Near Coherence of Filters III: 
A Simplified Consistency Proof

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Abstract In the model obtained from a model of the continuum hypothesis by iterating rational perfect set forcing \(^{\aleph_2}\) times with countable supports, every two nonprincipal ultrafilters on \(\omega\) have a common image under a finite-to-one function.

The principle of near coherence of filters (NCF) asserts that, for any two nonprincipal ultrafilters \(\mathcal{U}\) and \(\mathcal{V}\) on the set \(\omega\) of natural numbers, there exists a finite-to-one function \(f : \omega \to \omega\) such that \(f(\mathcal{U}) = f(\mathcal{V})\). This principle was introduced and studied in [1], and its consistency relative to ZFC was proved in [2]. Because [2] also contains the consistency proof for another statement (the existence of simple \(P_{\kappa}\)-points for two different \(\kappa\)), the model of NCF presented there was chosen to maximize the similarity of the two proofs. Although this approach is quite efficient for proving the consistency of both statements, there is a simpler consistency proof for NCF alone. The purpose of this paper is to present this proof.

By rational perfect set forcing, we mean the forcing introduced by Miller in [3]; a definition is given below.

Theorem NCF holds in the model obtained from a model of the continuum hypothesis by iterating rational perfect set forcing \(^{\aleph_2}\) times with countable supports.

The proof to be presented here can be viewed as the result of deleting, from the proof in [2], all references to (what is there called) depth. The observation that the consistency proof for NCF survives this deletion was made by Shelah shortly after he found the proof in [2]. Blass noticed that the resulting forcing was equivalent to Miller’s rational perfect set forcing.

The substitution of rational perfect set forcing for the forcing used in [2]

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considerably simplifies the analysis in Sections 2 and 3 of [2]. On the other hand, the parts of [2] that deal not with the specific forcing at hand but with general properties of iterated proper forcing, particularly Section 4, do not benefit at all from this substitution. Thus, a self-contained presentation of the new proof would include a verbatim transcription of these parts of the old proof. To avoid such unnecessary repetition, we simply quote here the general facts about proper forcing and its iteration that we shall need.

**Lemma 1** ([5], p. 81) Let $G$ be a $V$-generic subset of a proper notion of forcing. If $X \in V[G]$ is a countable (in $V[G]$) subset of $V$, then $X \subseteq Y$ for some countable (in $V$) set $Y \in V$. In particular, $\kappa_1$ is absolute between $V$ and $V[G]$.

For the next four lemmas, let $\langle P_\alpha, Q_\alpha : \alpha < \lambda \rangle$ be a countable support iteration of proper forcing with limit $P_\lambda$. That is,

- $P_0$ is the trivial notion of forcing (a singleton);
- $P_{\alpha+1} = P_\alpha \ast Q_\alpha$ for all $\alpha < \lambda$;
- $P_\beta$ is the direct limit of $\langle P_\alpha \rangle_{\alpha < \beta}$ for all $\beta < \lambda$ of uncountable cofinality;
- $P_\alpha$ is the inverse limit of $\langle P_\alpha \rangle_{\alpha < \beta}$ for all $\beta < \lambda$ of countable cofinality;
- $P_\alpha$ forces "$Q_\alpha$ is a proper notion of forcing" for all $\alpha < \lambda$.

Let $G$ be a $V$-generic subset of $P_\lambda$. For each $\alpha \leq \lambda$, we write $G_\alpha$ for the $V$-generic subset $G \mid P_\alpha$ of $P_\alpha$.

**Lemma 2** ([5], p. 90) $P_\alpha$ is proper for all $\alpha \leq \lambda$.

**Lemma 3** ([2], Lemma 5.10) Let $\mathcal{F} \in V[G]$ be a family of reals. There is an $\kappa_1$-closed unbounded set of ordinals $\alpha < \kappa_2$ for which $\mathcal{F} \cap V[G_\alpha] \subseteq V[G_\alpha]$.

**Lemma 4** ([5], p. 96) Suppose that the continuum hypothesis holds in $V$, that $\lambda = \kappa_2$, and that, for each $\alpha < \kappa_2$, $P_\alpha$ forces that $|Q_\alpha| \leq 2^{\kappa_0}$. Then, for each $\alpha < \kappa_2$, $P_\alpha$ has a dense set of cardinality $\leq \kappa_1$, $P_{\kappa_2}$ satisfies the $\kappa_2$-chain condition, and (therefore) cardinals are absolute between $V$ and $V[G]$.

