies, and let Conj_i be the conjunction of all the sentences A_j which lie immediately inside the same subproof and whose justification is Reiteration. Then Σ is the sequence $\langle\sigma(A_1), \sigma(A_2), \ldots, \sigma(A_n)\rangle$, where:

$$
\sigma(A_i) = (\mathrm{Ass}_i \,\&\, \mathrm{Conj}_i) \supset A_i.
$$

When A_i lies inside no subproofs and under no assumptions, clearly $\sigma(A_i) = A_i$.

Now suppose that all the members of Σ preceding $\sigma(A_i)$ are in NML; we shall show that $\sigma(A_i)$ is too. There are four cases to consider; first, when A_i has no justification, i.e., it is an assumption; second, when A_i is obtained by Reit.; third, when A_i is obtained by a proper inference rule; and fourth, when A_i is obtained by an improper rule.

First and second cases: If A_i is an assumption, or obtained by Reit., then A_i either is Ass_i or it is one conjunct of Conj_i. Either way, $\sigma(A_i)$ is a tautology, hence $\sigma(A_i) \in$ NML.

Third case: If A_i is derived using a proper rule, then it has the form $\Box^n B_i$, and there are preceding occurrences $\Box^n B_j$, $(\Box^n B_k)$, $(\Box^n B_l)$ which are in the same subproof and which satisfy the following conditions:

(i) $(\text{Ass}_i \,\&\, \text{Conj}_i) \supset \Box^n B_j \in \text{NML}$
(ii) $(\text{Ass}_i \,\&\, \text{Conj}_i) \supset \Box^n B_k \in \text{NML}$
(iii) $(\text{Ass}_i \,\&\, \text{Conj}_i) \supset \Box^n B_l \in \text{NML}$
(iv) $(B_j \,\&\, B_k \,\&\, B_l) \supset B_i$ is a tautology.

By RK, then, (v) is in NML:

(v) $(\Box^n B_j \,\&\, \Box^n B_k \,\&\, \Box^n B_l) \supset \Box^n B_i$

and hence from (i)–(iii) and (v), $\sigma(A_i) \in \text{NML}$.

Fourth case: In the case where the rule is $\supset$I, A_i has the form $\Box^n(A_j \supset A_k)$, and there is a preceding subproof flagged with $\Box^n$ whose assumption is A_j, and in which an occurrence of A_k lies. Thus the following condition obtains:

(i) $(A_j \,\&\, \text{Conj}_k) \supset A_k \in \text{NML}$.

By propositional logic and RK it follows that

(ii) $\Box^n \text{Conj}_k \supset \Box^n(A_j \supset A_k) \in \text{NML}$

where $\Box^n \text{Conj}_k$ is the conjunction formed by prefacing every conjunct of Conj_k by $\Box^n$. However, since every member $\Box^n C_m$ of $\Box^n \text{Conj}_k$ must occur as an earlier line in the same subproof as A_i, we obtain, by the induction hypothesis, for every C_m,

(iii) $(\text{Ass}_i \,\&\, \text{Conj}_i) \supset \Box^n C_m \in \text{NML}$.

Hence, from (ii) and (iii), by propositional logic, we obtain

(iv) $(\text{Ass}_i \,\&\, \text{Conj}_i) \supset A_i \in \text{NML}$.

The case in which A_i is derived by $\sim$I is essentially similar.

Theorem 3 Th(**NDK**) = **K**.

Proof: Since by Theorem 1 Th(**NDK**) is a normal modal logic, and by Theorem 2 it is contained in any normal logic, it is the least normal modal logic, which is **K**.

Having shown that **NDK** is a natural deduction form of **K**, I shall compare it briefly with Fitch's own natural deduction modal logics, and those that have grown out of them. Using the same starting point, the classical logic **FCL**, Fitch extended it by adding introduction and elimination rules for $\Box$, thereby treating $\Box$ exactly on a par with $\sim$, $\vee$, &, and $\supset$. The elimination rule was easy: $\Box A \vdash A$. The introduction rule required more elaborate methods, however. Fitch introduced a category of strict subproofs, flagged with a $\Box$, and an associated rule of strict reiteration; the $\Box$I rule then says that if A is established in a strict subproof, $\Box A$ can be inferred immediately outside it. Depending on the constraints on the rule of strict reiteration, different modal logics are obtainable. If the rule is as in **NDK**, of the eye for an eye, $\Box$ for a $\Box$ variety, then we get **KT**; if we allow the reiteration of $\Box A$ itself through a barrier we get **KT4**, and so forth. In Fitch [5] different reiteration rules are offered for the systems **T (KT)**, **B (KTB)**, **S4 (KT4)**, and **S5 (KT5** or **KTE)**. Coupled with either a variant of the $\Box$E rule of the form $\Box A \vdash \Diamond A$, or no $\Box$E rule at all, and distinguishing different components of the rules, it becomes possible to give versions of all the normal systems that arise from combining **D**, **T**, **B**, **4**, and **5** (or E).[6]

