Notre Dame Journal of Formal Logic
Volume 31, Number 1, Winter 1990

Book Review

John Bell. *Toposes and Local Set Theories, An Introduction.* Oxford Logic Guides 14, Oxford University Press, 1988. 267 pages.

Toposes come to logic from outside, notably from category theoretic methods in geometry, and the border crossing has occasioned its share of difficulties. Most of the present work aims to initiate the newcomer to toposes, connecting them to established concerns in logic. A classical logician and set theorist whose own initial response to toposes in the foundations of mathematics was skeptical (see [2]), Bell introduces topos theory in a way congenial to mainstream logicians and with his usual expository skill. Then he swings away from the project of initiation into a speculative epilogue. He goes beyond the conventional model theoretic content of the body of the book to suggest that abstraction in mathematics, including set theory, has led to a point where the "pluralism" of category theory must replace the "monism" of set theory (p. 235). His arguments here should provoke debate from all sides.

Requiring basic knowledge of model theory and set theory, the book covers the standard theorems of topos theory and some set theoretic techniques for constructing toposes. It requires no prior knowledge of category theory, although a reader might find it helpful to look at some of the other sources Bell cites or at the brief nontechnical treatment in [14]. The book gives more extensive and elementary treatment of logic in toposes than [9], and could serve as an introduction to that book's results on categorical methods in the lambda calculus and recursive functions. It could also serve a logician as a starting point toward understanding Lawvere's work, and also towards [1], [6] and other research literature in toposes and categorical logic, although for these latter it would have to be supplemented by more general category theory.

A topos can be seen as a kind of universe in which one can interpret higher-order logic and do mathematics. The universe of sets is an example and so are its Boolean extensions as used in independence proofs. In a less classical vein there is a well-known *topos of smooth spaces*: A universe which includes among its objects a line R, a plane R^2, and so on through all classical manifolds of differential geometry and more, including infinitesimal spaces. In this topos every object has a geometric structure and every function is continuously differentiable. One can work within this topos more or less as if working with ordinary sets and arbitrary functions, and yet be assured that all the functions one constructs will be differentiable and all the results one proves will apply to differentiable spaces.

There is an *effective topos*, which it would be premature to call "well-known" right now although it is the object of intense investigation. Its central feature is that in it every well-defined function from the natural numbers to themselves is recursive. Thus one can proceed more or less as if working with ordinary sets and arbitrary functions yet be assured that all the functions constructed will be recursive. One reason for interest in this topos is that in it there are objects with a certain peculiar property: There are nontrivial 'sets' A such that A is isomorphic to the 'set' of all functions from it to itself. (It is easy to see that a classical set A isomorphic to the set of all functions from it to itself is a singleton, with one function to itself!) These 'sets' are rich enough to provide interesting models for the lambda calculus—models in which every function from the 'set' of terms to itself is already one of the terms (see [13]).

Of course the phrase "more or less as if working with ordinary sets" glides over the novelty of topos logic. Classical logic, sound in the topos of sets, is not sound in all toposes, yet a fairly simple higher-order logic is. This is what Bell calls *local set theory*, and he presents toposes almost entirely in terms of it. He could even be faulted for too sharply minimizing the use of category theory; he claims "the merit of avoiding the difficult category theoretic arguments originally employed to establish the basic properties of toposes, replacing them instead with comparatively simple 'set theoretic' reasoning" (p. vii). In fact, categorical methods are no more intrinsically difficult than his. A topologist or algebraist will find the quick elegant category theoretic proofs in Barr and Wells [1] clearer than the syntax of local set theory. But logicians may welcome a book on toposes which, while it is demanding, is demanding in a logician's style.

