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Two Hypergraph Theorems Equivalent to BPI

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Abstract Techniques originally developed for establishing NP-Completeness are adapted to prove that two compactness theorems concerning hypergraphs are equivalent to the Prime Ideal Theorem for Boolean algebras (BPI). In addition, some possible connections between NP-Completeness and BPI are explored.

1 Introduction We introduce two combinatorial compactness principles and show them to be in the large class of statements known to be equivalent in ZF set theory to BPI, the Prime Ideal Theorem for Boolean algebras (see, for example, [1], [2], [3], [4], [10]–[20], and [22] for other statements in this class). Both are about hypergraphs and were suggested by two NP-Complete decision problems considered by Schaefer [21]. In fact there seems to be an intimate connection between BPI and NP-Completeness; a major aim of this paper is to explore this connection.

2 A logical compactness theorem One of the more useful versions of BPI is the Compactness Theorem for propositional logic, which states that a set of propositional formulas is satisfiable if every finite subset is satisfiable. The equivalence of the Compactness Theorem for propositional logic and BPI was first proved by Henkin [10]. Here we shall need a restricted version—when all the formulas are disjunctions of at most three literals (a literal is a statement letter or its negation). This restricted version of the Compactness Theorem will be referred to as 3-SAT and our first task is to show that 3-SAT is equivalent to BPI.

Theorem 1 3-SAT $\Leftrightarrow$ BPI.

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Proof: It suffices to show that 3-SAT implies the Compactness Theorem for propositional logic. Let S be a set of propositional formulas, every finite subset of which is satisfiable. We must show, using 3-SAT, that S is satisfiable.

We can first assume that all formulas of S are in conjunctive normal form (cnf); this is so because to each propositional formula a unique equivalent cnf can be associated (that is, the process of finding cnf's can be defined in a canonical fashion). Next we can assume that S consists entirely of disjunctions because each cnf can be replaced by its conjuncts.

Finally, let S' be the result of replacing each disjunction $D = (l_1 \vee \ldots \vee l_k)$ in S which contains more than 3 literals by the formulas in the set

$$E = \{(l_1 \vee l_2 \vee a_1), (\sim a_1 \vee l_3 \vee a_2), (\sim a_2 \vee l_4 \vee a_3), \ldots,$$
$$(\sim a_{k-3} \vee l_{k-1} \vee a_{k-2}), (\sim a_{k-2} \vee l_k \vee \sim a_1)\},$$

where the a_i are new letters for each D. It can be readily shown that each truth assignment satisfying D can be extended to an assignment satisfying the formulas of E and, conversely, any assignment satisfying E, when restricted to the l_i, satisfies D. We claim that any finite subset W' of S' is satisfiable; for if W is the (finite) set of formulas of S that generated the formulas of W', W is satisfiable and any assignment satisfying W can be extended, as indicated above, to an assignment for W'. The satisfiability of S' now follows by 3-SAT. However, any truth-functional assignment satisfying S', when restricted to the literals of S, satisfies S. Therefore S is satisfiable, as required.

***3 Hypergraphs and* BPI** We shall consider in this section two compactness theorems concerning hypergraphs. These two theorems were suggested by two finite decision problems considered by Schaefer [21]. The first problem, called ONE-IN-THREE-SATISFIABILITY, states: "given sets $S_1, \ldots, S_m$ each having at most three members, is there a subset T of the members such that for each i, $|T \cap S_i| = 1$?". The second problem, called NOT-ALL-EQUAL-SATISFIABILITY, states: "given sets $S_1, \ldots, S_m$ each having at most three members, can the members be colored with two colors so that no set is all one color?". Schaefer shows that these problems belong to the class of NP-Complete problems, and thus probably there are no algorithms for deciding them that run in polynomial time (see [7] and [8] for full treatments of NP-Completeness). To treat these problems in a uniform way we adopt the language of hypergraphs.