(The comments after (5.4) in [2] explain how to handle some technical differences between our version of this lemma and the version in [5].)

A nonprincipal ultrafilter $\mathcal{U}$ on $\omega$ is called a $P$-point if, whenever $X_n \in \mathcal{U}$ for all $n \in \omega$, there is a $Y \in \mathcal{U}$ such that $Y$ is almost included in each $X_n$, i.e., $Y - X_n$ is finite. A classical result of Rudin [4] is that the continuum hypothesis implies the existence of $P$-points.

**Lemma 5** ([2], Theorem 4.1) If $\mathcal{U}$ is a $P$-point (in $V$), if $\lambda$ is a limit ordinal, and if, for each $\alpha < \lambda$, $P_\alpha$ forces "$\mathcal{U}$ generates a $P$-point", then $P_\lambda$ also forces "$\mathcal{U}$ generates a $P$-point".

This completes the list of general facts about proper forcing that we shall need. We now turn to the specific forcing that we shall iterate to get a model of NCF, Miller's rational perfect set forcing.

Let $p$ be a tree of finite subsets of $\omega$; that is, $\emptyset \in p$ and if $a \in p$ then every initial segment of $a$ is also in $p$. These initial segments are the predecessors of $a$ in $p$. A node $a \in p$ is said to be infinitely branching in $p$, if there are infinitely many $n$ such that $a \cup \{n\} \in p$, or equivalently, if $a$ has infinitely many immedi-
ate successors in \( p \). The tree \( p \) is said to be \textit{superperfect} if every node \( a \in p \) has an infinitely branching successor. The superperfect trees are the forcing conditions in a notion of forcing \( Q \), ordered so that the extensions of a condition are its superperfect subtrees. (Miller's definition in [3] involved trees of finite sequences rather than finite sets, but there are obvious isomorphisms that replace sequences by strictly increasing sequences and then by the sets that they enumerate.)

We shall need three methods of constructing extensions of a given condition \( p \). The first is to select a node \( a \in p \) and to form the subtree \( p/a \) of all nodes comparable with \( a \). Clearly \( p/a \) is superperfect. This method of extension suffices to show that the set

\[
\{ p \in Q \mid p \text{ contains only one node of cardinality } n \}
\]

is dense in \( Q \). It follows that, if \( G \) is a \( V \)-generic subset of \( Q \), then the intersection of all the trees \( p \in G \) is a single path through the tree of all finite subsets of \( \omega \), so it defines an infinite subset \( \mathcal{G} \) of \( \omega \). We write \( f_G(n) \) for the cardinality of \( \mathcal{G} \cap n \), thus, \( f_G : \omega \to \omega \) is constant precisely on the intervals into which the members of \( G \) divide \( \omega \).

The second way to extend a given condition \( p \) is to select, for each infinitely branching node \( a \in p \), an infinite subset \( X_a \) of \( \{ n \in \omega \mid a \cup \{ n \} \in p \text{ and } n > \max(a) \} \) and to throw away all the immediate successors \( a \cup \{ n \} \) of \( a \) with \( n \not\in X_a \). The resulting subtree of \( p \) is

\[
q = \{ b \in p \mid \text{for every proper initial segment } a \text{ of } b, \text{ if } a \text{ is infinitely branching in } p, \text{ then } \min(b - a) \in X_a \}. 
\]

It is easy to verify that \( q \) is superperfect and is therefore an extension of \( p \). We refer to this construction as \textit{thinning} \( p \) with the sets \( X_a \).

The third method of building an extension \( q \) of \( p \) is called \textit{fusion} and is really a meta-method, a way of combining many extension processes into one. It proceeds as follows. Select an infinitely branching node \( a \in p \). Throw away all nodes incomparable with \( a \) (so \( q \) will be a subtree of \( p/a \)), but put into \( q \) the node \( a \), all its predecessors, and all its immediate successors in \( p \). For each of these immediate successors \( a \cup \{ n \} \), select an extension \( p_n \) of \( p/(a \cup \{ n \}) \) and select an infinitely branching node \( a_n \in p_n \). Throw away all nodes still present that are in no \( p_n/a_n \) (so \( q \) will be included in the union of the \( p_n/a_n \), a union which includes the nodes already put into \( q \)), and put into \( q \) all the nodes \( a_n \), all their predecessors, and all their immediate successors in the corresponding \( p_n \). For each of these immediate successors \( a_n \cup \{ m \} \), select an extension \( p_{nm} \) of \( p/(a_n \cup \{ m \}) \), select an infinitely branching node \( a_{nm} \in p_{nm} \), and proceed as before. Repeating this process \( \omega \) times yields a tree \( q \) which is superperfect and therefore an extension of \( p \). The infinitely branching nodes of \( q \) are precisely the nodes \( a_s \) (where \( s \) is a finite sequence of subscripts) selected at the various stages of the construction. For each immediate successor \( a_s \cup \{ k \} \) of such a node in \( q \), the condition \( q/(a_s \cup \{ k \}) \) is an extension of the \( p_{sk} \) chosen during the construction.

