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Set-Mappings on Dedekind Sets

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Abstract Hajnal's free set principle is equivalent to the axiom of choice, and some of its variants for Dedekind-finite sets are equivalent to countable forms of the axiom of choice.

As was observed by Freiling [4] in the case of $X = \mathbb{R}$, if one applies some heuristics about selecting elements of X at random, one obtains assertions of the form "Every set-mapping $f: X \to I$ ($x \notin f(x)$) has a nontrivial free subset H (i.e., $x \notin f(y)$ for $\{x,y\} \in [H]^2$)", where I is some ideal in $P(X)$ (e.g., I = Lebesgue null sets or I = countable subset). Here we consider the ideal $I = [X]^{wo}$, the well-orderable subsets of X, and we show that some "randomness" axioms with respect to this ideal are equivalent to variants of the axiom of choice AC. (The use of the term "randomness" in this context is justified since, for some sets X, I indeed is an ideal of null sets for some measure; cf. [6], p. 148.)

Theorem 1 *In ZF, the following assertions are equivalent:*
(i) *AC,*
(ii) *For every set-mapping $f: X \to [X]^{wo}$ there is a co-well-orderable free set H (i.e., $X \backslash H \in [X]^{wo}$),*
(iii) *For every set-mapping $f: \kappa \to [\kappa]^{<\lambda}$, $\lambda < |\kappa|$ (λ a well-orderable cardinal number, $|\kappa|$ the not necessarily well-orderable Scott cardinal number of κ), there is a free set H of cardinality $|\kappa|$,*
(iv) *If $S: X^{\leq\theta} \to P(E)$ is a ramification system, then for each $g \in E$ there is an f which is maximal (with respect to inclusion) in $\{h \in X^{\leq\theta} : g \in S(h)\}$.*

Proof: That AC implies (ii) is immediate from the well-ordering theorem; a proof of Hajnal's theorem (iii) is in [3], p. 276, and of the ramification lemma (iv) in [3], p. 83.

(ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i): Let θ be the Hartogs-number of $|\kappa|$ (the cardinality of X). The function $f: \kappa \times \theta \to [\kappa \times \theta]^{<\theta}$ defined by $f(x,\alpha) = \{x\} \times \alpha$ is a set-mapping such that the cardinality $|H \cap (\{x\} \times \theta)| \leq 1$ for all $x \in \kappa$, if H is

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free. Then $|H| \leq \kappa$ and $|\kappa| \leq |(\kappa \times \theta)\backslash H|$. Hence, if the complement of H is well-orderable, so is $\kappa = |X|$. Also, if κ is not well-orderable, then $\theta < |\kappa \times \theta|$ and by (iii) there is a free H such that $|H| = |\kappa \times \theta| > |\kappa|$, a contradiction.

(iv) $\Rightarrow$ (i): θ is the Hartogs-number of X, $E \subseteq X^{\leq\theta}$ is the set of injective functions, and $S(f) = \{g : f \cup g \in E\}$. This is a ramification system and a maximal f such that $\varnothing \in S(f)$ defines a well-ordering of X.

A variant of the following partial axiom of choice was first investigated by Kleinberg [5] in connection with infinitary combinatorics; cf. Blass [1].

$\mathrm{PAC}_{\mathrm{fin}}$: Every infinite family of nonempty finite sets has an infinite subfamily with a choice function.

As follows from [2], if one drops the finiteness condition in the above statement, then one obtains the countable axiom of choice AC^{ω}, and if one restricts $\mathrm{PAC}_{\mathrm{fin}}$ to countable families, then the resulting assertion $\mathrm{PAC}^{\omega}_{\mathrm{fin}}$ is equivalent to $\mathrm{AC}^{\omega}_{\mathrm{fin}}$, the axiom of choice for countable families of finite sets. Also, in ZF-set theory (without the axiom of choice) the following implications are valid: $\mathrm{AC}^{\omega} \Rightarrow \mathrm{W} \Rightarrow \mathrm{PAC}_{\mathrm{fin}} \Rightarrow \mathrm{AC}^{\omega}_{\mathrm{fin}}$; here W is the statement that Dedekind-finite sets are finite. The reverse implications are not provable in ZF alone.