These systems have all the virtues of a good natural deduction system: they are easy to work in, they have successful strategy rules, and they provide a feel for the deductive structure of the respective logics. The disadvantage is that these virtues begin to disappear as soon as one tries to extend their methods to other logics. The method of varying the rule of strict reiteration works only for axioms of the form $\alpha A \supset \Box\beta A$, where α and β are modalities, and the number of axioms that can plausibly be thought of as embodying a $\Box$E rule is extremely small. Very soon, one has to admit defeat, and just add axioms, as Segerberg advocated. What one loses thereby in insight into specific logics, one gains in the ability to work in arbitrary ones.[7]

Nevertheless, since one can as easily add axioms to the version of **K** derived from Fitch's own system, which I shall call **FK**, as to **NDK**, it is interesting to compare the two systems. They start from the same point, Fitch's classical system, but diverge in what they preserve in their extensions to modal logic. In **FK**, what is preserved is the connection between connectives and rules; each connective has its own rules. Of course, even here there is a change, since in **FCL** each connective has an introduction and an elimination rule, whereas in **FK** $\Box$ has only an introduction rule, but that is a small matter. In **NDK**, by contrast, $\Box$ has no rules peculiar to itself, but is instead pervasive; every rule has its $\Box$ characteristic. What is preserved instead is the structure; there is no new category of strict subproofs, since the only things that are flagged would have been ordinary subproofs in **FCL**. All the rules are modal generalizations of the classical rules, and nothing else is changed. In **FK**, however, strict subproofs need have no assumptions at their heads (indeed, in some versions they never have). Thus the relationship of subproof to assumption also changes in **FK**.

These differences lead to further differences in the overall emphases of the systems, and also in the proof strategies. Taking the latter point first, the continuity of $\Box$ with the other connectives is preserved in strategy in **FK**; if you want to prove a sentence whose main connective is $\Box$, open a strict subproof and try to prove the immediate subformula. In **NDK**, though, strategy is determined by the main propositional connective, with the modal operators added along the way. In terms of overall emphasis, the dominant effect in **NDK** is given by the remark from Segerberg above (p. 263): **NDK** is classical logic in a modal context, and it would only be a slight distortion to regard modal compounds as being formed not by a modal operator operating on, say, a conjunction, but instead by one of a number of modalized conjunction operators, a $\Box^n\&$. In **FK** a heavier stress is laid on the parallel between the quantifiers and the modal operators, particularly when the rules for $\Diamond$ are added. Further, it is very natural to interpret a strict subproof as another possible world, and the reiteration rule as governed by the accessibility relation.

Thus the two systems emphasize different aspects of **K**. There is, however, a big difference in the redundancy of the formulations. **FCL**, at least in most of its standard versions, has no redundancy as long as the connectives are thought of as independent, and the same is true of **FK**. **NDK**, however, has two dimensions of redundancy. The first, which is obvious enough, is that the use of $\Box^n$ throughout could be replaced by plain $\Box$ plus induction. The second dimension is that the $\Box$'s could be eliminated altogether on a number of the rules, and retained on only a minimal kernel. As far as I can see, the only two kernels using standard introduction and elimination rules are $\supset$I and $\supset$E, and $\sim$I and $\sim$E; these two kernels have both an improper introduction rule and a proper elimination rule, so those who wish to may regard the full set of proper and improper rules of **NDK** as derived from a more economical basis. However, if the classical reductio rule is allowed (derive A from a subproof whose assumption is $\sim A$ and which includes B and $\sim B$) then a modalized version of that alone, together with nonmodalized versions of all the other rules, provides a basis for **K**.[8] This last rule, however, breaks down the introduction/elimination duality even in **FCL**, so is normally introduced only as a derived rule.