If \( q \) is the condition just constructed by fusion, if \( r \) is any extension of \( q \), and if \( n \in \omega \), then \( r \) must contain one of the infinitely branching nodes \( a_s \) of \( q \) with \( s \) of length \( n \). It must also contain \( a_s \cup \{ k \} \) for some \( k \), and must there-
fore be compatible with \(q/(a_s \cup \{k\})\), since \(r/(a_s \cup \{k\})\) is a common extension. Thus, \(r\) is also compatible with \(p_{sk}\).

If we were given a countable sequence \(W_0, W_1, \ldots\) of maximal antichains (or predense sets) in \(Q\), then, at the stage of the fusion construction where the conditions \(p_s\) are being chosen for \(s\) of length \(n \geq 1\), we could choose each of these conditions to be an extension of a condition \(w_s \in W_{n-1}\). Then, by the preceding paragraph, every extension of \(q\) is compatible with \(w_s\) for some \(s\) of length \(n\); in other words, the countable subset

\[
W'_{n-1} = \{ w_s | s \text{ of length } n \}
\]

of \(W_n\) is predense beyond \(q\).

If all the arbitrary choices in this construction of \(q\), with a prescribed condition \(p\) and prescribed \(W_n\)'s, are made in some fixed manner, say in accordance with a specific well-ordering of (a sufficiently large piece of) the universe, then \(W'_n\) depends only on \(W_0, W_1, \ldots, W_n\). List the countable set \(W'_n\) in an \(\omega\)-sequence and write \(F(p, W_0, \ldots, W_n, k)\) for its \(k\)th member.

**Lemma 6**  \(\text{The notion of forcing } Q \text{ is proper.}\)

**Proof:** We verify the criterion for properness called \(\text{Con}_2(\lambda)\) in [5], p. 77, except that we dispense with the ordinal indexing used there. Suppose that \(s\) is a countable set containing all the natural numbers and closed under the function \(F\) defined above. Let \(W_0, W_1, \ldots\) be all the predense subsets of \(Q\) that are members of \(s\), and let \(p\) be any element of \(Q \cap s\). The fusion construction above yields an extension \(q\) of \(p\) beyond which the sets \(W'_n\) are predense. The assumptions on \(s\) imply that \(W'_n \subseteq W_n \cap s\), so each \(W'_n \cap s\) is predense beyond \(q\). Since the collection of all \(s\) that satisfy these assumptions is closed and unbounded, \(\text{Con}_2(\lambda)\) holds.

Notice that our definition of fusion is such that, in the resulting tree \(q\), the only nodes with more than one immediate successor are the \(a_s\), which have infinitely many immediate successors. Thus, trees in which every branching node is infinitely branching are dense in \(Q\) and we may, whenever convenient, confine our attention to such trees. It will, in fact, be convenient to perform some additional normalizations on our trees, as follows.

We say that a superperfect tree \(p\) has interval structure if \(\omega\) can be partitioned into (finite) intervals \([0,i_0),[i_0,i_1),[i_1,i_2), \ldots\) so that, if a node \(a \in p\) has more than one immediate successor \(a \cup \{n\} \in p\), then

\[
\begin{align*}
\text{(i) each interval } [i_k,i_{k+1}) \text{ after the one containing } \max(a) \text{ contains exactly one } n \text{ such that } a \cup \{n\} & \in p \\
\text{(ii) every immediate successor } a \cup \{n\} \in p \text{ of } a \text{ is as in (i); i.e., } n \text{ is not in the same interval as } \max(a) \\
\text{(iii) each } a \cup \{n\} \text{ as above has an infinitely branching successor } b \in p \text{ with } \max(b) \text{ in the same interval } [i_k,i_{k+1}) \text{ as } n.
\end{align*}
\]

(The trivial problem that \(\max(\emptyset)\) is undefined can be avoided by requiring \(\emptyset\) to be a nonbranching node or by agreeing that every interval is “after the one containing \(\max(\emptyset)\)”.) Notice that, in a tree with interval structure, every branching node is infinitely branching.
Lemma 7  The set of superperfect trees with interval structure is dense in $Q$.