Theorem 2 *In ZF, PAC_{fin} is equivalent to the assertion that every set-mapping $f: X \to [X]^{wo}$, $x \notin f(x)$, on a Dedekind-finite infinite set X has an infinite free subset. AC^{ω}_{fin} is equivalent to the modified statement that every such mapping has arbitrarily large finite free subsets.*

Proof: In order to obtain $\mathrm{PAC}_{\mathrm{fin}}$ from the combinatorial statement, we let $\mathfrak{F}$ be an infinite family of pairwise disjoint nonempty finite sets and set $X = \bigcup \mathfrak{F}$, $f(x) = F\backslash\{x\}$ for $x \in F \in \mathfrak{F}$. If $Y \subseteq X$ is an infinite free subset, then the cardinality $|Y \cap F| \leq 1$ for $F \in \mathfrak{F}$, whence $\mathcal{G} = \{F \in \mathfrak{F} : F \cap Y \neq \varnothing\}$ is an infinite subfamily of $\mathfrak{F}$ which has a choice-function. On the other hand, if X is not Dedekind-finite, then one defines such a family with the help of a countably infinite $Y \subseteq X$.

Next, we assume $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ and prove the weak free set principle. Let $f: X \to [X]^{wo}$ be a set-mapping on the Dedekind-finite but infinite set X. Then $f(x)$ is finite and we set $X_n = \{x \in X : |f(x)| = n\}$. By $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ some X_n is infinite, $n \geq 0$, for otherwise X could not be a Dedekind-set. We now show that there are arbitrarily large finite free subsets of X_n. If not, then each free subset has at most N elements, where $\mathrm{N} \geq 1$, and therefore each $x \in Y \subseteq X_n$, Y arbitrary, is contained in a free set which is maximal in Y. Therefore we can find finite sequences x_i, Y_i, $i \leq n + 1$, such that $x_i \in Y_i$ and Y_i is maximal free within $X_n \backslash \bigcup\{Y_j, f''Y_j : j \in i\}$. Then $x_{n+1} \in X_n \backslash Y_i$, $i \leq n$, whence by maximality $Y_i \cup \{x_{n+1}\}$ is not free and so $f(x_{n+1}) \cap Y_i \neq \varnothing$, contradicting $|f(x_{n+1})| \leq n$.

We now prove the strong free set principle. From n applications of $\mathrm{PAC}_{\mathrm{fin}}$ we get an infinite $Y \subseteq X_n$ and mappings $f_i: Y \to X$, $i \in n$, such that $f(x) = \{f_i(x) : i \in n\}$. In view of an obvious inductive argument it suffices to find an infinite set $Z \subseteq Y$ such that $f_i''Z \cap Z = \varnothing$; i.e., we may assume that $n = 1$, $Y = X$, and $f(x) = \{g(x)\}$ where $g(x) \neq x$. We define an equivalence relation $x \approx y$ iff $g^l(x) = g^m(y)$ for some $l, m \geq 0$. If C is an equivalence class, then we set $C_m = \{x \in C : g^m(x) = x$ and $g^k x \neq x$ for $1 \leq k < m\}$. Since X is Dedekind-finite, $C_m \neq \varnothing$ for some $m \geq 2$. Moreover, we observe that $C^1 =$

$\bigcup \{C_m : m \geq 2\}$ is finite, for if $x \in C$ then $C^1 \subseteq \{g^k x : k \geq 0\}$, a finite set ($g^0 = \mathrm{id}$). If there are infinitely many equivalence classes, $\mathrm{PAC_{fin}}$ provides us with an infinite $Z \subseteq X$ such that $|Z \cap C| = |Z \cap C^1| \leq 1$ for each class C and so $g''Z \cap Z = \emptyset$. Otherwise there exists an infinite equivalence class C. We define

$$C^{l+1} = g^{-1}(C^l) \setminus \bigcup \{C^k : k \leq l\},\ l \geq 1.$$

This defines a family of pairwise disjoint sets, whence in view of $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ either $C^l = \emptyset$ or C^l is infinite for some $l \geq 2$, for in the contrary case X could not be Dedekind-finite. The negation of the latter case is impossible since otherwise $C \subseteq \bigcup \{C^k : k \in l\}$, a finite set, while in the latter case we may set $Z = C^l$.

Finally, we show that the weakened assertion implies $\mathrm{AC}^{\omega}_{\mathrm{fin}}$. Let $\langle F_n : n \in \omega\rangle$ be a sequence of pairwise disjoint nonempty finite sets— and $X = \bigcup \{F_n : n \in \omega\}$ —which is a counterexample to $\mathrm{PAC}^{\omega}_{\mathrm{fin}}$. Then every free set for the set-mapping $f(x) = \bigcup \{F_k : k \leq n\} \setminus \{x\}$, $x \in F_n$, is a singleton and X is Dedekind-finite, whence $\langle X, f\rangle$ is a counterexample to the weak version of the free set principle.

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