Local set theory The first chapter surveys category theory, concentrating on just the features relevant to this book. Even so, a reader leery of categories need not actually absorb all this material before going on for a first reading. The second chapter introduces toposes in a form that leads naturally to the third chapter where the central logical tools are introduced: *local set theories*. The fourth chapter applies local set theory to prove the basic theorems of topos theory. The next two chapters apply local set theory to particular constructions (themselves based on Gödel–Bernays set theory) including sheaves over topological spaces, fuzzy sets, and Boolean extensions of the universe of sets. Chapter seven discusses natural numbers and real numbers in a topos. In this chapter Bell briefly describes the *free topos*, which he follows [9] in describing as "the *ideal universe*, for the constructively minded mathematician" (p. 233). There is an epilogue speculating on the significance of topos theory, and an appendix on *classifying toposes*. Briefly, the classifying topos of a theory is a topos containing a model of that theory such that any model in any topos is an image of it (in a specific sense).

The presentation of local set theory is remarkable for its paucity of primitives and axioms.[1] This axiomatization is worth seeing even if one is familiar with purely categorical proofs. A *local language* is a typed term language with no connectives or quantifiers among its primitives. It has a type Ω, seen as the 'type of truth values'. For any finite list of types $A_1, \ldots, A_n$ there is a product type $A_1 \times \ldots \times A_n$, and each type A has a power type PA. There are variables over each type, and typed function symbols. We say that f has signature $A \to B$, meaning that f is a function from type A to type B. Terms are defined in the usual way: each variable of type A is a term of type A, and for each function f of signature $A \to B$ and term τ of type A, $f\tau$ is a term of type B. For terms $\tau_1, \ldots, \tau_n$, of types $A_1, \ldots, A_n$ respectively, there is a term $\langle \tau_1, \ldots, \tau_n \rangle$ of type $A_1 \times \ldots \times A_n$. A formula is a term of type Ω. For any two terms τ and τ' of the same type there is a formula $\tau = \tau'$. For any formula σ and any variable x of type A there is a term $\{x \mid \sigma\}$ of type PA. And for any terms τ of type A and σ of type PA there is a formula $\tau \in \sigma$.

The axioms include the usual structural rules for a sequent calculus, axioms of equality, an axiom saying that an n-tuple $\langle \tau_1, \ldots, \tau_n \rangle$ determines and is determined by the n terms τ_1 through τ_n, and axioms of extensionality and comprehension for terms of any type PA. We will state the comprehension axiom scheme, since we discuss it below. It simply says, for every formula σ and variable x, that this formula is true:

$$x \in \{x \mid \sigma\} \leftrightarrow \sigma.$$

Since all variables are typed, this is in effect a bounded separation axiom scheme: For every typed formula σ and type A there is a collection $\{x \in A \mid \sigma\}$. From these Bell defines the usual sentential connectives and quantifiers, derives a non-classical logic which we discuss below, and proves an array of theorems of local set theory.

The set theory is "local" in two senses. First, it is in a typed language. All quantifiers in local set theory are typed; only outside of the set theory can we quantify over all types and so on. Bell does not explain why this is a good thing, and the conventional wisdom, until recently at least, was that set theory had the advantage over higher-order logic precisely because it was not typed. The weakest claim one could make for this typed set theory is that it works for all the things it does work for and the typing is never felt as a limitation in practice. But the stronger claim would be to point out that applications come typed. Untyped logic is artificial. There are the familiar objections that, for example, single typed set theory makes it meaningful to ask whether the number five is an element of the group of symmetries of the plane (and in some versions the answer is yes). Here I will only point out that computer languages are typed, and higher-level computer languages have more types, because typing facilitates applications.

The second sense in which these set theories are "local" is that not a single one of them is conceived as *the* universe for mathematics. Each is the universe of its own mathematics, or a "local reference frame" for mathematics in Bell's preferred metaphor, comparable to others through *geometric morphisms.* We return to this in discussing the epilogue.

Bell describes the logic of local set theory thus: "The basic axioms and rules of local set theory will be chosen in such a way as to yield as theorems precisely those of (higher order) *intuitionistic logic*" (p. 68). This description may be reassuring to many logicians but it is seriously misleading if it suggests that topos logic was designed to agree with some previously established higher-order intuitionistic logic. It is worth taking a moment to get the history straight, since this too-widely received view conceals a philosophical puzzle.