A *hypergraph* is an ordered pair, $H = \langle V, E\rangle$, where V is a set of elements called *vertices* and E is a collection of finite, nonempty subsets of V; elements of E are called *edges.* If E consists entirely of pairs, H is called a graph. If V and hence E are finite sets, H is called a *finite* hypergraph. We emphasize that the edges are always finite sets whether or not the hypergraph is finite. A hypergraph $K = \langle W, F\rangle$ is a *subhypergraph* of H if $W \subset V$ and $F \subset E$. Let $H = \langle V, E\rangle$ be a hypergraph. A subset W of V is *independent* if no two elements of W belong to the same edge of H. A subset W of V is a *vertex cover* if each edge of H contains at least one vertex of W. An *n-coloring* of H is a function $f: V \to \{0, \ldots, n-1\}$ such that $|f[e]| > 1$, for all e in E with $|e| > 1$, that is, not all members of an edge receive the same color, unless the edge is a singleton. We shall prove that each of the following statements is equivalent to BPI:

[$\mathbf{H_1}$] *Let H be a hypergraph. If every finite subhypergraph has an independent vertex cover then H has an independent vertex cover.*

[$\mathbf{H_2}$] *Let H be a hypergraph. If every finite subhypergraph is 2-colorable then H is 2-colorable.*

The proofs that BPI $\Rightarrow$ H_1 and BPI $\Rightarrow$ H_2 are straightforward; instead of using BPI directly in the proofs it is easier to use an equivalent form, say either the Tychonoff Theorem for compact spaces (see [15] and [16]), or a version of the Rado Selection Lemma (see [2] and [19]). We omit these routine proofs. We turn next to the converses. We shall prove somewhat more than required; let H_i^n, $i = 1,2$, $n > 1$, be the statement H_i restricted to hypergraphs whose edges contain at most n vertices. Then we will prove below that $H_i^3 \Rightarrow$ BPI, $i = 1,2$.

Theorem 2 $H_1^3 \Leftrightarrow$ BPI.

Proof: We prove that H_1^3 implies 3-SAT.

Let S be a set of propositional formulas, all of which are disjunctions of at most three literals. By repeating literals if necessary (say the first which occurs) we can assume that each disjunction has exactly three literals. For each disjunction, $d = (l_1 \vee l_2 \vee l_3)$, we take six new letters, $u_1^d, \dots, u_6^d$ and we define five new sets, $X_1^d, \dots, X_5^d$, as follows (we omit the superscript d on the u's and X's for clarity):

$$X_1 = \{l_1, u_1, u_4\}$$

$$X_2 = \{l_2, u_2, u_4\}$$

$$X_3 = \{u_1, u_2, u_5\}$$

$$X_4 = \{u_3, u_4, u_6\}$$

$$X_5 = \{l_3, u_3\}.$$

We observe (and ask the reader to verify) that any set T such that $|T \cap X_i| = 1$, $i = 1, \dots, 5$, must contain at least one l_i and, conversely, any nonempty subset of $\{l_1, l_2, l_3\}$ can be extended to such a T, by adding appropriate u_i's. For example, $\{l_1, l_3\}$ can be extended to $T = \{l_1, l_3, u_2, u_6\}$. We shall refer to the X_i obtained from the same d as *relatives*.

Let hypergraph H be defined as follows: its edges consist of the sets X_i^d, for each d in S, together with the sets $\{p, \sim p\}$, for each propositional letter occurring in the formulas of S. The vertex set V of H is the union of the edges.

Suppose now that every finite subset of S is satisfiable. We claim that, likewise, any finite subhypergraph, $K = \langle W, F \rangle$, has an independent vertex cover. We can assume, without loss of generality, that if any X_i belongs to F so do all its relatives, for any independent vertex cover for a hypergraph induces an independent vertex cover for each subhypergraph. Let S_K be the set of all d in S for which X_i^d belongs to F; then S_K is finite and hence satisfiable. Let I_K be an interpretation that satisfies S_K. If d belongs to S_K, $d = (l_1 \vee l_2 \vee l_3)$, at least one of the l_i must be true under I_K. Starting with the set of these true literals we can add appropriate u_i's and also one of $\{p, \sim p\}$, for each $\{p, \sim p\}$ in F, to obtain an independent vertex cover for K; this follows from the observation we made above and the fact that the u_i's belonging to different d's are distinct. Hence every finite subhypergraph of H has an independent vertex cover. By H_1^3, H has an independent vertex cover, I. Since $\{p, \sim p\}$ belongs to E, this defines an interpretation: let l_i be true if and only if it belongs to the independent vertex cover I. Again using the above observation, for each d in S, at least one l_i must be in I, that is, true under the interpretation; hence S is satisfiable.