A more radical divergence between **NDK** and **FK** appears when we consider which rules can be adopted for $\Diamond$. The Fitch rule, elegantly preserving the modality/quantifier parallel, permits the inferring of $\Diamond B$ from $\Diamond A$ plus a strict subproof in which B is inferred from A. So far, no rules have been offered for $\Diamond$ in **NDK**, a defect I shall now remedy. The controlling ideal remains that of preserving classical logic in every modal context; we have to broaden the notion of context to include $\Diamond$. Of course, we cannot take over the $\Box$ rules and replace the $\Box$'s by $\Diamond$'s, since a number of the rules thus formed would be invalid. But rather than setting up a separate set of rules for $\Diamond$, parallel to those for $\Box$, I shall give a single set of rules which reduce to the **NDK** rules in the case where the context is $\Box^n$, but which allow for more general modal contexts.

In what follows, all references to modalities will be implicitly restricted to affirmative modalities, and, in fact, for ease of discussion, to affirmative modalities which contain no negation signs.[9] In view of the equivalence of $\Diamond$ and $\sim\Box\sim$ etc., this latter restriction involves no loss of generality. We start with some necessary preliminaries; Greek letters will be used as variables for modalities.

Definitions A modality α *resembles* a modality β just in case

(i) α and β are the same length, and
(ii) α has a $\Box$ wherever β does.

A finite sequence Σ of modalities $\alpha_1, \alpha_2 \ldots \alpha_n$ *matches* a modality β just in case either

(i) Σ is empty and β is $\Box^n$, or
(ii) for all i, α_i resembles β, and wherever β contains a $\Diamond$, α_i contains a $\Diamond$, for exactly one i.

If A occurs in a proof as a result of a strict reiteration from an occurrence of αA, then α is the *tail* of that occurrence of A.

Note that resemblance is not a symmetric property, and that a string of null modalities is to be distinguished from an empty sequence. In the case of a single modality, $\langle\alpha\rangle$ matches β just in case α and β are identical. With these resources, we can define a new set of rules for **NDK**.

The System **NDK***

Proper Rules For each of the proper rules of **FCL**, A_1, (A_2), $(A_3) \vdash A$, the rule $\alpha_1 A_1$, $(\alpha_2 A_2)$, $(\alpha_3 A_3) \vdash \beta A$ is a rule of **NDK***, so long as $\langle \alpha_1, (\alpha_2), (\alpha_3)\rangle$ matches β.
Improper Rules If a subproof headed by A and immediately containing B (or B and $\sim B$) is flagged by a modality α, $\alpha(A \supset B)$ (or $\alpha{\sim}A$) is derivable in the (sub)proof immediately containing this subproof, so long as the sequence containing all the tails of the sentences strictly reiterated into the subproof matches α.
Reiteration If βA lies immediately outside a subproof flagged by α, then A can be written immediately inside the flagged subproof, so long as β resembles α.

Despite the liberalization of all the rules, Th(**NDK***) = **K**. To see this, we need to establish a lemma about the presence in normal systems of the rule RK*:

$$\text{RK*} \qquad \frac{(A_1 \ \&\ A_2 \ \&\ldots\&\ A_n) \supset A}{(\alpha_1 A_1 \ \&\ \alpha_2 A_2 \ \&\ldots\&\ \alpha_n A_n) \supset \alpha A}$$

for $n \geq 0$, iff $\langle \alpha_1, \alpha_2, \ldots \alpha_n\rangle$ matches α

Lemma *Let Σ be a set of sentences containing all classical tautologies, closed under classical consequence and the substitution of $\Diamond$ for $\sim\Box\sim$ etc. Then Σ is closed under* RK* *iff it is closed under* RK.

Proof: Left to right. For $n > 0$, RK* has RK as a special case, $\alpha_i = \alpha = \Box$, for all i. For $n = 0$, the matching condition requires $\alpha = \Box^m$, so RK holds too.