Higher-order intuitionistic logic Logicians through the 1960's rarely wrote about "higher-order intuitionistic logic". Rather they wrote about "intuitionistic analysis", which could refer either to something like choice sequences or to extensions of Heyting arithmetic in various versions of second-order or higher-order logic. And there were many different versions. Troelstra [19] offers 52 different named axiom systems, and many more unnamed variants, extending Heyting arithmetic to various fragments and kinds of higher-order logic. In part this was because he had metamathematical results on each system, but there is a deeper reason why so many systems had been studied. Through the 1960's an intuitionistic system was expected not only to use intuitionistic logic but to begin with 'intuitionistically acceptable' axioms. Few logicians believed in Brouwer's philosophy, but all took belief in it as part of the point of studying intuitionistic analysis.

There was, and still is, little agreement as to what might be intuitionistically acceptable. There was, however, fairly wide agreement that the full axiom scheme of comprehension was not. That is, it seemed intuitionistically unacceptable to assume that merely because a collection A was given and a formula σ could be expressed there must be a legitimate whole consisting of those things in A satisfying σ. So one would limit the comprehension scheme to certain intuitionistically harmless formulas σ. But again, there was no consensus as to which these should be, and logicians seemed curious to try out many variations rather than intent on establishing one.

At least one aspect of extensionality was also dubious. Dana Scott, for example, in describing quantification over intuitionistic predicates, or "species", once said that "we are being careful *not* to assume these species are extensional in the sense of the validity of

$$\forall x \forall y \forall X [x = y \wedge x \in X \Rightarrow y \in X]$$

because remarks in [8] and elsewhere indicate that non-extensional predicates may be of interest and even of importance" ([17], p. 210).

Toposes were created in the 1960's by Alexander Grothendieck as a highly successful tool in algebraic topology and algebraic geometry. Grothendieck later urged logicians to look into toposes for their similarity to the universe of sets, for he had seen that they 'lift' properties from the universe of sets the same way categories of sheaves of groups lift properties from the category of abelian groups. But he has not pursued this himself and he probably never saw any connection with intuitionism. Nor were Lawvere and Tierney thinking of intuitionism when they axiomatized elementary topos theory in 1969–1971. They wanted to develop further Grothendieck's methods, and Lawvere thought axiomatic topos theory could provide a context for simple foundations for differential geometry, which idea was confirmed by the smooth topos mentioned above. They did not approach toposes through logic though they knew logic would be interpretable in any topos. In fact both have always felt that overemphasis on formal logic distracts from a clear understanding of toposes. They succeeded at capturing the key features of Grothendieck's toposes using the central category theoretic tool of *adjunctions* (pp. 30 ff). Lawvere had known for some years that a natural approach to logic via adjunctions was likely to yield something like Heyting's intuitionistic logic, but for him that was a mathematical fact about adjunctions and not a philosophical desideratum for logic.

The 1970's saw two approaches to higher-order intuitionistic logic come to domination. The first was actually set theoretic, intuitionistic Zermelo Fraenkel set theory, IZF. This was produced by Friedman and Myhill, often working together, at a time when Myhill at least already knew of Lawvere and Tierney's topos theory.[2] The second was Scott's higher-order intuitionistic logic which first appeared in [18], an article asserting its soundness and completeness for interpretations in toposes. Topos theory was among the inputs to these theories, not a later device for interpreting them.

Thus the slogan that "topos theory is higher-order intuitionistic logic" is not a description of topos theory but a modern explication of "higher-order intuitionistic logic", and one which considerably reforms its explicandum. The resemblance between traditional intuitionism and this modern higher-order intuitionistic logic lies in the rules of inference. The modern versions agree with Heyting's rules of inference for connectives and quantifiers; the formal differences in a nutshell are as follows. Traditionally, the existence property is part of intuitionism. A statement $(\exists x)Fx$ is accepted only if for some constant c we accept Fc. The disjunction property is also a traditional part of intuitionism. We accept a disjunctive statement $A \vee B$ only if we accept either A or B. Neither of these properties holds in higher-order intuitionistic logic in the current sense, including Bell's local set theory. But they do hold in the logic of the free topos, which is why this topos has been called the ideal universe for the constructively minded mathematician (it remains to be seen whether actual constructively minded mathematicians will agree). The modern versions also use the full axiom scheme of comprehension, and their axioms of equality include the kind of extensionality that Scott avoided in the quote above.