Proof: Let $p$ be any superperfect tree; we wish to extend it to one with interval structure. By a preliminary fusion, we can assume that every branching node of $p$ is infinitely branching and that $\emptyset$ is not branching. We inductively define a sequence $(i_k)_{k \in \omega}$, which will provide the interval structure of an extension of $p$. Choose $i_0$ arbitrarily. After $i_k$ is defined, choose $i_{k+1}$ so large that, for each (infinitely) branching node $a$ with $\max(a) < i_k$, there exist $n = n(a,k) \in \omega$ and $b = b(a,k) \in p$ such that

- $b$ is a branching node of $p$,
- $b$ is a successor of $a \cup \{n\}$ (so $a \cup \{n\} \in p$),
- $i_k \leq n$, and
- $\max(b) < i_{k+1}$.

Since only finitely many $a$ have $\max(a) < i_k$, and since $p$ is superperfect, it is clear that such an $i_{k+1}$ can be found. Finally, the desired extension of $p$, having interval structure given by these $i_k$'s, is obtained by thinning $p$ with $X_a = \{n(a,k) | \max(a) < i_k\}$; i.e.,

$$q = \{b \in p | \text{for every proper initial segment } a \text{ of } b, \text{ if } a \text{ is branching in } p \text{ then } \min(b - a) = n(a,k) \text{ for some } k \text{ (with } \max(a) < i_k)\}.$$

If $p$ has interval structure given by intervals $I_k = [i_{k-1}, i_k)$ (where $i_{-1}$ is 0) and if we choose any subsequence of this sequence of intervals, say $J_k = I_{n_k}$, then we can thin $p$ with $X_a = \{n | a \cup \{n\} \in p \text{ and } n \in J_k \text{ for some } k\}$ to obtain an extension $q$ of $p$ in which every node is a subset of the union of the $J_k$'s and the first branching node $a_0$. The $J_k$'s fail to provide an interval structure for $q$ only because they are not in general adjacent in $\omega$; if we expand them to adjacent intervals, for example by adding to each $J_k$ the intervals $I_m$ between $J_k$ and $J_{k+1}$, then we obtain an interval structure for $q$.

For the next lemma, recall that if $G$ is a generic subset of $Q$, then $\mathcal{G}$ is the infinite subset of $\omega$ determined by $G$ (namely, the union of the nodes that are in every $p \in G$), and $f_G : \omega \to \omega$ is constant with value $n$ on the $n$th of the intervals into which $G$ divides $\omega$.

Lemma 8  Let $X$ be an infinite subset of $\omega$ and let $\mathcal{U}$ be a nonprincipal ultrafilter on $\omega$, both in $V$. Let $G$ be a $V$-generic subset of $Q$. Then there exists a $Y \in \mathcal{U}$ with $f_G(Y) \subseteq f_G(X)$.

Proof: Let $X, \mathcal{U}$, and a condition $p \in Q$ be given. We shall find a $Y \in \mathcal{U}$ and an extension $q$ of $p$ forcing that $f_G(Y) \subseteq f_G(X)$, i.e., that if $m < n$ are members of $\mathcal{G}$ and the interval $(m, n]$ meets $Y$, then it also meets $X$. This will clearly suffice to prove the lemma. By Lemma 7, we may assume that $p$ has interval structure, and by the construction following that lemma we may assume that each of these intervals $I_k$ meets $X$. $\mathcal{U}$ must contain either the union of the even-numbered intervals or the union of the odd-numbered ones; let $Y_0$ be whichever of these two unions belongs to $\mathcal{U}$. Applying once more the thinning construction given after Lemma 7, extend $p$ to a condition $q$ such that every node in $q$ is a subset of $a_0 \cup (\omega - Y_0)$, where $a_0$ is the first branching node of $q$. Let
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Y ∈ ℰ be obtained by removing from Y₀ all (finitely many) intervals Iₖ up to and including the one containing max(a₀). Since q forces "G ⊆ a₀ ∪ (ω − Y₀)", it also forces that, if m < n are in G and (m, n] meets Y then (m, n] contains a whole interval Iₖ ⊆ Y, and therefore meets X.