Right to left. By induction on the length of α. The base case, when α is a null modality, is trivial. Suppose RK* holds for any m-long α. An $m + 1$-long α' is either $\Box\alpha$ or $\Diamond\alpha$. In the first case, by the condition of matching, either there are no α_i, or all the α_i^{m+1} are of the form $\Box\alpha_i^m$. Either way, the conclusion follows by RK. In the second case, the condition of matching requires that exactly one of the α_i^{m+1} is of the form $\Diamond\alpha_i^m$, while the remainder, if any, are of the form $\Box\alpha_i^m$. Let the one of form $\Diamond\alpha_i^m$ be α_n^{m+1}; then the induction hypothesis yields, by propositional logic

$$(\alpha_1^m A_1 \ \&\ \alpha_2^m A_2 \ \&\ldots\&\ \alpha_{n-1}^m A_{n-1}) \supset (\alpha_n^m A_n \supset \alpha^m A)$$

and thus

$$(\Box\alpha_1^m A_1 \ \&\ \Box\alpha_2^m A_2 \ \&\ldots\&\ \Box\alpha_{n-1}^m A_{n-1}) \supset \Box(\alpha_n^m A_n \supset \alpha^m A)$$

by RK.

But the following is a theorem of any normal logic:

$$\Box(P \supset Q) \supset (\Diamond P \supset \Diamond Q).$$

Hence, by jiggling,

$$(\Box\alpha_1^m A_1 \ \&\ \Box\alpha_2^m A_2 \ \&\ldots\&\ \Box\alpha_{n-1}^m A_{n-1} \ \&\ \Diamond\alpha_n^m A_n) \supset \Diamond\alpha^m A$$

which yields the desired result:

$$(\alpha_1^{m+1} A_1 \ \&\ \alpha_2^{m+1} A_2 \ \&\ldots\&\ \alpha_n^{m+1} A_n) \supset \alpha^{m+1} A.$$

Given this rule, the proof that Th(**NDK***) = **K** closely follows the proof for **NDK**. The analogue of Theorem 1 (that **NDK*** is a normal modal logic) is proved by replacing $\Box^n$ by α_i etc. or α, as appropriate, replacing references to RK by RK*, and relying on the preceding lemma. The analogue of Theorem 2 (that **NDK*** is in any NML) mostly requires similar rewriting; the only case which becomes appreciably more complex is the last one:

Fourth Case: In the case where the rule is $\supset$I, A_i has the form $\alpha(A_j \supset A_k)$, and there is a preceding subproof flagged with α whose assumption is A_j, and in which lies an occurrence of A_k. Thus the following condition obtains:

(i) $(A_j \ \&\ \mathrm{Conj}_k) \supset A_k \in \mathrm{NML}$.

Let Conj_k be spelled out as $C_1 \ \&\ C_2 \ \&\ldots\&\ C_m$. By propositional logic and RK* it follows that

(ii) $\alpha_1 C_1 \ \&\ \alpha_2 C_2 \ \&\ldots\&\ \alpha_m C_m \supset \alpha(A_j \supset A_k) \in \mathrm{NML}$

for any $\langle \alpha_1, \alpha_2, \ldots \alpha_m \rangle$ which matches α. However, each C_n in Conj_k has a tail, α_n, such that $\alpha_n C_n$ occurs as an earlier line in the same subproof as A_i, and such that all the tails together match α. Hence we obtain (iii), by the induction hypothesis, for every $\alpha_n C_n$; and then from (ii) and (iii) we obtain (iv):

(iii) $(\mathrm{Ass}_i \ \&\ \mathrm{Conj}_i) \supset \alpha_n C_n \in \mathrm{NML}$
(iv) $(\mathrm{Ass}_i \ \&\ \mathrm{Conj}_i) \supset A_i \in \mathrm{NML}$.

In order to show some of the workings of **NDK***, here are two invalid attempts at a proof, and one valid one (for convenience the third uses, and the second attempts to use, the reductio rule, rather than a step each of ~I and ~E):

(a)	$\Diamond A$			Ass.
	$\Diamond B$			Ass.
	$\Diamond(A \ \&\ B)$			&I
(b)	$\Box A$			Ass.
		$\Diamond$	$\sim A$	Ass.
			A	Reit.
	$\Diamond A$			Reductio
(c)	$\Box A$			Ass.
	$\Diamond B$			Ass.
		$\Diamond$	$\sim A$	Ass.
			A	Reit.
			B	Reit.
	$\Diamond A$			Reductio

The first attempt fails because in the application of &I $\langle \Diamond, \Diamond \rangle$ does not match $\Diamond$, while the second fails because $\langle \Box \rangle$ does not match $\Diamond$ either, though $\Box$ does resemble $\Diamond$. The rather odd (c) succeeds because the modalities in the reiterations resemble the flag, and because $\langle \Box, \Diamond \rangle$ matches $\Diamond$. Of course, these examples are rather slight, but they give some flavor of the rules.