Scott ([18], p. 686) discussed concerns about intuitionistic acceptability of the axioms and then deferred them, proceeding for the moment without worrying about them. (Perhaps Friedman and Myhill said similar things though I have not found them.) The deferral, well over ten years old now, seems likely to become permanent.

Yet topos logic resembles traditional intuitionism strongly enough to be puzzling. Why should the most natural definitions of the logical connectives in terms of adjunctions agree so closely with Heyting's formalization of Brouwer's philosophy? Is there a link through topology? Perhaps the intuitions of Brouwer and of earlier constructivists were shaped by experience with continuously varying quantities. And it is certain that category theory was created and developed through its first 15 years as a tool for relating topological structures to others. To Lawvere, toposes are universes of continuously variable sets (see [11] and [12], *inter alia*). Perhaps Brouwer and others were more or less aiming at a logic stable under continuous variations and this led them in the direction topos theory would go. And perhaps not. The effective topos is far from topological; almost all it has in common with topological toposes are the adjunctions that define a topos and this is enough to make the logic of toposes apply to it not only in principle but quite usefully. So a link through topology may not be a deep enough explanation.

The resemblance could be mere coincidence, though it seems unlikely. At any rate it is not the result of design. The earlier topological models of intuitionistic logic, for example, were expressly designed to model that logic and so they did, but that is all they did. Toposes were designed for quite different purposes, which they serve well, and their logic forces itself on us quite naturally. The way the logic grows irresistibly out of the properties of a topos is not so clearly manifest in a treatment which derives the properties from the logic, but see [10] for an example.

Completeness theorems Bell shows how to interpret a local language in a topos. A *local set theory S* consists of a local language and a collection of axioms in that language. He proves that the rules of inference for local languages are sound for interpretations in toposes and he also proves a kind of completeness theorem: For any local set theory S, if every interpretation of the language of S that satisfies the axioms also satisfies a formula σ then σ is deducible from the axioms by the rules of inference. Here the reference to "every interpretation" includes quantification over toposes. It means "every topos and every interpretation of the language of S in that topos". But Bell proves more. For every local set theory S there is a topos CS and a canonical interpretation of S in CS such that the following holds: A formula σ is deducible in S if and only if it is true in that interpretation.

He also shows that for every topos there is a local set theory S such that the topos is equivalent to CS, and thus he can use deductions in local set theory to prove the standard theorems of topos theory. The fact that the topos CS is constructed out of the expressions in S may give an illusory feeling of nominalism to the whole procedure; but the theory S itself is apt to have a proper class of types and of terms. The construction uses linguistic devices, but is not limited to the kind of finitely or effectively specified languages a nomimalist would want. Bell gives credit to others who have used the logic of toposes this way, but this is the first publication to bring together all these proofs in logical form.

It must be said that the completeness theorem, also found in [9], [18], and elsewhere, is more a logician's theorem than a category theorist's, and a category theorist's view of it suggests logical questions. It is a logician's theorem in that it takes the local set theory S as fixed and allows the toposes to vary; we quantify over all toposes, or construct a topos CS suited to S. Of course the point of topos theory for a category theorist is also to let the toposes vary—to choose the most suitable one for any purpose—but generally not to vary the toposes to suit some formalized theory, and at any rate to take each single topos seriously. From this point of view we naturally want to know what theory if any is complete for a *given* topos.

It is obvious that local set theory as axiomatized here is not complete for interpretations in sets. The law of excluded middle, for example, is valid in sets but not provable in local set theory. A stronger point can be made. Arithmetic is finitely axiomatizable in local set theory (Chapter 7) so Gödel's incompleteness theorem shows that no consistent effectively axiomatizable local set theory with arithmetic is complete for interpretations in a topos of sets. The result extends easily to many other toposes, though no work seems to have been done to tell just how far it extends and by what methods.