If ℰ is an ultrafilter in V, then we write ℰ for the filter generated by ℰ in some generic extension of V. The context will always make it clear which extension is meant. In general, ℰ may or may not be an ultrafilter in the extension.

Corollary 1 If ℰ and ℰ' are nonprincipal ultrafilters on ω in V and if G is a V-generic subset of Q, then f_G(ℰ) = f_G(ℰ').

Proof: f_G(ℰ') is generated by the sets f_G(X) for X ∈ ℰ'. By Lemma 8, every such set has a subset in f_G(ℰ). So f_G(ℰ') ⊆ f_G(ℰ), and the reverse inclusion follows by symmetry.

Lemma 9 ([3], Claim 2.4) If the set of infinitely branching nodes of a tree p ∈ Q is partitioned into two pieces, then p has an extension q all of whose infinitely branching nodes are in the same piece.

Proof: We attempt a fusion construction in which the nodes aₙ, chosen to become the infinitely branching nodes of q, are all in the first piece. If we succeed, the fusion produces the desired q. If we fail, it is because at some stage of the construction the tree p/(aₙ U {n}), from which an aₙ had to be chosen, had no infinitely branching node in the first piece. Then this p/(aₙ U {n}) is the desired q.

The following two lemmas are due to Miller ([3], Propositions 4.1 and 4.2]; we give a different proof, parallel to ([2], Theorem 3.3]).

Lemma 10 If ℰ is a P-point in V and G is a V-generic subset of Q, then the filter ℰ generated in V[G] by ℰ is an ultrafilter in V[G].

Proof: Let A be a name in the forcing language associated with Q, and let p ∈ Q force that A ⊆ ω. We shall find a B ∈ ℰ and an extension q of p such that either q forces B to be a subset of A or q forces B to be disjoint from A; this will clearly suffice to prove the lemma. We shall extend p in several steps to obtain the desired q. (The argument takes place in V.)

We begin by performing a fusion construction on p, choosing the conditions pₙ at each stage so that, for each j ≤ n, pₙ decides whether or not j ∈ A. The resulting condition p' has the property that, for each of its (infinitely) branching nodes a, each immediate successor a ∪ {n} ∈ p', and each j ≤ n, p'/a(a ∪ {n}) decides whether or not j ∈ A. Notice that every extension of p' has the same property. By thinning p', we obtain p'' such that the decision by p''/a(a ∪ {n}) about whether or not j ∈ A depends only on a and j, not on n, once n is large enough. (Here "large enough" depends also on a and j.) For each branching node a of p'', set

A'(a) = {j | for all sufficiently large n such that a ∪ {n} ∈ p'', p''/a(a ∪ {n}) forces j ∈ A}.
Partition the branching nodes $a$ of $p''$ into two classes according as whether or not $A'(a) \in \mathcal{U}$, and apply Lemma 9 to extend $p''$ to $p'''$ with all its infinitely branching nodes in the same class. We may assume that $A'(a) \in \mathcal{U}$ for all infinitely branching nodes $a$ of $p''$, for the other case reduces to this one if we replace $A$ with its complement. Notice that $A'(a)$ is unchanged when $p''$ is replaced by $p'''$ (or any other extension of $p''$) in its definition, provided that $a$ is an infinitely branching node of $p'''$ (or of the other extension in question).

As $\mathcal{U}$ is a P-point, there is a $B \in \mathcal{U}$ such that $B - A'(a)$ is finite for every infinitely branching node $a$ of $p''$. By Lemma 7, we can extend $p''$ to a condition $p'(4)$ with interval structure, and by the discussion following that lemma, we can extend this condition further to a $p'(5)$ with interval structure $[0, i_0)$, $[i_0, i_1) \ldots$ and with the further property that, whenever $a$ is a branching node of $p'(5)$ and $a \subseteq i_k$ (i.e., all members of $a$ are $< i_k$), then

(i) $B - A'(a) \subseteq i_{k+1}$
(ii) if $a \cup \{n\} \in p'(5)$, $n \geq i_{k+1}$, and $j \in A'(a) \cap i_k$, then $p'(5)/(a \cup \{n\})$ forces $j \in A$.