Theorem 3 $H_2^3 \Leftrightarrow$ BPI.

Proof: We prove that H_2^3 implies 3-SAT.

Let S be a set of disjunctions, each with exactly three literals. Assume that every finite subset of S is satisfiable. Let b be a new propositional letter and for each d in S, $d = (l_1 \vee l_2 \vee l_3)$, we take two new propositional letters, a_1^d, a_2^d, and we define the sets $X_1^d, \ldots, X_9^d$, as follows (we omit the superscript d for clarity):

$$X_1 = \{l_1, a_1, b\}$$
$$X_2 = \{l_2, a_2, \sim a_1\}$$
$$X_3 = \{l_3, b, \sim a_2\}$$
$$X_4 = \{l_1, \sim l_1\}$$
$$X_5 = \{l_2, \sim l_2\}$$
$$X_6 = \{l_3, \sim l_3\}$$
$$X_7 = \{b, \sim b\}$$
$$X_8 = \{a_1, \sim a_1\}$$
$$X_9 = \{a_2, \sim a_2\}.$$

Observe that the hypergraph whose edges are $X_1, \ldots, X_9$ is 2-colorable with b assigned 0 if and only if at least one of l_1, l_2, l_3 is assigned 1.

Let H be the hypergraph whose edges are the sets X_i^d, for d in S. It is easy to show, using the above observation and our assumption that every finite subset of S is satisfiable, that every finite subhypergraph of H is 2-colorable. Hence, by H_2^3, H is 2-colorable. Let b's color be 0. Define an interpretation of S as follows: let p be true if and only if p is assigned the color 1. Then, as we have observed, for each $d = (l_1 \vee l_2 \vee l_3)$ at least one of the l_i must be colored 1, that is, at least one of the l_i must be true under the interpretation. Therefore S is satisfiable.

If in the statements H_i^n we now take $n = 2$, we obtain two theorems about graphs. These two theorems are equivalent since a graph has an independent vertex cover if and only if it is 2-colorable. However, H_2 for graphs is known to be equivalent to C_2, the Axiom of Choice for families of pairs (see [17]), and C_2 is weaker than BPI (see [14]); thus both H_1^2 and H_2^2 are weaker than BPI.

The proofs of BPI from H_1 and H_2 are "lifted" from those that establish that the corresponding decision problems are NP-Complete (see [21]); the same is true for our proof, in Theorem 1, that 3-SAT is equivalent to BPI. Other proofs of NP-Completeness can be successfully lifted as well; for example, by lifting Stockmeyer's proof [23] that GRAPH 3-COLORABILITY is NP-Complete, Mycielski has obtained a simpler proof of the Theorem of Läuchli [13] that $P_3 \Rightarrow$ BPI, where P_3 stands for: a graph is 3-colorable if every finite subgraph is 3-colorable. We give Mycielski's unpublished proof next.

Theorem 4 $P_3 \Leftrightarrow$ BPI.

Proof: We shall only prove that $P_3 \Rightarrow$ BPI (see [17] for the converse). Let $\mathbf{B} = \langle B, \wedge, \vee, ', 0, 1\rangle$ be a Boolean algebra. We claim first that a subset I of B will be a prime ideal iff: (1) $b_1 \wedge b_2 \wedge b_3 = 0$ implies $b_i \in I$, for some i, $i = 1,2,3$, and (2) exactly one of $\{b, b'\}$ belongs to I, for each $b \in B$. We shall prove only that conditions (1) and (2) imply that I is an ideal since the rest is rather obvious. Suppose $a, b \in I$. Since $(a \vee b) \wedge a' \wedge b' = 0$ and $a', b' \notin I$, by condition (2), it follows from condition (1) that $a \vee b \in I$. Suppose $a \in I$ and $b \in B$; since $(a \wedge b) \wedge a' \wedge a' = 0$ and $a' \notin I$, it follows that $a \wedge b \in I$.