Such incompleteness is to be expected from any set theory or higher-order logic, so we might look to first-order logic. Here Gödel's completeness theorem tells us that the first-order part of local set theory plus the law of excluded middle is complete for the topos of sets (eliminate the free variable restriction on the cut rule if you want to require that every type be interpreted by a nonempty set). This first-order part of local set theory could be extended more modestly by adding the DeMorgan law "$\sim(\varphi \,\&\, \psi) \rightarrow (\sim\varphi \vee \sim\psi)$" as an axiom. This does not imply the law of excluded middle (see Johnstone [7]) but is sound and complete for some toposes. Or we could add the law of excluded middle "$\varphi \vee \sim\varphi$" as an axiom for every formula φ with no free variables, but not for all formulas. This is also sound and complete in some toposes but not all. Some work, done by logicians not thinking of toposes, has shown that certain first-order logics are sound and complete in various specific toposes. Kripke's completeness proof for first-order intuitionistic logic shows it is complete in certain toposes; and some work on tense logic with intervals is related to completeness proofs for certain other logics. But the field has hardly been touched.

Examples Bell does not describe the topos of smooth spaces (he has written about it in [3] and [4]) nor the effective topos, but develops examples closer to standard concerns in logic: Kripke models, Boolean extensions of the universe of sets, and fuzzy sets. He shows how toposes provide a unifying framework for these constructions and alludes to their relevance to others, such as ultraproducts in nonstandard analysis and Cohen forcing. Logicians began speculating on the connection between Kripke's models for intuitionistic logic and Cohen's forcing models almost as soon as the techniques were known. Bell gives a good account of the correct answer to their relationship, correct in that it is natural, elegantly addresses the specifics, and yet embeds the two in a much broader context, namely that both are constructions in presheaf toposes.

The discussion of fuzzy sets stops just short of an interesting observation which argues for extending each universe of fuzzy sets to a topos. A *locale* is a partially ordered set in which every two elements have a greatest lower bound, every set of elements has a least upper bound, and lower bounds distribute over upper bounds. (Think of the open sets of a topological space: the intersection of any two is open, so is the union of any set of them, and intersections distribute over unions.) Given a locale H the universe of H-fuzzy sets is a universe in which the claim $x \in y$ has a truth value in H. That is, it may be "partially true" with partial truth explicated by H as a set of truth values (for details see pp. 210ff). Bell shows that the H-fuzzy sets form a topos if and only if H is a Boolean algebra.

But set theory rests on two primitives—membership and equality—and fuzzy set theory has only fuzzified membership. That is, a claim $x = y$ is either simply true or simply false in the universe of H-fuzzy sets. This is a serious limitation on fuzzy sets. It makes fuzzy powersets problematic, since equality of subsets ought to be as fuzzy as claims about their members. Furthermore, given an

H-fuzzy set S and an H-fuzzy equivalence relation R on it, there is no H-fuzzy set of equivalence classes for the equivalence relation, since equality of equivalence classes would have to be as fuzzy as the relation R.

Bell does not mention that extending the notion of an H-fuzzy set to include sets of equivalence classes for equivalence relations amounts to allowing arbitrary H-fuzzy equality relations—and the result is precisely the topos of H-sets as defined by Bell (p. 202). And of course in the topos all higher-order constructions are available, including powersets and more. All of this is in [16]. The H-fuzzy sets are just those H-sets on which equality is not fuzzy, so using the topos of H-sets allows us to restrict ourselves to those when we want to, but it also gives a naturally greater range of constructions.