To see this, it suffices to choose inductively the $i_k$ from among the endpoints of intervals involved in the interval structure of $p'(4)$, so that each interval for $p'(5)$ is a union of intervals for $p'(4)$. Once $i_k$ is chosen, $i_{k+1}$ can be chosen large enough to satisfy (i) and (ii) because only finitely many $a$'s and $j$'s are involved, each $B - A'(a)$ is finite (by definition of $B$), and all sufficiently large $n$ are as desired in (ii) (by definition of $A'(a)$).

It will be convenient to assume that the first branching node $a_0$ of $p'(5)$ has $\max(a_0) < i_0$; this can be achieved by combining into a single interval all the intervals up to and including the one that contains $\max(a_0)$.

Partition $\omega$ into four pieces, each containing every fourth interval $[i_k, i_{k+1})$; that is, the $m$th piece ($0 \leq m < 4$) is the union of the $[i_k, i_k + 1)$ for $k = m$ (mod 4). Being an ultrafilter, $\mathcal{U}$ must contain one of these pieces. Replacing $B$ by its intersection with this piece, we can ensure that $B$ meets only (at most) every fourth interval, while all our previous statements about $B$ remain true. Similarly, we can ensure that $B$ has no members smaller than $i_2$.

Thin $p'(5)$ to obtain an extension $q$ (still with $a_0$ as its first branching node) all of whose nodes $a$ have $a - a_0$ disjoint from the intervals $[i_k, i_{k+1})$ that meet $B$, as well as from the immediately preceding and following intervals $[i_{k-1}, i_k)$ and $[i_{k+1}, i_{k+2})$. Since $B$ meets only every fourth interval, there are infinitely many intervals that $a - a_0$ is permitted to meet, so the construction after Lemma 7 yields such a $q$. (We continue to use the notation $[i_k, i_{k+1})$ for the intervals associated with $p'(5)$, not the larger ones associated with $q$.) We complete the proof by showing that $q$ forces $B \subseteq A$.

Suppose the contrary. Then there exist a $j \in B$ and an extension $r$ of $q$ such that $r$ forces $j \notin A$. Let $[i_k, i_{k+1})$ be the interval containing $j$. By our normalization of $B$ two paragraphs ago, $k \geq 2$ and, for every node $a$ of $q$, $a - a_0$ is disjoint from $[i_{k-1}, i_{k+2})$. Furthermore, by our earlier normalization of the $i_k$'s, the first branching node $a_0$ of $q$ (and of $p'(5)$) has $\max(a_0) < i_0 \leq i_{k-2}$, so in fact every node $a$ of $q$ is disjoint from $[i_{k-1}, i_{k+2})$.

Extending $r$ to some $r/b$ if necessary, we can arrange that the first branching node $b_0$ of $r$ has $\max(b_0) \geq i_{k+2}$. Let $a$ be the last branching node of $q$ that
is a predecessor of $b_0$ and has $\max(a) < i_{k+2}$. ($a_0$ is a predecessor of $b_0$ and $\max(a_0) < i_{k+2}$, so $a$ exists.) Since $a$ is disjoint from $[i_{k-1}, i_{k+2})$, it follows that $\max(a) < i_{k-1}$. In view of the interval structure of $p^{(5)}$, $b_0$ is a successor of $a \cup \{n\}$ for some $n \geq i_{k+2}$, for if $n$ were smaller there would be a branching node $a'$, between $a \cup \{n\}$ and $b_0$, with $\max(a')$ in the same interval as $n$, hence smaller than $i_{k+2}$, contrary to the choice of $a$.

Since $\max(a) < i_k$, requirement (i) (with $k$ changed to $k-1$) in the definition of $p$ says that $B - A'(a) \subseteq i_k$. But $j \in B$ and $j \geq i_k$, so $j \in A'(a)$. Then, since $j < i_{k+1}$ and $n \geq i_{k+2}$, requirement (ii) (with $k$ changed to $k+1$) in the definition of $p$ says that $p^{(5)}(a \cup \{n\})$ forces $j \in A$. This is absurd, because $r$ is an extension of $p^{(5)}(a \cup \{n\})$ and yet it forces $j \notin A$. This contradiction completes the proof of Lemma 10.