Next we define a graph $G(\mathbf{B})$ as follows. For each ordered triple $t = \langle b_1, b_2, b_3\rangle$ such that $b_1 \wedge b_2 \wedge b_3 = 0$ we build a Stockmeyer "house" as in Figure 1. Note that: (1) if the house is 3-colored and vertices b_1, b_2, b_3 receive the same color, then t must also receive this color, and (2) if at least one of b_1, b_2, b_3 gets a certain color, then t can receive that color. $G(\mathbf{B})$ consists of all these houses, along with two new vertices c and d, and the following edges:

(1) $\{c, t\}$, $t = \langle b_1, b_2, b_3\rangle$, $b_1 \wedge b_2 \wedge b_3 = 0$
(2) $\{c, d\}$
(3) $\{d, b\}$, $b \in B$
(4) $\{b, b'\}$, $b \in B$.

We claim that a 3-coloring of $G(\mathbf{B})$ yields a prime ideal of $\mathbf{B}$ and vice versa.

Suppose that $G(\mathbf{B})$ is 3-colored with colors $\{0,1,2\}$ and assume d receives the color 2 and c receives the color 1. Then all the b's receive either the color 0 or 1. If $b_1 \wedge b_2 \wedge b_3 = 0$ then at least one of the b_i's must receive the color 0; for otherwise, as noted, t would receive the color 1, where $t = \langle b_1, b_2, b_3\rangle$, which is impossible since t is joined to c. Also, for each $b \in B$, exactly one of b, b' is colored 0, since $\{b, b'\}$ is an edge. Thus the set of b's which are colored 0 is a prime ideal.

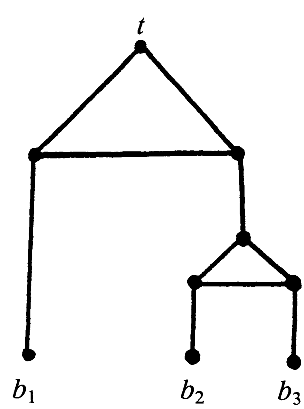

Figure 1

On the other hand, if I is a prime ideal, assign the color 0 to each b in I, and assign the color 1 to each b in $B - I$. Color 2 gets assigned to d and color 1 gets assigned to c. This can be extended to a coloring of $G(\mathbf{B})$ since at least one member of $t = \langle b_1, b_2, b_3 \rangle$, with $b_1 \wedge b_2 \wedge b_3 = 0$, receives the color 0 and hence, as noted, t can be colored 0 and thus c can be 1.

The Theorem now follows, since the Axiom of Choice is not needed to prove that finite Boolean algebras have prime ideals, and this readily implies that every finite subgraph of $G(\mathbf{B})$ is 3-colorable; for, as can easily be shown, a finite subgraph of $G(\mathbf{B})$ 'generates' a finite subalgebra of $\mathbf{B}$ and any prime ideal of this subalgebra leads to a 3-coloring of the subgraph in the manner indicated above. Therefore P_3 implies that $G(\mathbf{B})$ is 3-colorable and hence $\mathbf{B}$ has prime ideals.

A proof of $P_3 \Rightarrow$ BPI could also be obtained indirectly by proving 3-SAT as in Theorems 2 and 3; we leave the details to the reader.

4 **BPI** *and* **NP**-*Completeness* In the preceding sections we have used techniques developed originally for establishing NP-Completeness to prove that certain compactness theorems are equivalent to BPI (in ZF set theory without the Axiom of Choice). We wish to study the relationship between BPI and NP-Completeness more systematically and for this purpose we adopt the following uniform terminology.

Let R be a compactness statement; that is, R says that for a class of structures and a property P pertaining to these structures, if every finite substructure of a given structure in the class has property P, then the given structure has property P as well. Exactly what is meant by "structure" and "finite substructure" will be clear in each particular case. In addition we assume that R is provable in ZF + BPI. For example, if the class of structures is all graphs then finite substructure means finite subgraph, and if P is the property of being 3-colorable then R is just the statement P_3 which we considered in Section 3. On the other hand, if the class of structures is all sets of propositional formulas that are disjunctions of at most three literals and finite substructure means finite subset while property P is satisfiability, then R is 3-SAT. For a compactness statement R, by R^* we shall mean the decision problem with the question: "does the finite structure have property P?". We shall only consider statements R such that R^* belongs to class NP. In the case $R = P_3$, R^* is called GRAPH 3-COLORABILITY, which is NP-Complete (see [7] and [8]). It should not be assumed that every finite decision problem gives rise to a compactness result; therefore we have adopted notation which assumes a compactness result at the outset.