Bell's epilogue The soberly model theoretic tone of the book changes to brisk speculation in the epilogue "On the wider significance of topos theory". The epilogue begins with an account of developments in abstract algebra through the 1930's and 1940's, notably Bourbakiste structuralism, which preceded category theory. Bell describes the then increasing awareness of the importance of morphisms, structure preserving maps between structured sets, and points out that this fell short of the Bourbakiste ideal of structuralism since it did rely on the set theoretic make up of its structured sets. He says that category theory provided the first axiomatic framework for mathematics "which takes the notions of structure and morphism as primitive (as *objects* and *arrows* respectively) and which is indifferent to any particular set-theoretic construction that structures may possess" (p. 236). He offers an analogy:

> Category theory may be said to bear the same relation to abstract algebra as does the latter to elementary algebra. Elementary algebra results from the replacement of *constant quantities* (i.e. numbers) by *variables*, keeping the operations on these quantities fixed. Abstract algebra, in its turn, carries this a stage further by allowing the *operations* to vary while ensuring that the resulting mathematical structures (groups, rings, etc.) remain of a prescribed kind. Finally category theory allows even the *kind* of structure to vary: it is concerned with *structure in general.* (p. 236)

It is not clear whether Bell means to say that category theory evolved out of abstract algebra, but the reader might reasonably get the impression that he does. And, in fact, from the point of view of the history of ideas that is not an indefensible claim. Bell's model theoretic approach emphasizes the continuity between abstract algebra and category theory, and one could claim that that is the most important feature of category theory and thus the relevant part of its history. This reviewer, on the other hand, believes Bell understates the importance of geometry and topology, both in his history and in his treatment of toposes. Certainly, from the point of view of a detailed history of events, topology has occupied category theorists from the beginning much more than abstract algebra has.

Still on the same page, Bell remarks that "In category theory the morphisms (arrows) play an autonomous role which is in no way subordinate to that played by structures themselves. So category theory is like a language in which the 'verbs' are on an equal footing with the 'nouns.' In this respect category theory differs crucially from set theory in which the corresponding notion of *function* is reduced to the concept of *set*" (p. 236). Besides the linguistic metaphor this is interesting because throughout the book Bell treats arrows in a topos not by way of function symbols in local set theory but by way of functional relations. That is, he reduces functions to sets (see pp. 120–126 for example). Of course Bell began by promising to avoid category theoretic arguments in favor of set theoretic ones (p. vii, quoted above), but now we find that he is aware that this means neglecting the crucial feature of category theory, a feature which the epilogue seems to depict as a virtue of category theory in comparison with set theory.

Perhaps he considered this neglect a painful necessity in order to reach a wider audience. On the other hand, perhaps he himself does not find the equal treatment of 'nouns' and 'verbs' a virtue, but only means to say that a certain kind of structuralist would regard it as one. He makes no comment either way to explain the contrast between his procedure throughout the book and the opposite procedure he seems to praise in the epilogue.

Now Bell's strategy is not at all to oppose set theory to category theory. He sees category theory as the latest stage in the developments in abstract mathematics that included set theory; and no one can deny this view in broad terms. But it may be that Bell stays too close to set theory, which after all is not the whole of mathematics, and not the part that gave rise to category theory. He develops an argument based on the independence proofs of set theory, saying they show that the set concept is radically underdetermined so that "it becomes natural, indeed mandatory, to seek for the set concept a formulation that takes account of its underdetermined character, that is, one that does not bind it so tightly to the absolute universe of sets with its rigid hierarchical character. Category theory furnishes such a formulation through the concept of *topos*, and its formal counterpart *local set theory*" (p. 238). But the independence proofs show precisely that the ZF axioms themselves can accommodate an array of different set theories—with or without choice, with or without the continuum hypothesis, and so on. They can hardly represent more than the very thin edge of a wedge for arguing that ZF should be displaced in foundations by the vastly more general notion of a topos.

This brings up an impression one may have in considering the topos theoretic treatment of Kripke models, Boolean extensions, and fuzzy sets, namely that these things are not very hard to understand on their own. And in fact anyone who only wants to understand, say, Boolean extensions, would do best to study only them and skip toposes. The motivations for topos theory do not lie so close to set theory, nor do the applications topos theorists work on such as the smooth topos and more recently the effective topos. Apparently the examples Bell gives attracted him to topos theory and so they may attract other set theoretically trained logicians, but it seems that the more characteristically category theoretic applications would provide better motives for expanding from set theoretic foundations to category theoretic ones.