The advantage of this liberalization is that it gives us powerful tools for handling an extended class of normal modal logics. Consider the class of axioms of the form $\gamma A \supset \delta A$, where γ and δ are affirmative modalities, as before. We can define an extension of **NDK***, **NDK**(γ/δ), as follows:

The System NDK(γ/δ)

Proper Rules For each of the proper rules of **FCL**, schematically of the form A_1, (A_2), $(A_3) \vdash A$, the rule $\alpha_1 A_1$, $(\alpha_2 A_2)$, $(\alpha_3 A_3) \vdash \alpha\delta\alpha' A$ is a rule of **NDK***(γ/δ), so long as $\langle \alpha_1, (\alpha_2), (\alpha_3)\rangle$ matches $\alpha\gamma\alpha'$.
Improper Rules If a subproof headed by A and immediately containing B (or B and $\sim B$) is flagged by a modality $\alpha\gamma\alpha'$, $\alpha\delta\alpha'(A \supset B)$ (or $\alpha\delta\alpha'{\sim}A$) is derivable in the (sub)proof immediately containing this subproof, so long as the sequence containing all the tails of the sentences strictly reiterated into the subproof matches $\alpha\gamma\alpha'$.
Reiteration Unchanged: If βA lies immediately outside a subproof flagged by α, then A can be written immediately inside the flagged subproof, so long as β resembles α.

In terms of this formulation, **NDK*** is therefore **NDK(/)**, the null switch, and we should always permit null switch inferences even in logics permitting various switches. It might be possible to relax the reiteration rule after the manner of Fitch, though this is far from clear; but the complications of keeping track of the matching would outweigh the advantages. We can, instead, avail ourselves of a derived rule. Unsurprisingly, $\gamma A \supset \delta A$ is a thesis of **NDK**(γ/δ):

γA		Ass.
γ	$\sim A$	Ass.
	A	Reit.
δA		Reductio

Thus we can gain some of the effect of the liberal reiteration rule, while simplifying the accounting, by using the derived rule $\gamma A \vdash \delta A$ outside the subproof and then reiterating in whatever is possible. An example of this technique is the following proof in McKinsey's **S4.1**, which in our terms is **NDK**$(\Box/)(\Box/\Box\Box)$ $(\Box\Diamond/\Diamond\Box)$, the (γ/δ) forms given being those appropriate to **T**, **4**, and McKinsey's **M**, respectively. In this proof, 'PL' is used for obvious steps of classical propositional logic:

1	$\Box(A \vee B)$			Ass.
2		$\sim(\Diamond\Box A \vee \Diamond\Box B)$		Ass.
3		$\sim\Diamond\Box A$		2, PL
4		$\Box\Diamond{\sim}A$		3, Def.$\Diamond$
5		$\Box\Box(A \vee B)$		1, Reit.+ $\Box/\Box\Box$
6		$\Box\Diamond$	$\sim B$	Ass.
7			$A \vee B$	5, Reit.
8			A	6,7, PL
9			$\sim A$	4, Reit.
10		$\Diamond\Box B$		6-9, $\sim$I
11		$\sim\Diamond\Box B$		2, PL
12	$\Diamond\Box A \vee \Diamond\Box B$			2-11, Reductio

Another proof, this time from **NDK**($\Box\Diamond/\Diamond\Box$):

1	$\Box\Box(A \supset B)$	Ass.
2	$\quad \sim\Diamond\Box(A \supset B)$	Ass.
3	$\quad \Box\Diamond(A \vee \sim B)$	2, Def.$\Diamond$ + PL
4	$\quad \Box\Diamond A$	3, &E
5	$\quad \Box\Box(A \supset B)$	1, Reit.
6	$\quad \Diamond\Box B$	4,5, MP($\Box\Diamond/\Diamond\Box$)
7	$\quad \Box\Diamond\sim B$	3, &E
8	$\quad \sim\Diamond\Box B$	7, Def.$\Diamond$
9	$\Diamond\Box(A \supset B)$	2-8, Reductio

Naturally, for the many logics whose axioms are not of the form $\gamma A \supset \delta A$, the possibility of extending **NDK*** to **NDK**(γ/δ) is of only marginal interest. Nevertheless, it is an illustration of the power of **NDK***, and of some interest in its own right. Returning to the complaints of Segerberg with which this paper began, we can see that if deduction in modal logic does have to be treated in the way he suggests, with a natural deduction system for **K** and other logics introduced via axioms, **NDK*** is a useful foundation.