We now give a list of pairs R, R^* and discuss R's relation to BPI and R^*'s complexity. In some cases the status of R is unknown and we make a conjecture. Unless we indicate otherwise, it should be assumed that all statements we make about the complexity of R^* can be found in [8], and statements about R for which we don't supply references have been proved above. We write "$R <$ BPI" if R is weaker than BPI in ZF (without Choice).

(1) If R is n-SAT, then $R \Leftrightarrow$ BPI, and R^* is NP-Complete, if $n > 2$; however, for $n = 2$, $R <$ BPI, since it is a special case of binary consistent choice on pairs (see [11] and [14]) and R^* is solvable in polynomial time.

(2) If R is P_n and $n > 2$, then $R \Leftrightarrow$ BPI and R^*, called GRAPH n-COLORABILITY, is NP-Complete; however, for $n = 2$, $R <$ BPI (see [14] and [17]) and GRAPH 2-COLORABILITY is polynomial.

(3) If $n > 2$ and R is H_1^n or H_2^n as defined above, then $R \Leftrightarrow$ BPI and R^* is NP-Complete (see [21]); however, for $n = 2$, $R <$ BPI, as we have observed in Section 3 and the corresponding decision problems are polynomial.

(4) Let R be the statement: "a collection of finite sets has a system of distinct representatives (SDR) if every finite subcollection has an SDR". (This is equivalent in ZF to the infinite marriage problem of [9].) R^*, the finite marriage problem, is polynomial and it is routine to show that BPI $\Rightarrow R$; however, the exact status of R is not known; *we conjecture that* $R <$ BPI.

(5) Let k be a positive integer and let R be the statement: "an infinite partially ordered set can be partitioned into k chains if every finite subset can be so partitioned". Then, by Dilworth's Theorem (see [6]), R^* is easily seen to be polynomial. BPI $\Rightarrow R$ (see, for example, Theorem 14 of [4]); however, Howard has shown (in an unpublished communication) using a Fraenkel-Mostowski model that $R \not\Rightarrow$ BPI; whether $R <$ BPI in ZF remains an open question.

(6) If R is obtained from 3-SAT by imposing the additional condition that each disjunction have at most one negated variable, then R^* is polynomial (see [21]). Howard and Höft have shown that $R <$ BPI; in fact they proved that R is a theorem of ZF (in an unpublished communication).

(7) Let R be the statement: "a system of polynomial equations over the field $\{0,1\}$ has a solution if every finite subsystem has a solution". Then $R \Leftrightarrow$ BPI. This was stated and proved explicitly in [1]; of course it is implicit in [10]. R^* is NP-Complete. If, however, R is the result of replacing "polynomial" by "linear" in this statement, *we conjecture that* $R <$ BPI. R^* is polynomial since a finite linear system can be solved by Gaussian Elimination.

(8) If R is the compactness theorem for propositional logic with the added restriction that each propositional variable occurs in only finitely many formulas, Wojtylak has shown (in an unpublished communication) that $R <$ BPI; in fact he has shown that $AC_{\text{count}} \Rightarrow R$, where AC_{count} is the Axiom of Choice for families of countable sets, and since $AC_{\text{count}} \not\Rightarrow$ BPI follows easily from known results (e.g., ACW $\not\Rightarrow$ OP, BPI $\Rightarrow$ OP, see [12], p. 184), this suffices. However, R^* is NP-Complete since it equals the satisfiability of propositional formulas.

(9) A graph is *locally finite* if all its vertices are of finite degree. If $n > 2$, let R be the statement P_n restricted to locally finite graphs. Then Mycielski has shown (in an unpublished communication) that $AC_{\text{count}} \Rightarrow R$ and hence, as in (8), $R <$ BPI. R^* is NP-Complete.

The examples we have given above are of three types: (1) $R \Leftrightarrow$ BPI and R^* is NP-Complete; (2) $R <$ BPI and R^* is polynomial; and (3) $R <$ BPI and R^* is NP-Complete. We have not given any example, nor do we know of any, where $R \Leftrightarrow$ BPI and R^* is polynomial. We conjecture that this case does not occur; that is, if R^* is polynomial then $R <$ BPI. This implies in particular that $P = NP$ is false, since if $P = \text{NP}$ were true then, if $R = P_3$, R^* would be polynomial and hence, by the conjecture, $R < \text{BPI}$!

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