Bell's suggestion that "the topos theoretic interpretation of mathematical concepts bears the same relation to classical set theory as relativity theory does to classical physics" (p. 242) will catch criticism from both sides. Few confirmed set theorists will enjoy it, yet neither is it likely to satisfy proponents of category theoretic foundations. A continuous real valued function on a topological space X appears as a single real number in the topos of sheaves over X, $\mathrm{Sh}(X)$, and Bell gives a pretty description of that topos as a framework 'comoving' with the varying function so that within that framework the function appears to be a single number (p. 240). But it is not clear how far this can be pushed. Bell says that looking at the function in the topos "causes its variation not to be 'noticed' in $\mathrm{Sh}(X)$; it is accordingly regarded as a *constant* real number" (p. 240). But the variation has objective effects within the topos and the number is not "constant" in the fullest sense within the topos, unless it corresponds to a constant function on the space in the usual sense.

The view of a topos as a 'coordinate frame' and of geometric morphisms between toposes as 'coordinate transformations' breaks down in that geometric morphisms do not faithfully preserve and reflect information. The important examples, the ones Bell cites, are precisely not invertible. Nor is there, so far as anyone has found to date, some kind of invariant substrate that toposes can be regarded as 'coordinate frames' *on*.

Proponents of categorical foundations will particularly object to Bell's saying that toposes with set-like features "correspond to inertial coordinate systems" (p. 241), since such toposes certainly do exist. A physicist who believed in strictly inertial coordinate frames would *ipso facto* reject general relativity. So Bell's own analogy suggests rejecting topos theory in the foundations of mathematics. On his analogy toposes would be treated the way curvilinear or accelerated coordinates were in classical physics: as occasionally handy technical devices, but no challenge to the absolute truth of the classical theory.

In the end Bell does not seem to have made up his own mind concerning the status of toposes in foundations. The epilogue is curiously noncommittal. He chooses to "start with an account of the relationship between set theory and category theory, contrasting the 'monism' of the former with the 'pluralism' of the latter" (p. 235), and in this context "pluralism" seems a term of praise, but Bell never explicitly chooses between them. To say that category theory represents the latest in the advance of algebra, and that toposes stand to set theory as general relativity to classical physics is surely high praise for category theory and toposes. In the passage quoted above he said the independence proofs make it "mandatory to seek" an alternative to current axiomatic set theory, something like topos theory. But he did not say it was mandatory to *use* such an alternative and in fact he takes Gödel–Bernays set theory as the foundation for the whole book (p. vii). The toposes he invokes as examples in the comparison with relativity theory, as well as in his discussion at the end of the epilogue of Lawvere's idea of negating constancy, are all explicitly constructed out of sets. He does not adopt categorical foundations.

And yet his ambivalence may itself strike a chord in the reader. This book is a clear look at topos logic with set theoretic foundations, and a sometimes impassioned look at toposes in foundations, by a logician who does not always think in topos theoretic terms and is not entirely certain how far foundations should rest upon topos theory or upon set theory. This book contributes to the investigation of toposes in logic and in foundations, rather than to an argument for or against them.

NOTES

1. Bell's axioms for a topos are equally spare, precisely paralleling the primitives and axioms of this logic. In fact Johnstone discussed such an axiomatization, saying "it is really a set theorist's rather than a category theorist's definition of a topos, in that it subordinates the notion of 'function' to the notion of a 'subset'" (p. xviii in [6]). And Bell's practice bears this out. He defines function types and functional application in local set theory (pp. 84, 117) but never uses them, and his examples and the points he makes do run more toward set theory than category theory.

2. The proceedings of the 1971 Dalhousie conference on toposes, algebraic geometry, and logic list Myhill as a participant, although in fact he was unable to attend and [5] was presented there by Goodman. Shortly after that Lawvere became a colleague of Goodman's and Myhill's at SUNY Buffalo.

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