NOTES

1. He might have added that, unlike in the case of classical logic, the standard Hilbert systems of modal logic do not give a natural characterization of the (syntactic) consequence relation $\Gamma \vdash A$. This is because of the presence of rules such as Necessitation, which require their premises to be theorems, rather than simply prior items in the derivation.

2. The distinction, briefly put, is that inference rules characterize immediate inference for a system, but that deduction rules are needed to explain how these are combined into full deductions. Segerberg's use of the distinction appears to be slightly different from that in Prawitz [8], pp. 16–24.

3. The *locus classicus* is Fitch [4], but a more convenient source for present purposes is Fitch [5]. Any number of introductory logic texts give slight variations on Fitchian themes.

4. In many systems there is an improper $\vee$E rule, too; this can be replaced by the proper rule $P \vee Q, P \supset R, Q \supset R \vdash R$.

5. Originally, normal modal logics were classified model-theoretically; even among syntactic characterizations, the one given is not the most common, but it is convenient. It is borrowed from Chellas [3], to which the author is heavily indebted.

6. The systems which use either **D** or **T** and any or all of the rest are present more or less explicitly in Fitch [5]. The earliest trace I have found of systems using neither of those principles, i.e., no $\Box$E rule, are in Siemens [9]. I have serious doubts that all the claims in this latter paper about possible bases for systems are true, however. Versions of all the systems mentioned, and some more, are presented in Fitting [6].

7. I do not mean by these remarks to suggest that no other natural deduction systems exist. For a given logic, it is often possible to tailor the rules to produce a Fitch-like version; an example is the provability system which uses the axiom $\Box(\Box A \supset A) \supset$

$\Box A$, a version of which is given in Fitting [6]. But two things become clear. First, there is no systematic way in which the rule changes can be implemented to cover more than a few systems at a time, and second, there is a rapid fall-off in insight into the system which they afford.

8. In fact, a modalized reductio rule is the exact analogue in **NDK** of the strict subproofs of **FK**, in the following sense. Take any proof in **FK**, and for each strict subproof find the sentence B for which $\Box B$ is inferred from the subproof; insert $\sim B$ as an assumption to the strict subproof. The result is a proof in a version of **NDK** which uses reductio as its only modalized rule. A similar transformation works in reverse.

9. The definition of an affirmative modality, as a string of $\Box$'s, $\Diamond$'s and $\sim$'s in which the number of $\sim$'s is even, apparently goes back to Becker [1].

REFERENCES

[1] Becker, O., "Zur Logik der Modalitäten," *Jahrbuch für Philosophie und Phänomenologische Forschung*, vol. 11 (1930), pp. 497–548.

[2] Bull, R. A. and K. Segerberg, "Basic modal logic," pp. 1–88 in *Handbook of Philosophical Logic, Volume II*, edited by D. Gabbay and F. Guenthner, Reidel, Dordrecht, 1984.

[3] Chellas, B., *Modal Logic: An Introduction*, Cambridge University Press, Cambridge, 1980.

[4] Fitch, F. B., *Symbolic Logic: An Introduction*, Ronald Press, New York, 1952.

[5] Fitch, F. B., "Natural deduction rules for obligation," *American Philosophical Quarterly*, vol. 3 (1966), pp. 27–38.

[6] Fitting, M., *Proof Methods for Modal and Intuitionistic Logics*, Reidel, Boston, 1983.

[7] Hughes, G. E. and M. J. Cresswell, *An Introduction to Modal Logic*, Methuen, London, 1968.

[8] Prawitz, D., *Natural Deduction: A Proof-Theoretic Approach*, Almqvist and Wiksell, Stockholm, 1965.

[9] Siemens, D. F., Jr., "Fitch-style rules for many modal logics," *Notre Dame Journal of Formal Logic*, vol. 18 (1977), pp. 631–636.

Department of Philosophy
University of Hawaii at Manoa
2530 Dole Street
Honolulu, Hawaii 96822