**Lemma 11** Under the hypotheses of Lemma 10, $\mathcal{U}$ is a P-point in $V[G]$.

**Proof:** Let countably many sets $X_n \in \mathcal{U}$ be given (in $V[G]$); we seek a $Y \in \mathcal{U}$ almost included in every $X_n$. As each $X_n$ has a subset in $\mathcal{U}$, we may as well suppose that $X_n \subseteq \mathcal{U}$ for all $n$. By Lemmas 1 and 6, there is a countable (in $V$) family $\mathcal{F} \in V$ that contains all the $X_n$'s. As $\mathcal{U}$ is a P-point in $V$, it contains a $Y$ that is almost included in every member of $\mathcal{U} \cap \mathcal{F}$, hence in particular in every $X_n$.

Armed with all these lemmas, we are ready to prove the theorem. Assume the continuum hypothesis in the ground model $V$. Let $\langle P_\alpha, Q_\alpha : \alpha < \aleph_2 \rangle$ be a countable support iteration in which, for each $\alpha$, $P_\alpha$ forces $"Q_\alpha$ is the set of superperfect trees ordered by inclusion". Let $G$ be a V-generic subset of $P = P_\kappa$, the direct limit of the $P_\alpha$'s. For each $\alpha < \aleph_2$, let $G_\alpha$ be the $V$-generic subset $G \cap P_\alpha$ of $P_\alpha$, and let $H_\alpha$ be the $V[G_\alpha]$-generic subset of $Q_\alpha$ (the value in $V[G_\alpha]$ of $Q_\alpha$) such that $G_\alpha * H_\alpha = G_{\alpha+1}$.

Since the continuum hypothesis holds in $V$, there is a P-point in $V$. By Lemmas 5 and 11, it generates a P-point in each $V[G_\alpha]$ and in $V[G]$. We write $\mathcal{U}_0$ for the one in $V[G]$, so, for each $\alpha$, $\mathcal{U}_0 \cap V[G_\alpha]$ is the ultrafilter in $V[G_\alpha]$ generated by $\mathcal{U}_0 \cap V$.

Let $\mathcal{U}$ be an arbitrary nonprincipal ultrafilter on $\omega$ in $V[G]$. By Lemma 3, there is a $\kappa_1$-closed unbounded set of ordinals $\alpha$ such that $\mathcal{U} \cap V[G_\alpha] \in V[G_\alpha]$. For each such $\alpha$, $\mathcal{U} \cap V[G_\alpha]$ is clearly a nonprincipal ultrafilter on $\omega$ in $V[G_\alpha]$. Applying Corollary 1, with $V[G_\alpha]$ as the ground model, $H_\alpha$ as the generic set, and $\mathcal{U} \cap V[G_\alpha]$ and $\mathcal{U}_0 \cap V[G_\alpha]$ as the two ultrafilters, we find that $f_\alpha(\mathcal{U} \cap V[G_\alpha]) = f_\alpha(\mathcal{U}_0 \cap V[G_\alpha])$, where $f_\alpha$ abbreviates $f_{H_\alpha}$ and where the bars mean "filter generated in $V[G_{\alpha+1}]"$ by". Thus,

$$f_\alpha(\mathcal{U}) \supseteq f_\alpha(\mathcal{U} \cap V[G_\alpha]) = f_\alpha(\mathcal{U}_0 \cap V[G_\alpha]) \supseteq f_\alpha(\mathcal{U}_0 \cap V).$$

But $\mathcal{U}_0 \cap V$ generates $\mathcal{U}_0$, so $f_\alpha(\mathcal{U}) \supseteq f_\alpha(\mathcal{U}_0)$. Since $f_\alpha(\mathcal{U}_0)$ is an ultrafilter, it follows that $f_\alpha(\mathcal{U}) = f_\alpha(\mathcal{U}_0)$.

If $\mathcal{U}'$ is another nonprincipal ultrafilter on $\omega$ in $V[G]$, then it too satisfies $f_\alpha(\mathcal{U}') = f_\alpha(\mathcal{U}_0)$ for an $\kappa_1$-closed unbounded class of $\alpha$'s. Since any two $\kappa_1$-closed unbounded subsets of $\kappa_2$ intersect (here we use that $\kappa_2$ is preserved, by Lemma 4), there is an $\alpha$ that works for both $\mathcal{U}$ and $\mathcal{U}'$. So $f_\alpha(\mathcal{U}) = f_\alpha(\mathcal{U}_0) = f_\alpha(\mathcal{U}')$. Since $f_\alpha$ is finite-to-one, this completes the verification of NCF in $V[G]$. 
REFERENCES


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